**School mathematics as a special kind of mathematics**

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**Introduction**

The substance of this paper was presented originally at a working group on “Disciplinary Mathematics and School Mathematics’”[[1]](#footnote-1). It has long been an aim of mine that school students should be introduced to authentic mathematical activity such as is practised by professional mathematicians, and those forms of exploration that contribute to the development of the subject. Through this kind of activity, students get a sense of mathematics as human invention, as certain habits of mind, that is more engaging and meaningful than learning a procession of given facts, methods and question-types. For example, in one school well-known to me, all mathematics classrooms display a text derived from Leone Burton’s descriptions of professional mathematical behaviour (2004):

“Mathematicians…

* Have imaginative ideas
* Ask questions
* Make mistakes and use them to learn new things
* Are organised and systematic
* Describe, explain and discuss their work
* Look for patterns and connections
* Keep going when it is difficult

Together we can learn to be mathematicians”

For the working group I chose to explore how far it makes sense to expect that school students can act like mathematicians. Before I start this exploration, let me make it clear that my own experience as a teacher convinces me that nearly all students can indeed act in the ways described above, and can also generate mathematical ideas, identify and describe relationships between properties, construct effective algorithms. It is also my experience that those who have the opportunity to do this become better mathematical learners than those drilled in more traditional methods. This is now so well supported by research that it hardly needs stating. But to say that mathematical learning is better if learners ‘act like mathematicians’ is not to say that it is possible to do this in institutional settings. In this paper I take ‘school mathematics’ to be the institutionalization of mathematical knowledge for novices, and it is the relationship between this and the discipline of mathematics that I am setting out to describe.

I shall claim that school mathematics is not, and perhaps never can be, a subset of the recognised discipline of mathematics, because it has different warrants, authorities, forms of reasoning, core activities, purposes and unifying concepts, and necessarily truncates mathematical activity in ways that are different from those of the discipline. By ‘discipline of mathematics’ I mean the activities which advance mathematical knowledge: the forms of engagement, kinds of questions, and standards of argument which are accepted as contributing to the conventional canon of pure or applied mathematics. ‘School mathematics’ means the forms of engagement in mathematics in formal teaching contexts for the novitiate, including some undergraduates, or for those who do not even see themselves as novices but have mathematics thrust upon them.

In its worst form, school mathematics can be a form of cognitive bullying which neither develops students’ natural ways of thinking in advantageous directions, nor leads obviously towards competence in pure or applied mathematics as practised by adult experts. The relationship of school mathematics to adult competence is similar to the relationship between doing military drill and military leadership; between being made to eat all your spinach and becoming a chef; between being forced to practise scales and becoming a pianist. There are some connections, but they are about having a focus on a narrow subset of semi-fluent expertise in negative social and emotional contexts, without full purpose, context and meaning. That some people become effective military leaders, beautiful pianists or inspiring cooks is interesting, but what is more interesting is the fact that most people who go through these early experiences do not: instead they merely follow orders, or hate green vegetables, or give up practising their instruments.

The image around which I hang this paper is the cafeteria at the heart of the mathematics faculty at Cambridge. It is a large room with coffee at one end, small tables covered with papers and laptops, each surrounded by four chairs, students (mainly male), some undergraduate, some graduate, pure and applied, some alone, some in casual groups, some in self-organised study groups with their own internal disciplines and plans, taking time, talking, arguing. There are many furrowed brows, people leaning forward with arms and hands working away to express some thought; bodies, words and minds hauling together to communicate how they see relationships and properties, and offering each other handles to work with abstract objects and multi-dimensional extensions. No one is doing exercises or practising techniques; no one is interrupted by bells or instructions to change to the next task; no one is taking their work to teachers to be marked. Every now and then there are whoops of excitement or groans of frustrated realisation.

I recognise that behind such an apparently idyllic scene there are a variety of pressures and frustrations which arise from institutional demands and the lived experience of mathematicians and mathematics students. One might also say that Cambridge is not typical, or question the dominance of men and ask who is excluded, or simply laugh at how anyone could want to replicate such behaviour. Nevertheless, this picture is of people doing mathematics together, in ways which fit the description ‘acting like mathematicians’. It is also possible to replicate much of this scene in school classrooms. But even in these environments, where teachers may believe that their students are ‘working like mathematicians’, there are major significant differences between school and ‘the discipline’.

**Mathematical enquiry**

The students in the cafeteria are in various stages of transition between school maths and academic maths. With a few exceptions they have rarely worked like this at school and only a few may have been told explicitly that this is how they could be working now. Mathematics as a discipline includes the social and cultural characteristics which have contributed to its genesis. Most mathematical advancement does not arise in an isolated, independent way but is a product of its time, within the current paradigms, co-emergent with current technological and economic needs and tools, co-emergent with the *zeitgeist*. Locally and globally, it is a product of social interaction, even if that interaction does not take place until there is a result (Davis & Hersh 1981, Burton 2004).

In school things are generally different. I am not assuming any homogeneity about school practices – it is always possible to read such assumptions and argue to exclude your particular patch of the world, or particular teachers, or a particular curriculum. Instead, I focus on the mathematical practices in which, I presume, the students described above are engaged but which cannot be observed in the short-term, nor do they fit neatly into socio-cultural analyses of the learning environment. For me, the starting point is what it means to do mathematics, and to be mathematically engaged.

In the discipline of mathematics, mathematics is the mode of intellectual enquiry, and effective methods of enquiry become part of the discipline – so much so that mathematics theses do not have chapters explaining methodology and methods. This is not just about the availability of coffee, social groupings, the published discourse, or choices between Maple and Mathematica, although these are crucial parts of the picture. ‘Doing mathematics’ is predominantly about empirical exploration, logical deduction, seeking variance and invariance, selecting or devising representations, exemplification, observing extreme cases, conjecturing, seeking relationships, verification, reification, formalisation, locating isomorphisms, reflecting on answers as raw material for further conjecture, comparing argumentations for accuracy, validity, insight, efficiency and power. It is also about reworking to find errors in technical accuracy, and errors in argument, and looking actively for counterexamples and refutations. It is about creating methods of problem-presentation and solution for particular purposes, tinkering between physicals situations and their models, and it also involves proving theorems. Of course all these can be done in school, but it is the way in which these are coordinated and aligned that makes the overall activity different.

**Different kinds of mathematics**

The practices of mathematics just described are very far indeed from the concerns of psychologists who want to construct efficient methods of instruction, seeking for the fastest and most productive ways to teach students how to find answers to broadly isomorphic problems. Research and development in psychology and neuroscience do not even begin to offer ways to induct students into the practices listed above. My perspective on the discipline of mathematics is also different from the concerns of the socio-cultural take on classrooms and learners, which tells convincing and robust stories about the existence of differences in practice, and the process of development of identity in different cultural settings, but which cannot put detailed flesh on these stories in terms of the development of specific mathematical ideas.

Of course we would expect to see different kinds of learning and different classroom practices where there are different views of mathematics and different curriculum goals, and in particular we would expect to see a political dimension to this as countries connect the outcomes of education with their future workforce. As Luis Radford has said (2003)

While the humanist view of mathematics emphasizes the role this discipline bears in the development of logical thinking, abstraction, rigor and other highly prized faculties that were the clear marks of men of sophisticated spirit since the Enlightenment, the socialist trend stressed the applicability of mathematics. Its importance is seen in terms of the utilitarian dimension of mathematics to master nature in the interest of mankind.

If the curriculum takes a ‘humanist’ approach, rigorous argument and proof are taken to be important aspects of the subject, yet many teachers treat ‘proof’ as a topic within mathematics rather than as the way truth is examined and warranted throughout the subject, and argument as a specific focus for particular tasks or lessons. If this is combined with fallibilist-inspired teaching styles such as investigative work, ‘what if…?’ questions, and algebra introduced as an expression of generality, then in many classrooms two standards of argument are being offered. One is empirical, arising from specific cases, tables of values, inductive construction of formulae and testing special cases, while the other requires deductive reasoning and a willingness to engage in formal logical argument. The shift learners have to make between inductive to deductive argument is hard and demands some ‘giving up’ of forms of reasoning which serve students well elsewhere. For example, it is well-documented that students who can produce examples easily, in an inductive mood, find the production and understanding of counter-examples, with their deductive role, much harder (Zaslavsky & Ron 1998). To use mathematical argument as the natural and normal method of enquiry in mathematics it needs to be fully embedded in day-to-day lessons as part of classroom discourse. In reality, a curriculum written in accordance with what Radford calls ‘humanist’ principles is likely to be presented as a taught curriculum in which topics are presented in a developmental order and few students will see the role for logical thinking embedded in the development of mathematical understanding. If their own logical thinking *does* play a part in mathematics lessons, as it does with some teachers, these humanist curriculum aims might be realized in part.

Learners exposed to utilitarian mathematics have a different experience to those taught with a more abstract view, but solving realistic and everyday problems need not lead them to understand the role of mathematics beyond providing *ad hoc* methods for real problem-solving, or as a service subject which holds tools for moving forward in other domains.

**Mathematical shifts of understanding in school mathematics**

But the choice between different possible ‘mathematics’ is, to my mind and in my experience, not the core problem about mismatch between school mathematics and the discipline of mathematics, as I intend to show. I am going to describe several shifts of understanding which have to be made for learners to be successful within and beyond school mathematics, whatever the dominant view of the discipline:

*Additive to multiplicative reasoning:* A shift from seeing additively to seeing multiplicatively is expected to take place during late primary or early secondary school. Not everyone makes this shift successfully, and multiplication seen as ‘repeated addition’ lingers as a dominant image for many students. This is unhelpful for learners who need to work with ratio, to express algebraic relationships, to understand polynomials, to recognise and use transformations and similarity, and in many other mathematical and other contexts. Eventually another shift has to be made to exponential reasoning.

*Probabilistic reasoning:* The concept of probability, understood mathematically, offers a different warrant for truth than is associated with either deductive logic or induction from empirical evidence. To understand probability as a tool, or to see the world probabilistically, requires abstraction and imagination well beyond observable phenomena. Moreover, one cannot merely follow algorithms to get answers except in very simple contexts; often learners have to decide for themselves whether events are independent, exclusive or not. The shift here is from being told everything about a situation to having to identify characteristics and properties for oneself, before conceptually-based action.

*Integration:* In the UK, integration used to be the first context in the school curriculum where learners could not merely apply methods and be sure they would get some kind of answer. However students now are told not only what method to use, but also what substitutions to make if substitution is the method examiners wish to see. The shift from being told what methods and tools to use, to developing sensible selection criteria for what is possible, has to be made to become a mathematician.

*Geometrical reasoning:* Questions involving application of theorems can be avoided in UK national tests at 16+ and students still be awarded the highest grades. Theorems and proof of any kind, let alone geometrical contexts, do not play a part in higher school examinations. In countries where reasoning from axioms still has a place in the school curriculum these may be taught mechanically, as a kind of memory or question-spotting activity, rather than as a demonstration of deductive reasoning to explore phenomena and to establish a particular kind of truth. A shift from knowing what to look for, to selecting what to look at, then to deciding what to use and constructing multi-stage arguments for oneself, has to be made to become a mathematician.

*Getting answers to gaining insight:* a problem (in Polya’s sense) can be solved and nothing new be learnt, even about problem-solving. A mathematician will usually have a purpose in mind when solving a problem, so that the outcome is used to reflect on the context in which the problem arose, to decide if something unexpected has arisen, to raise further questions, or in some other way to enrich or extend knowledge. The answer, if there is one, is not the end of the process. A shift from getting answers to gaining insight or constructing arguments has to be made. In non-mathematical contexts, once the problem is solved there is no motive for extending the work hypothetically.

*Modelling:* Mathematical modelling of realistic or artificial situations is a feature of many mathematics curricula. While generalisation might take place in order to create the model, and this might be explored further to look for extreme cases or further variation, the modelling process itself does not require more abstraction or structural understanding than the situation being modelled.

The shifts of perception, attention and engagement just described do not take place without the learner being in an environment of tasks, language, representation, deliberately designed to support those shifts, that is – a schooling environment whose purpose is to ensure learners make these shifts.

**The discipline of school mathematics**

It is not mere coincidence that these are all inherently ‘hard-to-teach’ aspects of mathematics. They all offer epistemological obstacles which require shifts of reasoning, or new ways to act with imagery, or encapsulation of previous experience to be overcome.

Importantly, it is becoming more and more the case that, in an effort to ensure that more students can gain school mathematics qualifications, the difficult shifts which would have to be made for school mathematics to be a subset of the discipline of mathematics are being edited *out* of mathematics as a school subject, rather than edited *in* as the discipline itself becomes more complex, more post-modern and less certain. I am talking here of all kinds of classrooms: reform and traditional; classrooms in which students are expected to behave like little mathematicians and those where they are expected to behave like acquiescent cognitive machines; classrooms in the developing world and those in high-achieving countries. I am not arguing for or against particular kinds of school curriculum; I am saying that the task of schooling the mathematical mind to make the shifts which have to be made to become a mathematician is not, by and large, undertaken in schools.

In all the curriculum contexts I described above, and in many others, there are features which are peculiar to school mathematics, and the way it is generally taught, which are not part of the discipline as practised by adult experts. For example:

* there is a strong focus on answers and generalisations rather than structural insight and abstraction
* there is avoidance by teachers, tests, and curricula, of the need for uncertain choices
* curricula seem to cling to topics and approaches which can be represented experientially, diagrammatically or in concrete ways, rather than in abstract and imaginary ways
* in school, inductive, empirical, and ad hoc reasoning are privileged over deductive or probabilistic reasoning.

These characteristics, peculiar to school mathematics, can hinder the progress of students at university on pure and applied courses, alongside inappropriate work habits, challenges to identity, and unrealistic expectations of the subject (sometimes promulgated by popularisers). Mathematics as a discipline, by contrast to school mathematics, is concerned with thought, structure, alternatives, abstract ideas, deductive reasoning and an internal sense of validity and authority. It is also concerned with uncertainties about ways forward in its own realms of enquiry. To do maths includes holding nagging questions in the mind while carrying on with life, and not expecting answers to be found, problems to be solved, within the confines of a particular room or timescale. The concerns of school mathematics pull learners in directions which differ from these. The core activity in school mathematics is to learn to use mathematical tools and ways of working so that these can be used to learn more tools and ways of working later on.

Whichever approach is taken by the curriculum, most school students are taught and examined on mathematics of a kind which is done, both in the academy and in other workplaces, by machines. They are taught not as an apprenticeship to adult mathematics users, but as bottom-up preparation for future mathematical activity which might one day be meaningful, either as an intellectual or an economic activity. Furthermore, they are taught this in regimented settings, with short time-scales, by teachers who themselves have limited experience of the mathematical practices described above.

Those students who continue beyond school might be motivated by a so-far satisfied need to have right answers, or by getting a kick from the resolution of puzzles, or by discovering special features which intrigue them, or by anticipating the delayed gratification of getting a further qualification, or of finally being able to study at the cutting edge. They might also have what Krutetskii identified as common features of mathematically gifted students: propensities to see the world and organize their mental activity in certain mathematical ways (1976). The list he identified has little in common with the contents of formal taught lessons and assessment regimes, and much to do with the list of practices above, and what Cuoco and his colleagues have called mathematical ’habits of mind’ (1996). Neither a curriculum based on rigour, historical genesis and conceptual development, nor a curriculum based on modelling and authentic contextual questions, can achieve the education of the mind required to engage consistently, habitually, with the normal intellectual practices of mathematicians.

**The roles of unifying concepts**

A further striking difference is in the role of unifying concepts in mathematics as a discipline and in school mathematics. In mathematics as a discipline these orientate much of what is studied and researched. Such theories provide opportunities for links and connections within the discipline, new languages for discussing ideas, and new questions to be explored. In school mathematics, however, the shaping of a curriculum with unifying concepts could be unhelpful. A unifying concept makes sense to mathematicians precisely because it offers unification of previously disparate ideas of which they have a range of experience. For school students, to be told about a unifying concept before experiencing several widely differing examples of it reduces the concept to ‘something else to be learnt’ or ‘something else to be taught’ rather than an enabling encapsulation. Linearity, for example, crops up throughout school and could have been included in the ‘hard to teach’ list above, because it requires a shift from looking at functions as representations of relationships to comparing properties of functions. However, this only makes sense when a learner has experienced several different linear and non-linear situations so that there is a need for a word for those which behave in certain ways. Older mathematicians are supposed to have had a range of experiences to draw on when they meet, for example, vector spaces so that they have examples which might be activated by hearing definitions, theorems and questions (Watson & Mason 2002). To use the idea of a unifying concept successfully in school mathematics would require vertical coordination of teaching across years by teachers who have constructed a shared understanding of how that concept will be approached and learnt. In the discipline of mathematics, unifying concepts arise through the published and broadcast insights of mathematicians drawing on the processes of historical cultural genesis – they move the field on.

**Constraints on teaching which alter the discipline**

It certainly is the role of school mathematics to provide a range of experiences of various kinds so that students understand the usefulness of mathematics, and can do various necessary calculations and estimations. It would also be helpful for students to understand the purpose and value of practice, and of basic standards of accuracy. School could introduce students to ways of working on mathematics, to the kinds of questions mathematicians ask and the subject matter worked on, but would have to include in that some of the practices of number systems, algebraic manipulation, and ways of using diagrams, which are assumed within the discipline. At the most extreme, schools use entirely different kinds of questions, enquiry, warrants and work habits than those of the discipline. At best, they introduce what is to come, while moulding it to fit the institutional constraints, rather than to fit the development of mathematical ideas.

Many teachers and projects try to make their classrooms more and more like mathematical workshops, but the school context overlays these with purposes which are not about the development of mathematics, but are more about learning what to do and how to be. In the best of these classrooms, teachers are explicit about appropriate forms of statement and discussion in ways which are not made explicit in mathematics faculties, but even the best teachers cannot be present at every student’s side to be explicit about the many practices of professional mathematics listed above. Mathematics as a discipline includes discussion and critique of its own modes of enquiry, but does not include explicit teaching of these modes of enquiry, or practising core techniques, as school mathematics has to do. Mathematicians decide for themselves what they need to practise, and how to practise it, and how to validate their techniques.

In school the truncation of mathematical activity takes place when enough has been learnt for now, or at the end of a lesson, or when the curriculum requires moving on, or a syllabus has been ‘covered’, or students leave school. In maths the truncation of enquiry happens when a problem has been solved, a proof has been accepted, a model has been produced, or everyone is tired of the enquiry – but all of these can be the starting points for new enquiry. Of course temporary truncation takes place because of teaching, administrative and family commitments, but the mathematician is not expected simultaneously to worry about academic problems in geography, language, science, citizenship as well as mathematics.

Another aspect of school mathematics which differs from the discipline of mathematics is the nature of authority. Vergnaud (1997), Freudenthal (1973) and others have emphasised that mathematics includes its own methods of validation, and a good teacher can enable students to use the structures of mathematics to verify their own work and ideas. In the education system as a whole, however, validation and authority are highly structured through textbooks, examinations, and curriculum systems rather than by mathematicians working as a community. Formal axiomatic proof is inaccessible for most school-age children, although many children of all ages can develop robust arguments with appropriate scaffolding. Even so, for most of what is taught in school the correct application of procedures and checking answers using inverse procedures, or checking reasonableness, are available as warrants for truth, and the prevailing warrants in classrooms are usually teacher’s approval, or checking answers.

**Conclusion**

In conclusion, therefore, I find myself claiming that school mathematics is necessarily not a subset of the discipline of mathematics, whatever the nature of mathematics being taught, whatever the way it is taught. Essentially, this is for the following reasons:

* Schools have to prepare students for future study and/or employment and this requires them to be explicit about ways to develop recall, fluency, accuracy, and ways of working; these constitute the major part of the goals of school mathematics
* Mathematics requires shifts away from the everyday thinking of mathematics lessons to special forms of mathematical thinking. Someone has to do this work, which requires knowledgeable teachers in school and university. We have to recognise that many school teachers of mathematics do not have personal experience of what it means to do mathematics over time, exploring questions which have intellectual purpose, not pedagogic purpose.
* Limited time slots, curricular pressures, and assessment regimes constrain or prevent the development of the kinds of questions and ways of working which characterise the discipline
* Authority in school mathematics lies with teachers, textbooks and assessment regimes, not with mathematical argument.

It is not only ways of working and goals that are different between school and disciplinary maths; it is the way that these shape the available forms of mathematical enquiry that makes school mathematics a different discipline, with its own rules, purposes, authorities and warrants.

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1. Symposium on the Occasion of the 100th Anniversary of ICMI

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