The role of teachers’ knowledge of functions in their teaching: a conceptual approach with illustrations from three cases

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### Abstract

In this paper we develop our understanding of how teachers’ personal mathematical knowledge informs their teaching. We do this by analysing the development of the function concept in secondary mathematics and identifying the necessary contributory concepts. We observe how the function concept is treated in various textbooks to show that personal knowledge can, and possibly should, augment or replace them. We then report on some instances in the pedagogy of three teachers who have high levels of personal mathematical knowledge, and describe how these manifest their personal knowledge. Finally we introduce a theory of continuous pedagogy for which their practice provides illustrations.

**Running head: The role of teachers’ knowledge of functions**

**Keywords: teachers’ mathematical knowledge; mathematical functions; teachers’ personal subject knowledge**

### Introduction to perspectives on mathematical knowledge in teaching

In this paper we aim to understand more about how teachers’ personal mathematical knowledge informs their teaching. Research takes various approaches to this issue. There is no straightforward correlation between qualifications and teaching effectiveness because many other factors, pedagogic and systemic, also contribute (Harris & Rutledge, 2007). However, a recent comprehensive synthesis of the literature on the link between teacher’s quality and student outcome highlights a critical finding: the quality of the teacher’s knowledge of mathematics and of pedagogical mathematical knowledge is positively correlated with student achievement in all grades but *particularly in the secondary level* (Goe, 2007). There are two main methods of categorising teachers’ mathematical knowledge: the first being the production of taxonomies of knowledge types which teachers use while planning and during teaching (e.g. Ball, Thames & Phelps 2008), the second being overlapping sets of knowledge-informed pedagogic actions (Rowland et al. 2009). A different approach is to see the interplay of knowing and teaching as holistic and complex (Davis & Simmt 2006; Davis & Mason this issue; Handa 2011), depending on how mathematical knowledge becomes available to teachers during their teaching. One entailment of this latter approach is that it allows for how teachers can themselves learn more mathematics through attentive and responsive teaching (Leikin & Zazkis 2007). Another is that teaching can be seen as a kind of mathematical enquiry in which the effects of tasks on learners’ mathematical conceptualisations are observed (Watson & Barton 2011). In Handa’s view, teaching is a manifestation of the relationship between the teacher and the subject matter.

At stake here are not only different views of the nature of mathematical knowledge, nor only the different ways in which it is ‘held’ by individuals, but also whether it is manifested in practice as discrete events or as a continuous background (Warburton, 2012). Mason (1998) expands on this in his explorations into levels of awareness and ‘awareness of awareness’ or, as Damasio describes it (2000), our consciousness of something happening, and what that ‘something’ might be. If indeed teachers’ knowledge is held as a continuous presence of mathematical awareness, then we would expect a teacher who is working ‘in flow’ (Csikszentmihalyi,1992) to exhibit evidence of that knowledge not only in discrete, deliberate, declarative forms but also in the spontaneous acts and utterances that connect their planned acts and utterances. ‘In flow’ implies action which, to the actor, totally engages the mind and body in a continuous, subconscious, coherent sequence that is not interrupted by conscious or deliberate intervention. The use of personal mathematical knowledge when ‘in flow’ is inherent and automatic.

Evidence of personal mathematical knowledge contributing throughout teaching was suggested by Watson & De Geest (2011), who found some differences between specialist and non-specialist teachers who were skilled, committed and collaborative. The discussions of the implications of work done, the affirmations of the value of new mathematical knowledge, and the integrations of mathematical ideas were, with the specialist teachers, often about mathematics. For non-specialists they were never about mathematics, but drew on applications outside mathematics, cultural status (e.g. ‘now you know what people mean when they talk about Pythagoras’) or the requirements of upcoming assessments. It is possible for a teacher whose knowledge is limited to help students to pass a limited test on the subject, and it is possible for a teacher who knows a lot to teach ineffectively, but only a knowledgeable teacher can have this ‘in flow’ knowledge as a component of their own mathematical identity.

We chose ‘functions’ as an arena to explore this idea of teachers’ in-flow knowledge further, for several reasons:

* Understanding functions well requires significant shifts in focus, that is significant changes of meaning and levels of objectification, that might occur throughout the curriculum (Harel, 2009; Leinhardt, Zaslavsky & Stein 1990)
* To develop these levels of understanding requires coordination of elementary knowledge and consolidation of the function concept (Hershkowitz, Schwarz & Dreyfus 2001)
* A strong understanding of functions makes a difference to students’ ability to study mathematics and related subjects further, for example by seeing the solution of equations as the intersection of functions (Yerushalmy 1991).
* We have both found weaknesses in textbook treatment of functions (Harel 2009).

In this paper we first analyse the conceptual imperatives associated with a good understanding of functions. The key question is then: can teaching afford the later development of these advanced understandings? Our discussion takes place in two rather different contexts. We set out to answer this question by looking at (a) textbook treatment of functions in a textbook-orientated culture (US) and (b) ‘in flow’ manifestations of knowledge in a culture of relative teacher autonomy[[1]](#footnote-1) (UK).

**Learning the concept of function**

While not all students will go on to study functions at an advanced level, nevertheless the teaching they experience should provide the groundwork for advanced understanding. O’Callaghan (1998) identified five areas of expertise with functions that need to be developed: modelling, interpreting, translating between representations, treating them as objects, and hence being able to act on them. To achieve such expertise, students have to move between seeing functions as processes to conceptualising them as objects (Tall, 1992); from acting on them as objects to thinking about what is invariant under such actions; and from using prototypes to developing a wider repertoire of examples (Schwarz & Hershkowitz, 1999; Curcio,1987).

A lot can go wrong: emphasis on mapping discrete sets of data to each other can create the view that a function is a set of discrete data, and that anything that can be mapped is a function (Spyrou & Zagorianakos, 2010); limited experience can create an impression that all functions are continuous, smooth and calculable (Leinhardt et al., 1990), so that the later appearance of non-calculable functions is some kind of aberration. Function notation presents problems (Sajka 2003), in particular if for students whose previous experience of use of letters in mathematics has led to the limited meaning that letters stand for numbers.

Working ‘in flow’ requires deep mathematical knowledge. For example, teachers who are assigned to teach calculus are expected to possess at least the ability to translate flexibly from an equation to function and vice versa. Additionally, they need to conceive composition of functions as a description of related variations, and to understand that physical phenomena usually do not occur in isolation but are often related. Such related variations are represented by composition of function and their rate of change is accordingly represented and investigated through the Chain Rule. The literature on the concept of function has addressed extensively what such an understanding might look like and we have summarised some aspects above. To this, we add two advanced understandings, not previously explicitly addressed in the literature. The first is the ability to differentiate between a function and its representation. A manifestation of this ability is the understanding that characteristics of a function are invariant under a change of the coordinate system in which the function is represented; for example, that the form of the graph of a function is independent of the coordinate system used to represent the function; that if a function is continuous at a particular point in one coordinate system, it is continuous at that point in any other coordinate system; that if a function is differentiable at a particular point in one coordinate system, it is differentiable at that point in any other coordinate system. The second is the ability to differentiate between a function as a mapping satisfying the ‘single value’ property from its instantiation; that is, the difference between  and . The latter is an instantiation of the general concept of function through a rule:. On a spectrum ranging from the process-conception of function to object-conception of function, this way of understanding is very close to the latter conception. It allows the student to understand, for example, differentiation as an operator: given a function  the operator, say , returns the no more than function .

**Current approaches to the concept of function and its role in the curriculum**

In the UK, curriculum content is governed according to a national curriculum. At the time of writing[[2]](#footnote-2) the national curriculum requires that students between 11 and 14 years should learn about: polynomial graphs, sequences and functions based on ‘simple rules and relationships’. From 14 to 16 years students should learn about graphs of exponential and trigonometric functions and transformation of functions (QCDA, 2007). After that, functions arise extensively in trigonometry and early calculus, and the language used in documents distinguishes between functions and their representations. Functions as a separate formal and abstract topic of study is included only for 17-18 year olds specialising in mathematics, for example they are expected to:

(a) understand the terms function, domain, range, one-one function, inverse function and composition of functions;

(b) identify the range of a given function, and find the composition of two given functions;

(c) determine whether or not a given function is one-to-one, and find the inverse of a one-to-one function in simple cases;

(d) illustrate in graphical terms the relation between a one-to-one function and its inverse.

Within this framework, it is up to teachers how functions are taught. Some textbook series avoid the word ‘function’ and talk only of equations, graphs and relations, except for particular functions such as exponential or trigonometric, until the post-16 specialist courses of study. Others introduce the word ‘function’ much earlier, and use it in the context of graphs, algebraic generalisations from sequences, using inverse flow diagrams to solve linear equations, and also to relate simultaneous equations to the intersection of functions. Teachers in the UK tend to use textbooks with some flexibility as one among many resources so the content of the textbooks in UK, while it often relates closely to the format of high stakes assessments, cannot be regarded as indicative of actual teaching.

Harel has undertaken an extensive analysis of the treatment of functions in popular US textbooks (2009). At this level, teaching and learning mathematics in the US is highly dependent, far more so than in UK, on textbooks which provide the curriculum so this analysis gives a ‘most likely’ picture of students’ experience of functions.

In general, the introduction of the concept of function in many widely-used textbooks in the US can be characterized as being abrupt, flat, and intellectually aimless. Typically, in these textbooks, the chapter on functions begins, abruptly and without adequate motivation, by defining the concepts of relation, function, domain, and range in ‘one breath’—often on the same first page of the chapter. In one of these texts, these definitions are preceded by the following ‘goal’ and ‘motivation’: *We learn to identify functions in order to determine whether a relation is a function*.

This introduction also suffers from ‘flatness’—any concept appears to be as important as any other. Teachers using these textbooks tend to devote the same attention to ‘relation’ (a concept that has little or no value for secondary mathematics) as to ‘function’ which is perhaps the single most important concept in all branches of mathematics. Some textbooks go even further and define *equivalence relation*, relation that is *reflexive, symmetric,* and *transitive.* The concept of relations, in general, and equivalence relation, in particular, is of importance in advanced college mathematics. Many high-school textbooks teach students that *equality* is a relation and that statement such as  must be justified by the axiom that this relation is reflexive. Such a treatment is superfluous for the majority of high school students, and in our experience for many college students as well. This focus on ‘relation’ distracts students from the conceptual development of function. The absence of intellectual motivation for ‘function’ is also clearly evident since what we have described is a sequence of terms and definitions which do not all appear to be associate closely with the function concept.

When examining the content further, one discovers many other problems (see Harel, 2009). For example, in most textbooks problems dealing with general forms of functions are rare, and the advantage of general algebraic approaches over the other approaches (the use of tables, graphs, and calculators, for example) in logical deduction is not clear. Except for the quadratic formula, fundamental theorems on linear functions and quadratic functions are not justified. This includes important theorems, such as: ‘A line in the plane is represented by a linear function, and the graph of a linear equation is a line’; ‘The graph of the function is symmetric and the shape of its graph is determined by the coefficients,, , and .’ Such theorems are often demonstrated empirically without being backed by reasoning about connections between the form of the function expression and its graphical shape.

While we cannot generalise about early textbook treatment of functions in UK, nor about how they are taught in the pre-specialist phase, most textbooks for the post-16 phase in the UK are closely tied to the assessment requirements and exhibit similar features to those in the US, although we have not analysed them in detail and the association with relations seems less frequent. We shall look at a particular UK textbook treatment of functions later. The exceptions to this generalisation introduce functions in the context of modelling realistic situations, a utilitarian approach which does not necessitate studying them as a class of objects in their own right but does develop a repertoire of functions to use in modelling situations (Dickinson et al., 2012). The onus is therefore on teachers to provide an adequate preparation through what we have chosen to call ‘function-aware teaching’ to develop the ‘intellectual need’ (Harel, in press) that underpins the place of functions in mathematics, and to avoid over-simplification that limits future understanding.

**Teachers’ tacit use of mathematics ‘in flow’**

We now turn our attention to three knowledgeable UK teachers using their knowledge of functions ‘in flow’ while teaching other aspects of mathematics and conjecture about how this prepares their students for the ‘intellectual necessity’ of functions. We also look at the textbooks are available for two of the teachers in their classrooms.

Earlier we conjectured that ‘in flow’ teaching manifests personal subject knowledge. If this is true in the case of functions, then opportunistic observations of highly-qualified teachers should yield examples of how their knowledge of functions influences their ‘in flow’ teaching. Three examples of ‘in flow’ function-aware teaching are given here: two from our intentional collection of data, and one from serendipity. This latter example will be described after the following report of two teachers who were deliberately observed.

Anne gained permission to observe two qualified mathematics specialist teachers for four days, teaching a range of topics to a range of age groups aged between 12 and 18. Mathematics teaching in UK is taught as one compulsory subject up to age 16, students usually being grouped according to prior attainment, and after that some students elect to follow a further course of study until 18. Paul and Linda (pseudonyms) both have honours degrees in mathematics, both are trained experienced teachers, working in a non-selective school in a neighbourhood of mixed socio-economic backgrounds and teaching across the whole age range from 11 to 18. They were told that the purpose of observation was to identify how their own mathematical knowledge influenced their teaching, but to avoid influencing the content of lessons they were not told that the specific focus was their knowledge of functions[[3]](#footnote-3). Observation was in the form of field notes about what they said to the whole class and to individuals, what they wrote on the board, and what they presented using digital and other technologies. Discussion took place with individual teachers after each lesson to clarify the data and to find out more about their actions and intentions. Analysis was low-inference, being restricted to identifying the mathematical content of what had actually been said, written, displayed or enacted.

### Paul

Paul was observed mainly teaching younger students in the 12 to 14 age group. There were consistent themes about the nature of mathematics and mathematical enquiry in all his teaching. An example of such a theme was the importance of derivation, for example with one class he was observed telling them a formula for the area of a trapezium in the context of an investigation about areas of quadrilaterals. He said to them:

I would never normally give you a formula without showing where it comes from but this would take a long time and you need it now so we are going to focus on 'how to use it' first and 'where it comes from afterwards'.  You should be cross with me - everything in maths should follow from each other so you should be questioning me, not just believing me... we have to work from definitions.

More relevantly for developing good foundations for functions, in every observed lesson there was an emphasis on structural understanding and the value of inverse reasoning, such as the role of inverse operations when checking calculations.  These themes seemed embedded in his teaching and from students' spontaneous remarks in several lessons it was clear that they had been enculturated to look for and use inverse reasoning as a normal part of their mathematical work; for example, in one lesson a student called out: ‘it doesn’t check back’ in response to an answer suggested by another student.  With 12 and 13 year old students he emphasised the word 'operations', getting them to use it deliberately where appropriate in whole class discussions; later with 14 year old students he and they were using the word ‘operations’ fluently while talking about solving linear equations algebraically, and in talking about a ‘function machine’ as a sequence of operations.  The textbook available for that class stated a rule that: when solving linear equations, if there are brackets you should always expand them first.  However, he wanted students to read algebra in terms of the meaning of the operations, not to merely follow specious rules about order. He wrote 6(x + 2) = 18 up twice in the board asked them for solution methods. With some encouragement two methods were proposed: first solving it by expanding brackets and then by dividing by 6. The whole class then discussed which method they preferred and why. In this discussion it was clear that he expected them to regard 6(x + 2) as a representation of a structure and not merely as an instruction to operate on symbols, since they had a choice about how to act. It seemed as if he wanted them to know what operations were, but also to recognise that they combined to make structures which had to be looked at holistically. He was also countering any tendency to cling to a ‘left to right’ assumption which, paradoxically, could be encouraged by a function machine approach.

These instances illustrate foundational understandings that are necessary for a formal understanding of, and tools for engaging with, functions as whole objects constructed from arithmetical operations. They also illustrate how he built intellectual need into his teaching.

In discussion after the lesson he confirmed that he saw operations as the building blocks of polynomial functions. In informal discussion a few weeks later after he knew the focus he commented that he sees a progression in his teaching towards the idea of function, and functions of functions, in the way he handles sequences of operations with younger students. He had been teaching 16 year olds during the intervening weeks and noticed how fluently they talked about their use of operations in the context of polynomial functions. By avoiding simplistic rules for order of operations, and engaging with their meaning instead, he avoids limiting their understanding of more complex expressions later on. However, he said he had only just thought about this, having been stimulated to think about it explicitly by our post-observation discussion. As Eraut says ‘implicit learning may eventually lead to explicit knowledge’ (Eraut, 2000 p.118).

Paul’s self-report provides evidence for our conjecture that higher-level knowledge can act ‘in flow’ at lower levels of teaching and hence we suggest it is a component of the tacit knowledge for teaching described by Eraut (1995). However, writing on the use of tacit knowledge in the workplace tends to assume that it develops alongside other workers through informal learning. By contrast, higher-level mathematical knowledge is usually achieved by formal learning and teachers in classrooms are most often working on their own so a different model of knowledge-in-action is required. Mason (Davis & Mason, this issue) describes this as ‘knowing-to’ in the moment, yet Paul says that this is more than ‘in the moment’ – it is the usual pattern of conceptual progression in his teaching because it reflects his understanding of how the concept of function is built from other concepts. It would be easy to critique this and suggest that his approach only supports understanding of algebraic functions, but as we have not witnessed his explicit teaching about functions, say in trigonometric contexts, we do not have evidence about what he does about this.

### Linda

Linda’s teaching of all classes included mathematical challenges which were appropriate for the students and glimpses of mathematics a long way beyond their current focus. For example, in a lesson with 12 year olds on converting fractions to decimals some students volunteered the information that *pi* was a non-repeating infinite decimal number. She extended this event to talk about the existence of other such numbers, named them as irrationals, and showed them the classic proof by contradiction of the irrationality of √2. This took place in a lesson that was only observed from half way through, and she had not been expecting the observation, so we are certain that this was a typical segue.

Linda was also observed, among other lessons, teaching two advanced classes of 17 year olds. In a lesson for advanced students on the Poisson distribution, Linda asked them what the probability would be if the desired number of occurrences in a given time period (x) doubled: . One student suggested that P would double. This was a response Linda expected and it conforms to research about the dominance of assumptions of linearity (De Bock, Verschaffel & Janssens 1998). She then asked them to work out a few cases.  She then went back to the formula and pointed out where x was ‘If you looked at this formula you wouldn't expect to get that scaley thing ‘if you double this you double that’; you wouldn't double factorial x’.  We were struck by her referring to 'that scaley thing' to indicate a special linear behaviour that could not be generalised and was not relevant in this case. She had drawn their attention to the effects of varying the relation itself, i.e. thinking of the formula as a function. It would be possible for students to carry out the associated examination questions by merely using the formula as an instruction to calculate, so her segue was, as in Paul’s work, an example of drawing on higher level knowledge.

She also taught a lesson in which advanced students were finding roots numerically.  They were encouraged to construct or find (using the internet) some unusual functions and then find their roots by: (i) rearrangement and iteration; (ii) Newton Raphson method; (iii) zooming in to see where signs change; and (iv) algebraic methods where possible. They had to compare methods for the various functions they had chosen and say whether and why different methods might work better with different functions. She offered probing questions: 'why are you calling this exponential?' and 'so how do your assumptions relate to what it looks like?' and 'can you find one that the graph plotter cannot cope with?' It would have been possible for them to complete the task without these questions but she was determined to get them to (a) think of mathematical properties rather than visual characteristics and (b) avoid the simplistic assumption that every function can be graphed and all the methods work for all graphs. In this case the task had been specifically designed by an external task designer to develop knowledge beyond polynomials and other algebraic functions, as well as to treat more familiar functions as objects with properties, so this is not so much an example of using knowledge ‘in flow’ as of how higher-level knowledge enables a teacher to scaffold and redirect students’ thinking in an environment of open enquiry about functions.

### Simon

The final example was found in a draft assignment from a masters’ student, Simon, who already has a doctorate in theoretical physics. He was aiming to help his 13 year old students understand the need for proof. He had been using Moser’s circle problem (http://mathworld.wolfram.com/MosersCircleProblem.html [[4]](#footnote-4)) to challenge students’ tendency to reason inductively from the first few terms of a sequence, but he was unhappy about presenting something for which the actual value, 31, is hard to explain or demonstrate – it could be thought to be merely a pathological case. He therefore created a software environment which, using Lagrange polynomials, generates functions which take the values 1, 2, 4, 8, 16, k ... for successive integer values of x and various values of k. This demonstrates to students that any number can be substituted for k and a suitable function can still be found. The underlying function cannot be directly deduced from patterns in a sequence. To do this Simon had to know (a) that there are always families of such functions (b) the characteristics of such families and (c) how to construct generators as well as how to develop the software.

Obviously developing such a tool is not possible without knowing the underlying maths, but it would also be possible to access the tool from the internet if Simon posts it there. However, to look for such a tool requires a teacher to know that there is a family, that ‘31’ is not a pathological case but is one of an infinite class of functions, as is ‘32’. There are pedagogical decisions to be made about the mathematical and epistemological purpose of using such a tool with students, and how it is used, but such decisions cannot be made in an informed manner by someone who does not know the underlying mathematics and hence be able to imagine how to exploit it and how it relates to the curriculum. In developing this tool Simon, who was in his first year of teaching, was addressing one of the crucial problems that arise from an inductive approach to functions common in early algebra, that of generalising with the nearest available ‘plausible’ function (Stylianides, 2008).

None of the three teachers above depend on textbooks to decide the order and content of their teaching, but there are textbook sets in school to provide supportive material. The textbook series available for ages 12 to 14 in Linda and Paul’s school is one which attempts to be explicit about the word ‘function’ as it arises in the context of other areas of mathematics, such as deriving expressions for relations, solving equations and understanding graphs. Students do not need to know about functions as this age for the UK curriculum so it is likely that authors made a well-meaning attempt to introduce the idea of ‘function’ alongside relevant early experiences. However, as well as having the symptoms of US textbooks as described above, its attempts at clarity lead to some statements which are at best misleading, particularly if students took them to be definitions. For example: the statement ‘every function has an inverse’ appears in a chapter about solving linear equations with the unknown on one side. ‘Functions describe events in everyday life’ appears on a page about situations which embody direct proportionality. Elsewhere the same book says: ‘a function is a way of expressing a relationship between two sets of values’ which confuses the function itself with the ways in which it can be represented. Elsewhere again, in the same series, students are told that there are two methods for finding the relationship between ‘a set of numbers that map to one another: method 1: look at inputs as term numbers of a sequence and the outputs as the sequence terms and find the general term; method 2: look at inputs as x-coordinates and outputs as y-coordinates, plot a graph and hence find the function’[[5]](#footnote-5). This confuses discrete and continuous data, and also confuses a point-wise view of functions with the underlying relations. Even if this is considered unimportant, the inputs would have to be sequential for the outputs to be seen as a sequence. The implication of method 2, that finding a function follows simply from a graph, is also oversimplistic.

In the discussions after the whole observation process had been completed Paul and Linda were told that the focus had been their knowledge of functions.

About the treatment of functions in the textbooks in their school, which they hardly ever used, Linda said: 'this makes the fundamental confusion between discrete and continuous data and mappings' and Paul said: 'this is why I don't ever talk about mappings' and explained that mapping diagrams focus on discrete data points and lines joining related points give misleading visual images about the relationship. They thought that introducing the idea of 'function' when all you are going to do is look at linear functions was ‘a bit pointless’ because you do not need to discuss domain and range. Linda said she does not use the word ‘function’ until it is necessary to describe a collection of graphs or even until they being to talk about 'functions of functions’. This means that by the time she uses the word ‘function’ so if they have a variety of experiences which give it some meaning.

## Function-aware teaching

Do Paul, Linda and Simon demonstrate the capability to prepare students well for their future understanding and avoid typical problems? Firstly, even with a small set of observations we saw evidence of functions being interpreted and treated as objects with their own properties, and of coherent foundations being laid in anticipation of such treatment. Elementary concepts which would be needed later to understand concepts were mainly algebraic, and in Paul’s teaching there was a consistent emphasis on the meaning of symbols in expressing underlying structures. Paul and Linda made explicit comments about possible confusions between discrete and continuous situations. We saw evidence of older students being pushed beyond well-behaved functions, and also of functions being transformed and used purposefully in Linda’s and Simon’s work. We also saw evidence of students being exposed to uses of functions beyond their current expertise.

In relation to the issues we raised earlier from an advanced perspective, we were not looking at their teaching of functions *per se*, but mainly at how their advanced function knowledge informed their other teaching. Nevertheless, we saw Linda working explicitly on pushing older students towards ‘ungraphable’ functions, thus separating the function from its graphical representation. Paul and Linda both mentioned composition of functions as a topic which needed a clear understanding of the component functions. Paul’s approach to early algebra was based on the development of a holistic understanding of simple equations – later to be related to polynomial functions.

We conjectured earlier about a need for ‘function-aware’ teaching, that is teaching by teachers who have an advanced knowledge of functions and can use it ‘in flow’. These three teachers demonstrate this kind of knowledge in the above examples and were articulate about it later. However, in some cases it appeared to be used without immediate awareness – a combination of holding tacit knowledge about conceptual progression, such knowledge being based on formal knowledge, and this knowledge being available all the time in their teaching, but rather than being explicitly drawn on in specific moments.

We also conjectured earlier about the intellectual necessity of functions. We saw this badly done in a textbook series, but well articulated by Linda in putting off the use of the word ‘function’ until it was necessary and made sense. We also saw it in Simon’s task construction where functions were the appropriate tool for disrupting inductive reasoning about sequences.

Having briefly explored how teachers’ advanced mathematical knowledge of functions can inform their teaching at a more elementary level, we should ask whether, in a different curriculum regime, their knowledge could be put to more explicit use. For example, it has been claimed that very young children have an intuitive understanding of exponential growth, and could be introduced to representations of this, with interpolation and extrapolation, by function-aware teachers (Ebersbach et al., 2008). However, while the curriculum is restricted to well-behaved algebraic functions for most students (mainly linear and quadratic) it is hard to imagine how intellectual necessity can be generated. By contrast, the curriculum suggested by Yerushalmy (1997), in which functions with a variety of behaviours, including two variables, are introduced to younger children, might be adequately embraced by the personal subject knowledge of these three teachers.

In all three cases the teachers’ personal knowledge exceeded by far what was in the available textbooks, and two of them critiqued the textbook treatment legitimately without much prompting.

**Implicit principles for mathematics ‘in flow’**

There appear to be two underlying beliefs about learning exemplified in the teaching described above, and also in the reflections on teaching:

1. Students come to a learning situation with a set of ways of thinking (practices, dispositions, beliefs, etc.), some desirable and some undesirable, that inevitably affect the way they will understand concepts and skills we intend to teach them, and
2. Students develop desirable ways of thinking only through proper understanding of concepts and skills.

Linda’s teaching included a good example of (a), namely her awareness of a possible linearity assumption. Paul’s teaching illustrates intuitive and knowledgeable understanding of (b), namely the development of desirable ways of thinking. The two statements taken together form what Harel has called the ‘duality principle’ (D) (2008, a; b & c), and Linda and Paul show that it can inform even small actions. The principle entails that long-term curricular planning is essential, and absence of such planning can have harmful consequences, because the ways of thinking students acquire now will affect, by limiting or enabling, the quality of the concepts and skills they will learn later. The principle also entails the need to take into account students’ current ways of thinking in designing curriculum and instruction, because these determine what students can and cannot learn and the quality of what they will learn. In a curriculum that is based on the duality principle, desirable ways of thinking do not wait until students take advanced mathematics courses.

In Paul’s case, continuity of the teacher is also seen to be important. There is no guarantee that students who have a different teacher next year will build on the ideas which Paul has seeded for them. On the other hand Paul may have older students who have not had the benefit of establishing foundational concepts with him at an earlier age. Recent research about successful mathematics students has shown that continuity of teacher is a beneficial factor (Rodd et al., 2010). This could be about strong interpersonal relationships, but it might also be an about the continuous, coherent, development of a particular disposition towards mathematical ideas. Paul and Linda’s own mathematical knowledge leads them naturally towards repetition of forms of reasoning taking place over time as students move through school years. Paul’s statement about his mathematical statements usually needing to be questioned is evidence of such continuity. Research has shown that repeated experience is a critical factor in developing the cognitive habits that are beneficial in doing mathematics (Cooper, Heirdsfield, & Irons, 1996). Repeated reasoning, not mere drill and practice of routine problems, is essential to the process of internalization—a conceptual state where one is able to apply knowledge autonomously and spontaneously—and reorganization of knowledge. A curriculum sequence, such as a sequence of problems, can continually call for reasoning through the situations and solutions, responding to the students’ changing intellectual needs. This observation goes beyond the notion of scaffolding students’ thinking to a higher level within one situation as a need arises. Rather it calls for new forms of reasoning to be embedded and habituated through the effective fading of support for new ways of thinking. This is addressed by the repeated-reasoning principle (R) (Harel, 2008 a; b & c):

Students need to practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking.

A third principle that underpins the teaching described above is that of intellectual

necessity (N), a need for a new concept in order to handle new situations.

For students to learn what we intend to teach them, they must have a need for it, where ‘need’ refers to intellectual need.

Intellectual need is different from motivation. Motivation has to do with people’s desire, volition, interest, self-determination, and the like. Intellectual need, on the other hand, has to do with disciplinary knowledge born out of people’s current knowledge through engagement in problematic situations conceived as such by them, and structured by teachers who have the responsibility for creating situations that promote such need. Relevant to curriculum design, the necessity principle entails that new concepts and skills should emerge from problems understood and appreciated as such by the students, and these problems should demonstrate to the student the intellectual benefit of the concept at the time of its introduction.

The following example is relevant to thinking in terms of functions. When seeking a function to model a natural phenomenon, the data which are typically available to us consist of how the phenomenon changes. Thus, one of the main purposes of examining rates of change is to use some information about rate to gain information about a function, a purpose which is often masked in traditional calculus courses. Guershon designed an instructional unit consistent with this purpose which necessitates ways of thinking critical to understanding rates of change. The unit begins with a set of problems on functions, in particular problems in which the objective is to describe a physical situation; for example: At any time, what is the population?; At any time, what is the distance between two moving objects? The purpose of these problems is to advance students’ understanding of the concept of function as a *dynamic input-output process* by intellectually necessitating thinking of a dependency rule between two varying quantities. As research has shown, thinking of a function as a *dynamic input-output process* is abstract and expected to be difficult because the student has to attend simultaneously to three objects, *input* (e.g., *time*), *output* (e.g., *distance),* and a *dependency* *rule* (e.g., a formula) that connects them. Even when the dependency rule is given, the student must think of the input and output in general terms: for *any* given input the dependency rule determines its corresponding output (if it has one).

One of the primary goals is that students understand functions as models of reality, and of course that they develop a more sophisticated understanding of functions. In these problems, attending to rate of change is necessary for determining a model. This, in turn, necessitates an in-depth study of rates of change, in particular, an exploration of average rate of change which leads naturally to an intuitive notion of instantaneous rate of change. The *need for communication*—in this case the need to communicate to others a precise definition of ‘approaches’ and ``arbitrarily close’—demands the formalization of our intuitive notion, i.e. the definition of the derivative. With the definition of the derivative in hand, we may prove properties of functions which follow from properties of their derivatives. Many of these properties are intuitive, but the *need for certainty*—to know that something is true—demands formal proof. Truth alone, however, is not our only aim, and we desire to educate students to know *why* something is true—the cause that makes it true—a need referred to as the *need for causality*.

The three principles delineated above are the components of a theoretical framework for mathematics curriculum design known as DNR: duality (D); necessity (N); and repeated-reasoning (R). The full system includes other principles and elaborations organized around these three principles (Harel, 2008 a; b & c). Learning in DNR is viewed as a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium (Piaget, 1985; Thompson, 1985). The system can also be viewed from a Vygotskian perspective in the interactions and actions generated by Paul, Linda and Simon who scaffold shifts to new ways of thinking through tasks and, especially in Paul’s case, engineer moves towards internalisation through dialogue. Even if concepts and skills are intellectually necessitated through suitable tasks, there is still the need to ensure that students internalize, organize, and habituate this knowledge by repeated –reasoning over time to activate the duality principle.

The DNR system was developed to aid curriculum design, but seems to capture characteristics of how personal mathematical knowledge is enacted in flow, over time. It is obvious that a lack of description of complex knowledgeable behaviour keeps it tacit and unavailable for others. It does not follow, however, that articulation of the behaviour of Paul, Linda and Simon makes it possible for others without their knowledge to enact similar DNR merely through pedagogic planning. In the absence of personal mathematical knowledge that can act automatically, deliberate pedagogy and planning may not fulfil the principles. We also speculate that new forms of reasoning scaffolded by Paul, Linda and Simon might dissipate with changes of teacher.

### Conclusion

This paper was conceived as an investigation into how personal mathematical knowledge at a high level can impact on teaching at a lower school level, and beyond particular topics. A small observational project of two mathematics graduate teachers, and knowledge of a task planned by a third teacher, indicated that it can and also yielded some insights into how this can work. The particular context for this investigation was the teaching of functions, which are often be treated superficially in our textbooks.

‘Function-aware’ teachers, that is teachers who have personal experience of studying functions at a high level and using them in their own past study, can teach throughout school in ways that lay foundations for later understanding of functions, rather than ways that would hinder that development. Further, they can prepare students for the intellectual need for functions, so that students can respond to situations as if seeking a function to describe and explain mathematical and physical phenomena. They can embed repeated experience, because they themselves see such phenomena as functions, rather than seeing ‘functions’ as an isolated curriculum topic. This teaching is based on a continuous understanding of functions which informs both their in-flow pedagogic actions and also their continuous pedagogy over time.

This paper therefore supports the importance of mathematics teachers having personal mathematical knowledge beyond the level at which they are teaching.

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1. Some would contest whether all UK teachers work in a culture of autonomy. We use this phrase to indicate autonomy in relation to textbook use. [↑](#footnote-ref-1)
2. A new national curriculum is being produced [↑](#footnote-ref-2)
3. For compliance with ethical standards they were informed that there was a focus, and they agreed to be told about it only after observations. [↑](#footnote-ref-3)
4. If n dots on a circle are joined by straight line segments, how many regions are created? An inductive approach produces the sequence: 1, 2, 4, 8, 16 and students expect the next term to be 32 when in fact it is 31. [↑](#footnote-ref-4)
5. For legal reasons we cannot give a full reference for this series and have paraphrased where possible without losing the mathematical sense (or nonsense). [↑](#footnote-ref-5)