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| WORKING MATHEMATICALLY ON TEACHING MATHEMATICS |
| Preparing graduates to teach secondary mathematics |
| ANNE WATSON and LIZ BILLS |
| *University of Oxford Department of Education, Oxford, UK**anne.watson@education.ox.ac.uk* |

*Abstract:* We illustrate our approach to using mathematical tasks with prospective teachers to promote complex thought about what it means to do and learn mathematics. Prospective teachers are enculturated into ways of thinking about teaching mathematics which persist, by and large, when they are in school. We offer tasks organised to challenge their instant responses, and support the development of habits of probing mathematical meaning as the starting point for thinking about teaching.

Key words: Teaching prospective teachers, preparing to mathematics, shifts of understanding, multiplicative relations, algebra

# Introduction

In working with prospective teachers our practice is to start with mathematical tasks, so in this chapter we describe three tasks which we presented to them, the way in which they responded to the tasks, and our interpretation of their learning through these tasks.

We saw the chance to write this chapter together as an opportunity to examine for the first time the way we are working together as teacher educators. We knew already that both of us had a strong commitment to the view that the shared experience of working on mathematical tasks is at the heart of learning mathematics, learning about mathematics, and learning about learning mathematics. Although we have been colleagues in a number of different contexts for many years, this year is the first time that we have worked together with a group of prospective teachers over the period of a year. Our common approach has developed over the period of this year through shared planning and teaching and through observation of each other’s teaching. We have spent time discussing the responses of our students to our teaching, especially the way in which we see them working in schools, but had not explicitly compared our teaching approaches.

In this chapter we have written about the mathematical tasks we present to prospective teachers and the work which they have done with the tasks. Most of our taught sessions with the group start from mathematical tasks and move on to pedagogical questions. Occasionally their experience as teachers is used as the starting point. The work we present here is typical of our teaching sessions rather than illustrative of an occasional approach.

The prospective teachers we teach are taking a one year post graduate course which will give them Qualified Teacher Status (necessary for teaching in state funded schools in the UK) as well as academic credits at masters level. Teaching is still not a popular career choice for mathematics graduates in the UK, which means that admission to our course is competitive but not highly so. We usually attract around 60 applications for 30 places each year. Nevertheless, our students tend to be well qualified mathematically. They have a good first degree which includes at least fifty percent mathematics (this might be an engineering degree, for example, where experience of rigorous pure mathematics is limited) and a significant number of them have higher degrees as well. They have mostly been very successful in mathematics at school, but may more recently have felt that they reached some kind of wall in their own learning. Because most of our applicants are academically qualified for the course, we are able to select on the basis of other requirements, which include strong interest in mathematics and evidence of the ability to think critically about teaching and learning.

# some theoretical background on using tasks to learn to teach mathematics

The approach we take to mathematics teacher education is to offer a sequence of complex mathematical experiences which are designed to expose and bring to articulation ambiguities, distinctions, alternative conceptions, of teaching and of school mathematics. In each session we work on what Thompson and colleagues call ‘coherent mathematical meaning’ (Thompson, Carlson & Silverman, 2007) through bringing what is coherent for our students alongside what might be seen as coherent for their learners. In this way, we ask them to appreciate learners’ experiences, and to see ‘coherence’ from the learners’ perspective.

This is a delicate task, because as we have worked for many years as teacher educators, some distinctions and constructions are very obvious to us – but this does not mean that they will be helpful for our students. It is a classic temptation in education to teach unifying theories, which make sense to those who already have a lot of relevant knowledge, to novices who do not know what is being unified. Instead, we use their existing mathematical knowledge and experience as learners as a starting point for developing language and realisations about their experience, and then applying those realisations in their teaching. Even with high level qualifications, there is always enough variety in ways of understanding the tasks we give to use diversity, comparison, analysis of implications, and relationships to school mathematics as structuring devices for interactive sessions.

We rarely offer easy closure by giving ways to teach topics, or ways to use ideas. We do not give generalisations about teaching and learning. Instead we work together on tasks, we use their responses to expose pedagogic and didactic details and choices, and we reflect on what is afforded for learners in imagined situations. It is a characteristic of our work that we do this through mathematics, so that the thinking required at every stage is mathematical, that is, concerned with presentation, exploration and perception of variation in questions, examples, diagrams, and other mathematical artefacts. Yet the atmosphere is about pedagogy. For example, in an early session on fractions, several different representations were used, each for a different task for which they were well-suited. The final task was intended to evoke criticism of reliance on limited images. All the representations which had been used so far were offered as a list:

|  |  |
| --- | --- |
| Fraction walls | Folded rectangles |
| Squares in rectangular arrays | Folded strips |
| Congruent parts of shapes | Area representations |
| Shaded parts, not congruent | Shaded elements of set |
| Slices of pizza | Division sums |
| Points on a number line | Decimal number  |
| Conventional symbolic form |  |

The task was:

Decide the uses and limitations of each representation, bearing in mind that secondary school students have to work with objects which have a ‘fraction’ structure such as “sine = opposite/hypotenuse”.

This end-of-session task provided more complexity than closure, prompting one prospective teacher to say that he thought this was why some teachers only taught procedures – working with images and understanding took a lot longer. Another announced that he was confused, but this is not a problem for us – a sense of confusion reduces as they realise there are no ‘right’ answers. What we aim to achieve is a shift from an approach characterised by the question ‘how shall I teach so-and-so?’ to one of ‘what does it take to learn so-and-so?’

We report on some tasks we have used, and how we use them, seen within the holistic nature of our course. School-based experience, mentoring, and university-based teaching are integrated to support the development of complex understanding of teaching mathematics. Key ideas about mathematical pedagogy are raised in practice, in formal sessions, or in small-group tasks or assignments. Within a student’s individual trajectory there are opportunities to recognise structures and distinctions, through talking about experiences, which will inform future thinking about teaching. In the task sequences described below, some of these themes can be seen as threads that run through several sessions. Distributivity emerges in work on mental arithmetic and in algebra. Representation is explicit in the session using a line segment, explicit in a session on fractions, and implicit in other sessions. Ratio arises as an example of a shift to be made from additive to multiplicative thinking, but is given a full session of its own later. In a session on ‘student errors seen in school’ our students find that they learn even more about arithmetic, and we find that they apply a view based on alternative conceptions rather than ‘mistakes’. All of this is enacted in schools through observing experienced teachers and by prospective teachers being supported through mentoring. In this way, we manifest many of the practices which are taken-as-shared internationally (Watson & Mason, 2007). Where we might differ from others is in the established, integrated, relationship between all aspects of our course (McIntyre, Hagger & Burn, 1994; Furlong et al., 2000). It would be wrong to give an impression that there is a finely-detailed advance plan underneath what we do. Each prospective teacher teaches different years, groups and topics in school, so mentoring is responsive to individuals. Our teaching focuses on coherent mathematical meaning, and is influenced by the ‘preparing to teach’ frameworks developed at the Open University in the 1980s (e.g. Griffin & Gates, 1989). This framework (which is still evolving) offers three dimensions, cognitive awareness, behaviour and emotional engagement, to think about teaching a topic (see Figure 1).

In our teaching, therefore, we offer opportunities to do some mathematics and talk about it, to articulate their responses to it, and to think about how these would be contextualised for their students in school. For this chapter we observed each other teaching and identified common principles of how we do this. Since we are teaching teachers, we often state openly to them and each other how we have planned our sessions, but what we had not realised until this shared observation and analysis was that we also adhere to similar methods of putting these into practice, using prospective teachers’ comments to develop a critical atmosphere.



Figure -. *Preparing to teach framework (taken from Mason, 2002)*

Typically, we offer a mathematical task or set of tasks which relates directly to the school curriculum, and which can be tackled by all prospective teachers. Often, this task will trigger experiences they have had in school, either as teachers, supporters, or as learners. Soon after this, we give a new task which develops from the earlier one, but which is unexpectedly harder for some reason. It might demand comparisons between tasks or methods. We may have asked an unexpected question in a familiar context, or pushed a mathematical commonplace into an unfamiliar arena, or gone beyond the usual range of numbers or shapes, or questioned something which is often assumed. An example of this might be to ask prospective teachers if it is valid to join the points of a curve which has been generated from integer data. The introduction of such shifts and comparisons generates uncertainty, debate, intrigue, disturbance, which is not publicly resolved but becomes more comfortable through shared perceptions and thought about pedagogical implications.

In the next three sections we present accounts of three teaching episodes and relate them to this theoretical perspective.

# Working with a line segment to think about shifts of understanding

Static image 1 was projected on the board as prospective teachers entered the room. They were asked to say what they saw. Initial comments were: ‘65%’ and ‘golden ratio’ and ‘a black line with blobs on’. The diagram was then animated by moving the middle dot while maintaining the overall length. I[[1]](#footnote-1) then asked them to say more about what they were seeing.

Image1

Image 2

Image 3

Sandy: a line of set length which is divided into two sections – two variable lengths – well one is variable and the other is fixed to the variable

Pat: there are two or three lengths, which is the starting length?

I commented that they had shifted from trying to guess what the diagram meant to reporting what they had seen, and that this shift appeared to have come when I animated the diagram and asked them to say what they saw.

Don: part of the line has a variable length

I asked them to write down something which represented this variable. Eventually someone offered:

*t = kp + (1-k)p*

Someone else observed that this simplifies to *t = p*. The next offering was:

*x + y = L*

and I queried the status of each term. *L* was said to be a constant, or given; *x* represented one of the lengths, and *y* was therefore a dependent variable. The letters therefore had three different uses in this statement of a relationship.

Someone then offered two further versions of the same relationship:

*L – x = y*

*L – y = x*

I said it was important for learners to have a sense of these three representations as a package, as three ways to represent the same relationship. I then described shifts of understanding that had been demonstrated so far:

* from guessing to being analytical about what they were shown
* from descriptive comments to analysing in terms of variables
* to interpreting what is free to move and what is constrained
* from variables to conjectures about variables
* from variables to relationships

I announced that we were about to shift from additive to multiplicative ways of conceiving relationships. I pointed out that *x + y = L* seemed to be an attempt to record an additive relationship, where the earlier attempt using *k* seemed to be trying to express a multiplicative relationship. Someone said ‘it is like probability’.

I then animated image 2, the length being extended but the blob which was positioned on the line stayed in the same place. This animation creates a different invariant, but is still additive. Finally I animated the line again in the way shown in image 3. Could they all try to express this as a multiplicative relationship? Eventually this was offered:

*x* = *k* x total length

*y* = *(1-k)* x total length

where *x* and *y* are the two parts of the total segment. I had been hoping for an expression of direct proportionality such as *x = ky.*

I commented that this group had ‘gone into algebra’ straight away, but I was not sure that everyone was able to ‘see’ the relationships they were describing. What question could they ask learners to help them shift from seeing the lines additively to seeing multiplicatively?

The prospective teachers suggested:

How much of …?

What fraction of …?

How many times does this bit go into that bit?

What proportion of…?

What is the ratio of x to y?

Tell me the length of this bit in terms of this bit?

At this point it became clear that one of our students had not noticed *how* the point positioned on the line had moved, so I repeated the animation, asking ‘what stays fixed and what changes in each diagram?’

I finished by exemplifying with 7 = ? x 3, asking for three different expressions:

7 = *k* x 3

= 3

= *k*

I repeated the earlier comment about having a ‘package’ of three ways to express one relationship.

Superficially, the session was about how diagrams can be used as images for algebraic relationships, and how focusing on invariance and change in dynamic representations can trigger new ways of seeing. The line segment image is particularly powerful because, seen as a statement about lengths, it carries semantic meaning about addition and multiplication, and it also acts syntactically, in that the three ways of transforming the key algebraic relationships can each be constructed from the diagram itself, rather than only from manipulating the formal expression.

However, this session contained far more about mathematical awareness than ‘just’ this. For example, someone referred to earlier work about how giving diagrams in particular orientations could be misleading for learners. I also hoped to initiate new awareness which would be revisited later on, and these were:

1. that learners have difficulty in shifting from additive to multiplicative understandings of change – and in that respect this session was precursor to considerations of ratio later.
2. that there are alternative ways to express relationships – and this signalled an approach to algebra as expressing generality, and transforming equations as constructing equivalent expressions
3. that letters have various roles
4. that shifts from thinking about variables to thinking about relationships are important.

Also there had been opportunities during the session for those who were not sure about the mathematics themselves to work alone or with others, either on the direct mathematical tasks or the related pedagogical issues.

# Working with mental calculations to explore links between algebra and arithmetic

The following calculations appear on the screen one at a time without comment, with time for our students to consider each before the next appears.

   

  

They were asked to work on each individually and make notes about what they did. Next they were asked to compare their methods with their neighbours (there were six per table) and to consider whether they could draw out any mechanisms or principles.

After a few minutes they were asked for comments. The first contribution was about calculating  by ‘multiplying by 100 and dividing by 4’. They called this ‘compensation’. I asked for further examples of compensation strategies and these were offered:

Do  by subtracting 200 and adding 25

Do  by multiplying 5 by 20 and then subtracting 5

Tutor: When is it helpful to use compensation?

Val: To break a complicated sum into something easier you can do in your head

Tutor: Let’s be fairly specific about this. How do you recognise what is going to be easier?

A series of responses to this mentioned: multiples of 10 and 100, multiplication by single digits, single digits used as ‘the adjustment’, dealing with decimals by multiplying and dividing by powers of 10, familiarity and ‘roundness’ and ‘splitting things into chunks of some bits that work’.

The discussion continued and touched on the usefulness of powers of ten, the use of ‘known’ facts and converting between percentages, decimals and fractions. After a few more minutes I asked them to take a few moments to consider whether they could link what had been said so far with things they had read or discussed earlier in the course.

Tutor: Anyone got anything to say?

Will: There is implicit use of the distributive law

Tutor: I thought I might have stopped you from seeing that by choosing 

Will wrote on the board



Andrew: When you thought it, did you think ‘bracket ten plus ten plus five’?

Will: I didn’t think ‘bracket’

Tutor: Did anyone see that idea in any of the others?

Madena wrote on the board:



Tutor: What if we had used Andrew’s approach of 10 + 9 is 19?

They nodded. Tansy wrote:



commenting “but it’s not nice to write percentages inside brackets”.

Sally wrote:



Caroline added ‘I started with 25 squared’.

Tutor: So you are using known facts.

Andrew: There are ‘known known facts’ and ‘recently known facts’. For example in the 17% example from 10% you get 5% - it’s recent knowledge. This is different from ‘knowing’ 25×25.

At this point I referred the prospective teachers to a government publication about strategies for mental calculation and we moved on to consider written calculation.

One of the main purposes of this session was to offer the possibility to see algebraic structure in arithmetic. I did this by

1. generating mathematical activity (asking them to do the calculations themselves)
2. focusing on sameness and difference (by comparing similar methods for different calculations and different methods for the same calculation)
3. prompting prospective teachers to connect their recent experience with past experience

As a result many, perhaps all, were able to see a relationship between distributivity as a property of the number system and a variety of informal methods they had used to calculate. The examples presented made it possible to see the wide application of this structure, not just as the distribution of multiplication over addition.

Beyond this a number of ideas arose from individuals, thus becoming available for the group to work with in this session and subsequently, for example:

1. Andrew’s question to Will about what he thought when doing the mental calculation (‘Did you think “bracket”?’) enabled a distinction to be made between informal use and formal expression of structure;
2. the importance of ten and its powers in the arithmetic of our decimal system;
3. the usefulness of being able to shift from one representation to another (here from percentage to fraction).

The prospective teachers also had the opportunity to experience the variety of valid approaches to the same calculation and learning with and from each other by comparing different ways of seeing.

# Exploring the meaning of algebra

The prospective teachers were seated at tables in threes or fours. Each table was given a collection of slips of paper on which the following items (questions or expressions) were printed:

|  |  |  |
| --- | --- | --- |
| A |  |  |
| B |  |  |
| C | Find the next term in the sequence **1, 4, 7, 10, ...** | Find the 100th term in the sequence**1, 4, 7, 10, ...** |
|  | Find the *n*th term in the sequence **1, 4, 7, 10, ...** |  |
| D | I asked my grandma to tell me how old she was. She replied, 'if you multiply my age by 3 and then subtract 100 you get the same answer as if you took my age and added 34'. Find grandma's age. | Alan thinks of a number, multiplies it by 7, then adds 13 to the result. The final answer is 69. What number did Alan think of? |
| E | The spreadsheet formula**= A3\*0.15 + 12.50** produced by typing on the keyboard | The spreadsheet formula**= A3\*0.15 + 12.50** produced by clicking on A3, and typing |
|  | The spreadsheet formula**= A3\*0.15 + 12.50** produced by dragging a formula from a higher cell |  |

They were asked to negotiate with each other in order to separate the slips into two piles ‘Algebra’ and ‘Not Algebra’. As they talked they tended to consider the items in the pairs or threes in which they are presented above. They also often preferred to talk about items being more or less algebraic rather than ‘algebra’ or ‘not algebra’.

After about ten minutes of group discussion I asked for comments. The first offered was about the expressions in set A above and asserted that the second is algebra but the first is not.

Veronica: It depends on how you present it

Tutor: Does putting a letter in make a difference?

Veronica: No, it’s because it is an unknown

Tutor: So what makes it algebra or not?

Andrew: Without the box the first one looks just like a problem

Tutor: If you think of it as something to rearrange it is algebra, but otherwise it is just a ‘sum’?

Soon after this Veronica summarised the discussion by saying that algebra is not marks on paper, but an approach that is taken to what is written.

The next comment was on the pair of expressions labelled B. Madena said that the first is a realisation of the axiom expressed in the second, so that neither of them is algebra if algebra is seen as something you do. This allowed us to contrast two meanings of algebra, that is the task of manipulating symbols according to certain rules (manipulative algebra) as opposed to the study of the rules themselves (abstract or axiomatic algebra).

The next remarks were about set C. Saidah said that the first is not algebra because you can ‘just add three’. The second is algebra because in order to find the 100th term you need to know how to find the nth term.

Tutor: If you do it by keeping on adding three is it algebra? Or if you add 96 times three is that algebra?

Madena: It depends whether you see algebra in the structure or as something you do. 10×7+6×7 is algebra depending on what you focus on.

David: If they are adding 96 times three they might just see it like that – it’s not necessarily algebra – it depends on what they do with what they see

Tutor: So you mean that having structure is not enough. Can you say a bit more?

David: Algebra is about letters, unknowns, generalising, so for me the nth term is algebra, but not the 100th term.

Tutor: You are talking about how they express the generality. Would the person who says that you add 96 times three be able to give the 102nd term or the 99th? This is a test of whether they see the generality?

In discussion of the ‘word problems’ (row D) Alastair said ‘the one you can “undo” is not algebra’. Madena added that writing down the expression (perhaps as a function machine) is the same thing as doing it in your head. In the continuing discussion we agreed that neither of the problems ‘is algebra’ but that algebra provides methods to solve either of them.

The discussion of E was curtailed by shortage of time, only allowing for a brief mention of the difference between using a ‘label’ consciously or unconsciously.

Later in the same session our students were offered experiences through which to consider the differences between uses of letters as unknowns, variables and generalised numbers. They also were introduced to the six uses of letters identified by Kuchemann (1981)[[2]](#footnote-2), to research on understanding of the ‘equals’ sign and to Gray and Tall’s (1994) notion of ‘procept’. The next day was spent considering some curriculum materials for teaching and learning algebra. They were asked to work in groups to comment on the materials using the mathematical distinctions they had developed the previous day.

The main intention of the card sorting activity described above was to broaden the prospective teachers’ understanding of what might be meant by algebra. Madena and David presented two points of view, namely that algebra (at least at school) is about manipulation of expressions involving letters and that expression using letters is the distinguishing feature of algebraic activity. During the discussion these ideas were explicitly challenged by several assertions that algebra is not what is written, but the way in which we think about what is written. The idea that the structure of a problem, relationships between quantities, and generalisation, can be the drivers for algebra was made available. In addition common mathematical experiences were offered from which a language of distinctions could be derived. Our students were also offered another opportunity to experience the usefulness of looking for similarity and difference between mathematical entities.

# Coda

The three examples above illustrate our general approach to using mathematical tasks to promote complex thought about what it means to do and learn mathematics. Because this approach is sustained by us throughout our teaching, prospective teachers are being enculturated into ways of thinking about teaching mathematics which persist, by and large, when they are in school. At the start of our course, it is usual for them to want to exchange stories about what they have seen teachers and learners doing in school, whatever the task we give them and whatever we hoped the focus would be. By offering tasks organised to challenge their own instant responses, we support the development of habits of probing mathematical meaning as the starting point for thinking about teaching, rather than trawling memory for associated stories.

We are not claiming that all our students sustain this approach all the time – that would be too hard. However, when we observe them teaching in school and ask about their planning and in-the-moment decisions it is clear that the majority start from wondering about how their students are going to learn and structuring what they do to support this, rather than adopting ‘tricks of the trade’. We have little knowledge about how many of them sustain this once they are in their first posts, but we do know that in some of our partnership schools the culture of the mathematics department is to think first about learning, and then about tasks, sequences of tasks and ‘coverage’.

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1. In these descriptions of episodes the word ‘I’ refers to the one of us who was teaching at the time. [↑](#footnote-ref-1)
2. The book from which this comes is a set text for the course. [↑](#footnote-ref-2)