Locating the spine of mathematics teaching

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At PME in Prague 2006 Stanislav Stech threw down a challenge to our community, asking how and when the scientific concepts of mathematics can be learnt if teaching focuses on everyday reasoning and realistic contexts. The second part of this challenge has been met in the work of the Freudenthal Institute, but I have felt that more needs to be known about shifts from everyday to mathematical reasoning. I describe my trajectory through practice, academic collaboration, and research towards some understanding of this problem, and towards ways of describing teaching that transcend organisational aspects. In doing this I attempt to keep faith with Illich, Freire and Gramsci from whom I drew early inspiration.

Background

In my first career as an academic bookseller I learnt a form of knowledge which I called ‘spinal’. This name is deliberately ambiguous – it sounds fundamental, structural and pervasive, but actually it meant merely the titles and authors on the spine of the book. Later on I started teaching mathematics in an upper secondary school, 14 to 18, without any training , so I used my superficial ‘spinal’ knowledge to direct me towards some books on education: Illich’s *Deschooling Society* (1973 edition) and Freire’s *Pedagogy of the Oppressed* (1972 edition). This was all I had read when, with a degree in pure mathematics, I began to teach. When Peter Gates, ex-acting president of PME, came to the school as head of mathematics I started to glimpse how Illich and Freire’s thinking might be put into practice. For the following twelve years I tried to teach maths in ways that make a social and political difference.

Underpinning my teaching and subsequent research are key issues which arise in the work of these authors. ‘Deschooling’ is not about avoiding planned teaching, it is about challenging the assumption that ‘behaviour which has been acquired in the sight of a pedagogue is of special value to the pupil and of special benefit to society’ (Illich 1973, p.71). For Illich’s what can be learnt in school is not the only learning of value. But he also says ‘the fact that a great deal of learning […] seems to happen casually and as a by-product of some other activity defined as work or leisure does not mean that planned learning does not benefit from planned instruction and that both do not stand in need of improvement’ (1973, p.20). There is a role, therefore, for schooling, but the knowledge he describes as suitable for planned learning is ‘skills’ such as reading and computation. These statements foreshadow the tensions mathematics teachers face between assessment regimes, curriculum coverage, and the recognition that complex non-routine problems are of major value. Now that many of us educate students in complex ways of being mathematical, not just narrow skills, do Illich’s deschooling arguments still apply? For me, the issue is not to throw out narrow skills simply because they have been over-valued, but to work out what skills are critical, and whether there is anything else about mathematics which is best met in the planned way that Illich describes.

Freire describes institutional education as oppressively controlling students. His list of its characteristics includes:

3. the teacher thinks and the students are thought about

6. the teacher chooses and enforces his choice, and the students comply

7. the teacher acts and the students have the illusion of acting through the action of the teacher

9. the teacher confuses the authority of knowledge with his own professional authority, which he sets in opposition to the freedom of the students (1972, p.46-7)

I have omitted other well-known issues to focus on those that indicate a change in the social nature of classrooms. Freire is, of course, not talking about learning at the level of how to add fractions or solve quadratics; he is talking about critiquing themes which impact on the lives of ordinary people, but the way he talks applies to mathematical understanding, for example the need for ‘perception of previous perception’ and ‘knowledge of previous knowledge’. These stimulate new perceptions and new knowledge which should be ‘systematically continued with the inauguration of the educational plan’ (1972, p.87). He is not suggesting anything haphazard, but a systematic plan of critical comparison and reflection, leading to new ideas through providing experiences, not through controlling minds.

Investigation and problem-solving

To follow Freire’s advice I had to find ways for students to engage in mathematically authentic practices, where mathematical structures, not the teacher, are the authority. Students would think; students would choose; students would reflect critically on prior perceptions and knowledge. Students are not going to meet the hegemonic mathematical culture outside school, they are likely to meet only situationally-limited practices, but since the hegemonic culture is economically, politically and socially empowering, they need to meet it in school. Gramsci (1971) argued that people from all social backgrounds need to have access to the intellectual tools available to the ruling classes. This includes mathematics beyond everyday lives and the workplace, such as mathematics that engages deductive reasoning; models abstract concepts; predicts financial risk, epidemics and climate change. How is this to be done without imposing on learners an unstable pile of methods, rules and facts?

Our answer in 1980s UK was similar to many other attempts - to complexify classroom activity in two ways. The first was based on the methods of John Wallis (1616-1703) who inductively conjectured relations from sequences of examples. We also drew on the work of Polya (1945) who described mathematical exploration in detail. These environments were called ‘investigations’ and would trigger group work over several lessons. This approach, pioneered by the Association of Teachers of Mathematics, had a major influence internationally. In the UK however it was soon absorbed into a national assessment system and used as a quantifiable tool. It lost the original aim of generating mathematically authentic engagement; teachers and students used it to achieve grades rather than to explore ideas.

Another kind of activity was contextual problem-solving, such as: making a conical holder for a certain number of potato chips; ordering different quantities of paving slabs that had to be laid in a particular pattern. The problems were seldom real, and most could be solved by *ad hoc* methods of everyday reasoning. Application of formal mathematical ideas was artificial and did not necessarily lead to improvements in understanding either the situation or the mathematics. In our mainly working class school we did very little traditional teaching – mostly students worked with extended tasks of these two kinds. We changed the social structures in which we taught, having different groupings and different forms of participation. Students chose what to do and how to do it. We used holistic assessment methods, and valued a range of aspects of mathematical work. More students from diverse backgrounds enjoyed the subject and succeeded in it. We had harnessed students’ everyday reasoning to the cause of school mathematics, and unsurprisingly their achievements increased. Freire’s statements 6 & 7 were successfully contradicted.

Although our students became adept at reasoning in spontaneous ways in numerical and spatial contexts, and did better in national tests than similar students in traditional setups, only a few were used the formal mathematical tools that they could only meet in school. Introduction to new ways of thinking and abstract conceptualisations was through *ad hoc* interactions with the teacher and we did not know how to do this systematically for everyone. We failed to act on the work done by, for example, Brousseau, on structuring the content and the methods of our students’ thinking.

Challenge

At PME in Prague, Stanislav Stech challenged our community on these points. He said that the connection between mathematical concepts and real-life problems ‘exists only as highly-mediated’ (2006, p. 46). Moreover he emphasised the particular role mathematics has to play in the development of a students’ capacity for ‘intellectualisation’, that is to move from performing a mental function as itself, to perceiving it as an example of a wider class, or an instance of a principle, or as dependent on an idea. Development occurs when students move from being embedded in action at one level to seeing what they do as a relation within an overarching generality. Although there are several ways in which these moves have been described in general terms our literature, the problem for me is how teaching can help students to make critical shifts, and indeed what those critical shifts are.

Vygotsky’s distinction between spontaneous and scientific concepts points to some necessary shifts. Spontaneous concepts ‘emerge from reflections on everyday experience’ and scientific concepts ‘originate in the highly structured and specialized activity of classroom instruction’ (Kozulin 1986, p. xxxiii). These are not skills in Illich’s sense, but logical concepts which are not empirically-inferred from material phenomena, physical actions and social interaction, but which have been conceptualised as formal knowledge structures. Students’ understandings need to be drawn out, extended, or systematically challenged by scientific understandings. The work of Dougherty (2004) and Schmittau (e.g. 2003) demonstrates what is possible when you take seriously the need for transformation between spontaneous and scientific understandings of relations between quantities. In their work, typical difficulties of understanding proportion and symbolic representation turn out to be products of the educational environment; not immutable obstacles that require maturity to overcome. Dougherty and Schmittau help very young students understand these supposedly-difficult ideas through situations in which they spontaneously reason. They then provide tools of language, symbolisation, material and graphical artefacts, and questions, which scaffold qualitatively different reasoning that does not arise without such mediation. This idea is similar to the ‘vertical mathematisation’ of Realistic Mathematics Education , in which models-of situations change purpose in a sequence of similarly-structured situations to become models-for mathematics.

IAMP and CMTP

I decided a few years ago to seek out teachers who make a difference to students’ learning in the context of our current orthodoxies and get beyond the surface features of their teaching.

In two projects with Els De Geest we identified teachers who intended to change their practice to make a difference for their students. In IAMP (Watson & De Geest 2005) we observed 10 teachers and in CMTP we chronicled the work of three school mathematics departments. These were neither innovation nor evaluation projects. The only interventions were the provision of off-site meetings for teachers to meet each other, and questions to clarify our observations. We observed, interviewed and collected documents. Nearly all the teachers achieved their intention of making some kind of difference for their students relative to comparable groups, but the identification of similarities and differences in their work was more dependent on grain size than perspective. In IAMP we did not find many similarities among the 10 teachers that could be turned into instructions for teachers on how to act, but we did find similarities at the level of principles – it was the way those principles were acted out that varied. For example, to enact the principle that students needed time and space to learn, one teacher might extend their wait time for answering questions in whole class episodes, while another might introduce extended tasks.

In the CMTP data I was puzzled about how to report lessons. Every lesson was different with overarching similarities, and we did not want to reproduce ‘typical’ descriptions. When the project started a particular style of teaching - controlled whole class teaching and measurement of micro-learning objectives – was nationally embedded. It may have contributed to a rise in our rankings in TIMSS but this is with a 25 percentage-point drop in enjoyment in secondary school (Sturman, Ruddock, Burge et al. 2008). This ‘style’ on its own could not account for the successes that the schools achieved in learning and enjoyment (see www.cmtp.co.uk). All the lessons we observed, about 40, included sequences in which the whole class was engaged in exposition of a key idea through a shared task, or worked examples, or some animation, or discussion of features of objects, and in all of these a range of students participated and the teacher wove their ideas into the lesson, to provoke debate, or to provide raw material for a point the teacher wanted to make, or to build up conceptual understanding from students’ prior knowledge. All lessons also included students discussing what they were doing within a pair or smaller group, and the teacher talking with individuals and smaller groups of students.

At this level, lessons were similar in structure and organisation, but this account gives no sense of how the teachers engaged students in the ‘scientific conceptualisations’ of school mathematics. To get at this, I needed to focus on how concepts were handled in lessons – what was available in the public realm for students to make sense of – and how students’ spontaneous and quasi-spontaneous conceptualizations were brought face to face with the formal concepts of mathematics. I wanted to know whether and how ordinary students could learn to use the intellectual tools of the elite in mathematics; tools that afforded access to scientific conceptualisations; ways to perceive their own perceptions and have knowledge of their knowledge; formalisations that do not arise in everyday activity. Teachers’ descriptions of their practice included phrases like ‘developing thinking skills’ or ‘asking higher order questions’ but lacked mathematical specificity.

Shifts

The shifts of conceptualisation that have to be made in mathematics are not generally described in the literature in ways that are useful for analysing teaching; how, for example, do teachers help learners encapsulate or reify mathematical ideas? Currently I attempt to answer two questions: what are the key shifts that are necessary for students to learn mathematics at secondary level? How do successful teachers help students recognize and make such shifts?

In this paper I focus on the second of these, although a great deal of work is needed on the first. For example, shifting between discrete and continuous understandings of number; between additive and multiplicative views of relations; between empirical and deductive reasoning; between performing a procedure and reflecting on its outcomes; between performing operations and understanding relations; and so on can easily be agreed to be important but are hard to find in our literature except in early development, or where failure to make such a shift leads to error or obstacle.

Finding conceptual development in lessons

I came across some work by David and Lopes in which they identified classroom moments that had the potential for particular shifts of mental action. Manuela David and I observed some lessons together and tried to learn more about those moments of potential. What is it that makes mathematics happen in classrooms? We worked within a sociocultural perspective, that is, we wanted to know what it was about the situation that would shape different mathematical practices, and what these could be. The evidence of ‘shaping’ we saw as episodes in which students’ and teachers’ activity appeared to be aligned. Peter Winbourne and I have called these ‘local communities of mathematical practice’ in time and place, which might anticipate a future community of practice, or might dissipate. We characterised them thus:

pupils see themselves as functioning mathematically and, for these pupils, it makes sense for them to see their ‘being mathematical’ as an essential part of who they are within the lesson;

through the activities and roles assumed there is public [from the participants] recognition of developing competence within the lesson;

learners see themselves as working purposefully together towards the achievement of a common understanding;

there are shared ways of behaving, language, habits, values, and tool-use;

the episode is essentially constituted by the active participation of the students;

learners and teachers could, for a while, see themselves as engaged in the same activity. (Winbourne & Watson 1998)

These are not necessarily ‘good’ - Manuela and I found an episode which had all the characteristics, but the meanings of ‘functioning mathematically’ and ‘engagement’ were about reproducing algorithms in exams, and not the development of conceptual understanding. In a classroom that was socially similar, the LCMP was developing powerful methods for multiplying decimals (David & Watson 2008). The LCMP criteria identified episodes of joint understanding, but Stech’s challenge directs us to ask if it is about scientific conceptualisation. Our work convinced me that the social aspects of classrooms might be a distraction from the question of whether the teaching affords understanding of mathematical concepts that require shifts of perception, attention and language. The alignments apparent in the LCMPs we described afford shared engagement in different kinds of mathematical practice. Pushing this further, suppose that we take two teachers who both have the same stated aims for conceptualisation and are operating in similar social contexts. What can we learn from the differences in the potential for mathematical conceptualisation in their lessons? Helen Doerr and I watched some video lessons of classes with broadly parallel prior attainment, the same curriculum, the same school, but different teachers who had planned together (Watson 2008). They were teaching a first lesson on loci, and had agreed which loci to use; to avoid the word ‘locus’ until students understood that the required line represented all the points that ‘obeyed a rule’; and to compare physical enactments of loci, using students to represent points, to constructions with straight edge and compass. This took place in a school which had common practices of working in groups, whole class interactions, discussion, inclusion, and valuing mathematical thinking.

We saw that language and gestures used by teachers, and the order in which tasks were presented, afforded different kinds of generalisation and abstraction. The difference was mainly due to the general mathematical relationships within which the teacher saw the task as being embedded. These two groups of students would therefore be differently prepared for future mathematical activity, because their attention had been drawn to, and they had been encouraged to work with, differently perceived relationships. The conceptualizations available to be learnt were different. For example, in one lesson explicit emphasis was on sets of points and the phrase: ‘same distance’. The teacher talked to us later about how hard it is for students to understand the two-way implications of loci, that all points following a rule can be spatially represented (in these cases by lines) and that any point on these lines must therefore follow the rule. Rather than being explicit about this she had chosen to emphasise ‘same distance’ in each context and hoped that the continuity of language across the two situations would help students connect these ideas. They started with the physical representation, with students standing somewhere which was the same distance from a point, or from a line, or a line segment, or from two lines, or from two points. Later they reproduced these with paper and pencil constructions, using compasses to make ‘same distances’. The difference between standing so that your distance from something is the same as your neighbour’s, and standing so that you are the ‘same distance’ from two things, was not explored explicitly.

For another teacher, the emphasis was on isomorphism between two actions. The first was the dynamic production by their own methods with compasses of the desired loci (the same loci as above). The second action produced lines as a set of static points, each represented by a student standing according to a positional rule, static in an open 2-D space. They discussed in pairs what had been the same and different about these processes, each of which produced mathematically identical loci, and to feed back the result of their discussions in a plenary session.

To infer how these approaches shape students’ knowledge of loci we asked “what is or is not afforded by this approach?” It is important to recognise that they are different approaches: one being geometric, depending on the property ‘same distance’, and one about isomorphism of structures, but both ultimately are about the two-way relation between points representing specific relations and lines representing the class of such points. What we brought to bear on our observations was our mathematical knowledge and experience rather than a particular research paradigm. We looked at language and gesture, used these to construct our own mathematical sense of what was going on.

Describing how lessons afford conceptual development

In CMTP we first analysed at too large a grain size to find out what teachers did that made a significant difference to students’ understanding. Indeed, one of the findings was that even in departments in which teachers were open with each other about much of their teaching – discussion of teachers’ own mathematical meanings did not often take place. The two teachers of loci thought that they were teaching the same mathematics, but the understandings they scaffolded are different.

The mathematics quality analysis group (MQAG) of the TIMSS seven-nation video study (Hiebert, Gallimore, Garnier et al. 2003) attempted to describe the mathematical affordances of lessons in ways which transcended differences in lesson structure, topic, task-type, cultural context or social organisation. They categorised whether lessons included deductive reasoning and other kinds of mathematical rationale; whether they contained opportunities for conceptual and notational learning as well as procedural; whether they contained generalisations, use of counter-examples and other kinds of reasoning; whether the components of the lesson were mathematically interrelated or not; whether ideas were developed, connected and justified. Their purpose was to compare countries, so categories needed to be chunky, a bit like my spinal knowledge of academic books when I was a bookseller. I could organise them according to the author’s name, or publication date, but not according to the ideas within. But their focus on mathematical quality helped me see that thinking about possible mathematical responses to what was available was a way to understand mathematics teachers. The grain size of lesson analysis at which we can learn about the mathematics has to enable questions such as: What is publicly available to be learnt? What are the mathematical affordances of a lesson? At a crude level, you cannot learn geometrical theorems, or multiplicative number relations, or mathematical habits of mind unless they are available to be learnt, whatever the lesson is like. What do teachers do that enables students to shift from spontaneous conceptualisations (including those derived from earlier teaching) to specific mathematical conceptualisations and ways of thinking that they would not undertake on their own? In Gramsci’s terms, how do they give students access to the intellectual tools of experts rather than merely get better at using the tools they already have?

A step forward came from the work of Marcia Pinto and her colleagues, who were worrying about why students from different previous schools had very different, but competent, understandings of the tangent line. They realised that ideas which are mathematically equivalent can be understood differently because of the community in which they were learnt. This was not about informal knowledge being formalised within mathematics classes, it was that formal mathematical ideas were expressed, emphasised, represented differently in different contexts (Pinto & Moreira 2008). So for highway engineers tangency was taught and understood as about constructing an arc of a circle tangential to a straight line - thus ‘tangent’ meant a line perpendicular to the radius of a ‘regular arc’, and ‘irregular curves’ had to be seen as sequences of circular arcs. The scientific notion of tangent is concretised differently for different purposes, although these ways as being equally valid mathematically.

I showed earlier that lessons can be co-planned and similar in every social aspect, but different mathematics is available to be learnt. Marcia and Valeria’s data point to the role of situated intention. Their lessons were different because the teachers were thinking of different workplace practices. With Manuela, the difference was between practices of passing exams, and the practices of constructing conceptual understanding. With Helen the difference was between similar features of situations and the instantiation of abstract structure. The mathematical intentions and understandings of the teacher, whatever the lesson is like, structure the learners’ experience of the content.

In the approach of Marton and his colleagues, the ‘space of learning’ has to contain an ‘intended object of learning’ but the interactions between intention and enaction mean that the ‘lived object of learning’ experienced by students is different from what was intended. In his work, intentions are deliberate; the planning that relates intention to enaction makes significant difference to what is available to be learnt – the opportunities to learn (Marton, Runesson & Tsui 2004). Therefore teacher intentions are not as informative about the potential for students’ learning as seeing how they are enacted. Greeno, who sees individual learning and the social milieu as interacting systems, uses Gibson’s ideas of affordances and constraints to which individuals become attuned (1994). I find these useful because learning is constrained by the mathematical affordances structured by the teacher. These, along with learners’ prior experience to which we have little access, shape the space of mathematical learning. The interactions within the space make mathematics lessons different from each other in terms of the students’ lived objects of learning.

This perspective accords with the informal conversations I have with students in classrooms. When I ask individual students in lessons what they are doing, which I do often, they do not talk in generic terms about ‘learning to learn’, nor in terms of specific mathematical targets such as ‘I am learning to solve quadratics with real roots’. They say things like: ‘I have to do this ...’ or ‘I am trying to see if ....’ or ‘she showed us how to ...’ In other words, they talk in terms of their perception of what has been offered to them during the lesson. It then takes very little prompting to get them to talk about their interpretation of the task, the written, verbal and diagrammatic resources available to them, their prior experience of similar tasks, and the meaning to them of what they are doing. Even in the most mundane of tasks, students’ descriptions of what they are doing can have elements of what Pratt and Noss call ‘the microevolution of mathematical knowledge’ (2002).

Microevolution of knowledge requires several elements to be coordinated: naive knowledge – which enables us to understand phenomena and to describe relations by fitting what we see to what we know; a setting – which invites conjectures, surprises, thinking about consequences; and new knowledge – the potential for new language and actions. They offered sequences of mathematically-related tasks to learners in settings that afforded particular constructions. Students working on the concept of randomness achieved an abstract understanding by working on a sequence of problems that had similarities, but which gradually broadened the field of application of their methods. The shifts of response learners made did not happen immediately it would have been mathematically appropriate. It took several similar, but slightly different, experiences before students began to think of using methods that had worked for an earlier example – and even then it was as if they were casting around blindly to try anything in the hope that it would work, rather than seeing similarities in the mathematical structures. Learners’ naive responses were eventually replaced with some new ideas which were coordinated and predictive: it was important that the tasks, the talk and the materials supported the shift towards these new ideas becoming dominant in students’ responses to subsequent tasks. If every new mathematical concept requires such a structure of supported task sequences for microevolution – what does teaching have to be like? Well it doesn’t mean that it has to consist of what textbook authors call ‘carefully graded exercises’ which lock students into performing instrumental, procedural operations correctly and accurately without overarching meaning, but neither does it mean that it can consist of empirical explorations with peers using existing understandings.

In Pratt and Noss’ study learners started by using what they knew, and the tasks drew them towards situations in which what they already knew was no longer effective and extendable, but which required them to use different available resources, or use familiar resources differently. Ryan and Williams (2007) give many examples of situations in which very young students respond to new kinds of question which reorientate their understanding through using the materials which are first used to model familiar situations, and then extended to model formal mathematical ideas. Simon (2007) shows how Erin can be enticed towards abstraction by being given numbers that are too big to calculate with – she has to resort to the pattern and structure of her earlier work rather than its calculations. These research studies hint at general features of the kind of teaching that changes perception. That is - teaching has to start with a situation about which learners have some informal understanding, naive conceptualisation, or situated intuition, enacted and represented iconically or symbolically, in words, diagrams, physical artefacts or moveable screen objects. What features of lessons would tell us whether this is happening or not?

Further analysis of lessons.

I now analyse lessons in terms of how mathematical concepts are developed in the interactions between teachers and students. I do this not to compare lessons, or make qualitative differential judgements about them, but to understand how teachers do this. I ask: how are mathematical ideas made available for students to learn in this lesson? This approach ensures that my knowledge of mathematics teaching is ‘spinal’ in the strong sense: fundamental, structural, grounded and pervasive. What I research, think and write is always based on what real teachers do, and how their real students can learn with them. I have seldom seen a lesson with no opportunities to learn mathematical concepts in it, although there have been a few, and often opportunities are not taken up for a variety of reasons. I have seen many lessons in which the construction of conceptual understanding, if it is to happen at all, has to be a private enterprise rather than a shared concern because there are no public affordances for such construction. I have also seen many lessons, some purporting to be ‘progressive’ or ‘reform’ or ‘problem-solving’ in which all that is encouraged is the re-use of old ideas in some new context. But when teachers make a deliberate effort to engage their students in constructing meaningful new mathematical understandings, some remarkable teaching can take place – and when I looked at lessons for how they did this, I found that the teachers in IAMP and CMTP nearly all did this in complex, fluent, and mathematically interesting ways.

I have focused for most of this talk on why taking this perspective is the strong spine of mathematics teaching, whatever the overt teaching style or lesson structure, and crucially important for making real differences to students’ mathematical understanding. That has not left much space left for the ‘how’ – but this was a deliberate choice because it is work in progress for the community. Instead I offer some approaches which are revealing and informative. They hinge on identifying what mental actions on mathematics are afforded in lessons.

Exemplification and variation

Focusing in the examples available to students, either on the board, in a book, on a screen or from their own constructions, has proved to be a good starting point (Watson & Mason 2005). What generalities are students likely to infer by deductive or abductive reasoning from one example, or from inductive reasoning on a set of examples? How does the teacher help them make these inferences? Further, sequences of examples can invoke progressively more structural understanding by creating conflict, providing counter examples to emergent conjectures (Watson & Mason 2006). For instance, when introducing reasoning about sequences suppose we start with this:

2,4,6,8... What do I know already? What is likely to come next? Can I generate a pattern?

2,5,8,11 .. What is the same or different compared to the previous example? What varies and what stays the same? How does the choice of examples draw attention to particular variables?

2,6,10,14 ... Does this fit with my previous ideas? What is the generality in this set? What properties does this class of objects have?

John Backhouse, one of my predecessors at Oxford who wrote well-planned textbooks, might have put one of these next:

2, 2+a, 2+ 2a, ... to provoke expressions of generality

2, a + 2, 2a + 2, 3a + 2 ... to focus on multiplicative thinking

2, -2, -6, -10 ... to extend meaning to new domains

2, 2, 2, 2, ... to include extreme or special examples.

John Mason and I have applied our mathematical understanding to Marton’s work on dimensions of variation, and think that the range of change within a dimension is as important as the dimension itself. For example, in the sequence work above I have only used small integers, yet if this is all we do the learners’ experience will be limited and, what is more, it is likely that they will develop ad hoc understandings which only have limited application. Thus it is important to extend the new idea into new domains so that the common feature is the structure rather than the arithmetic.

Following key ideas from variation theory Al-Murani compared the algebra learning of students whose teachers who were introduced to variation theory with some who were not (2007). The most effective teachers in his study did more than just offer certain dimensions of variation, teachers in the experimental and control groups both used variation, the difference was made by teachers using deliberate kinds of variation, and using students’ perceptions of variation as the raw material for further discussion. He called this ‘exchange systematicity’ thus indicating the importance of exchanging possible variations in public and exploring them systematically.

Microtask sequences

My recent focus has been on how teachers sequence micro-tasks to lead students towards new conceptualisations. By microtasks I mean anything students are asked to do: answer questions, copy methods, think about an object, compare objects, do some worked examples etc. Using this approach on the CMTP lessons I found that nearly all the teachers constructed lessons like conversations around meaning: showing and eliciting examples, or collections of examples; asking students to construct their own examples and make their own conjectures; displaying variation of methods, representations, and verbal descriptions; making comparisons; asking for or offering new classifications; asking for students’ views and using them to direct the discussion. These microtasks offer differing ways to engage with the same class of objects, or similar ways to work on a sequence of slightly different objects, or coordinations of different representations and ways to work. Interactions were led by teachers through sequences of questions, prompts and instructions, but the content of these interactions was shaped according to learners’ responses. Individual lesson structures were dependent on the nature of the topic and varied between lessons, with mathematical conceptual development driving lesson structure rather than the other way round.

Here are two such lesson structures. I am not making judgements about which is ‘better’ but operating at a descriptive level of ‘these are examples of lessons that we know made a difference to students’ learning’. We know this teaching made a difference to students’ learning and enjoyment within an overall enabling, inclusive context, and this is what we saw in their teaching. Key microtasks which afforded changes in conceptualisation in these lessons include: making new classifications, naming new ideas, generalising from examples, constructing examples, comparing methods and results, and discussing the mathematical implications of the work done.

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| **Lesson A**  T offers a mathematical word & asks students to 'brainstorm' possible connections to it  T identifies properties of what students offer  T and students describe, define and redefine this new concept together  Students use their own words, to describe & define, then create objects to illustrate the new idea  T asks for clarification & examples of what they say  Students and T draw on knowledge, techniques, facts  T introduces conventional formal notation  T classifies students' examples.  T summarises what has been done.  Worksheet to find answers with and without a known procedure - seems to be discontinuous with earlier part of lesson but the observer knows that this work relates to the new word  Students adapt procedures & identify relationships by deduction from results  Students explore variation in methods on the sheet, classifying methods  T leads discussion of implications of their ideas and methods, varying notation.  Students are expected to informally deduce meaning.  T shows how their generalisation links the work the earlier part of the lesson  T tells how this applies to harder maths  Final example from teacher has new features, and is left for students to think about |
| **Lesson B**  T introduces 'learning about equivalent equations'  T introduces one example and then asks them for examples with certain characteristics  T summarises so far, identifies variables in their examples, and compares selected examples, choosing so that the comparisons become more and more complex.  Students solve some equations made by other students and compare methods  T leads public deduction of how methods relate to each other, with explanation and adaptation.  T summarises ideas, and shows application to equations with more variables  Students work in groups to express in own words what sets of equivalent equations say about the value of the variables. |  |

These lesson structures transcend overt methods of classroom organisation. I witnessed them mainly in whole class interactive sequences, but the microtasks might also be made available to learners individually or in small groups, or on a screen, or in a teacher-led exposition. For example, students can be asked to ‘find what is similar or different’ in groups and report back, or a teacher can collect views about what is the same or different in a formal whole class setting, or students can be asked individually to compare methods with someone on Facebook, or on their own as a private piece of work. It is comparing mathematical examples and methods to decide which is useful, and when, that is important for the change of perception. The conceptual affordances of microtasks constitute the spine of mathematics lessons in terms of learners’ conceptual development.

So I have got some way towards finding answers my question about how teachers can help all students make necessary shifts of mathematical understanding, within an understanding of the social context of classrooms, while keeping faith with Illich, Freire and Gramsci, and answering Stech’s challenge.

I have lot more work to do, especially on becoming more articulate about the shifts necessary to learn the mathematics of the elite, but for me these shifts are the structural and pervasive spine of mathematical learning.

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