# The interplay between mathematics and pedagogy: Designing tasks for mathematics teacher education

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In our work with prospective secondary mathematics teachers we encourage interplay between mathematical and pedagogical ideas. In our experience, helping prospective teachers to explore connections between mathematics and pedagogy facilitates the development of a deeper understanding of the complexities of mathematics teaching. Task design and implementation is at the core of our approach to this issue. We shall illustrate the role of tasks in promoting interplay between mathematics and pedagogy with two examples of task sequences from our work. We shall also report on the immediate feedback we received from our students.

## INTRODUCTION

In this paper we use the distinction suggested by Christiansen and Walter (1986) between ‘task’ and ‘activity’. ‘Task’ refers to operations undertaken within certain constraints and conditions (Leont'ev, 1975) and ‘activity’ refers to the subsequent mathematical or pedagogical actions that emerge from interaction among the students, the teacher, the resources and the environment around the task. The teacher’s work is to engineer the milieu (Chevallard, 1999) in such a way that the activity is likely to be directed purposefully towards the teacher’s intentions for learning.

At the school level, ‘tasks’ in mathematics classrooms are generally equivalent to ‘mathematical tasks’ and so tasks can be seen as the mediating tools for teaching and learning mathematics. At the teacher education level, which is our focus in this paper, the meaning of the term ‘task’ becomes more complicated. Some tasks may start as being just about engaging prospective teachers in mathematical activity (‘mathematical tasks’), others may start as being about engaging them in issues of pedagogy (‘pedagogical tasks’), and others may be somewhere in-between these two viewpoints. A major point we aim to make and illustrate in this paper is that, when we design tasks for use with prospective secondary teachers, even if these tasks on the surface look purely ‘mathematical’ or ‘pedagogical’, there is value during the implementation of these tasks to not limit the discussion to mathematical or pedagogical issues, respectively, but to explore with prospective teachers issues that the tasks can raise in the interplay between mathematics and pedagogy. In our experience, helping prospective teachers to explore connections between mathematics and pedagogy facilitates the development of a deeper understanding of the complexities of mathematics teaching.

The students with whom we work as teacher educators are in a continuous internship relationship with schools (see Hagger & McIntyre, 2000; Thompson, 2014). Throughout their one year course they are encouraged to integrate, through problem-solving and reflective approaches, their experiences of teaching and observing in school, reading and participating in university sessions. The teacher educators who teach these sessions are the same people who visit them and observe their work in schools. Experience over time has taught us that at times during the university sessions, some prospective teachers will focus more on the mathematics and others will focus more on similar teaching situations they have experienced in school. Thus, our students' rich mathematical knowledge and recent school experiences come into play in how they react to different kinds of tasks in the university sessions. In our planning and teaching of these sessions, we take account of this background knowledge and experiences in the design and sequencing of tasks.

In our university-based work with prospective secondary mathematics teachers we use tasks as mediating tools. For example, through engaging them in mathematical tasks we promote reflection on their own mathematical activity and also on pedagogical implications of this activity. This interplay between mathematical and pedagogical issues helps to illustrate, and to give them experience of, the subject specific didactics of mathematics with its layers of increasingly general and abstract mathematical meaning explored through the use of mathematical modes of enquiry. In this paper, we make a distinction between general pedagogic issues that could apply to any subject and aspects of teaching that are specific for learning mathematics. The latter we call ‘didactics’ and one way in which our students encounter these ideas is by taking a pedagogical stance towards the work we do with them.

Possible pedagogical issues include the selection, modification, design, sequencing, and evaluation of tasks, thus requiring two levels of engagement with mathematical tasks: firstly as a prompt to mathematical action, and secondly as examples of what they themselves might produce or use in the classroom. So, although these prospective teachers are generally strong mathematicians (with at least a bachelor’s degrees in mathematics or the equivalent), they get opportunities to reflect on their own learning of mathematics and also to (explicitly or implicitly) become more insightful and articulate about the didactics of mathematics. Because we are hoping they will reflect not only on their own actions but also on the milieu that shapes those actions, our teaching aims to scaffold their shifts between different reflective perspectives, gradually introducing them to different issues that are involved in mathematics teaching and learning.

There is often a tacit assumption among publishers of school textbooks and between inexperienced teachers that the implementation of mathematical tasks chosen or designed by the teacher in a school classroom will lead to the intended student learning. This view is persistent despite extensive research that indicates that this is not a direct relationship (e.g., Henningsen & Stein, 1997; Margolinas, 2004, 2005; Stylianides & Stylianides, 2008). Seemingly minor differences in the design or implementation of tasks can have significant effects on learning (e.g., Runesson, 2005) as can differences in aspects of milieu; we try to make prospective teachers aware of the effects of subtle differences, for example, by asking them to reflect on their own mathematical activity and on their classroom experiences. In secondary school mathematics teaching, it is particularly important to understand that tasks on their own cannot scaffold the shifts in perception and cognition required to understand abstract mathematical ideas. In the two task sequences we present below, prospective teachers are explicitly invited to consider differences in task formulation. To understand how tasks can be linked in order to support teaching, it is important to understand the nature of the transformation of knowledge from the implicit knowledge-in-action used by confident students (see Vergnaud, 1982) to the knowledge which is explicitly formulated, formalised and memorised that they must handle as teachers. This thinking underpins sequences we present in our sessions.

There is significant recent interest in the multilayered roles of tasks in mathematics teacher education, including tasks that help teachers think about classroom tasks. Zaslavsky and Sullivan (2011) have coordinated the diversity of approaches to tasks for mathematics teacher education, some of the goals being pedagogic, some being about development of professional habits of mind, and some about developing didactic awareness. Watson and Mason’s (2008, pp. 207-8) overview of tasks focuses on the development of didactic awareness through engaging teachers in mathematics, thus:

* bringing aspects of mathematical knowledge to the fore in ways that promote deep understanding;
* bringing aspects of mathematical thinking to the attention of teachers and providing a context for articulation of them;
* developing awareness of multiple perspectives through task-centred discussion; and
* focusing on didactic decisions relevant for specific topics.

In our work, in common with many examples discussed in Watson and Mason (2008), we prioritise the didactics of mathematics, namely specific modes of enquiry, reasoning and behaviour such as questioning, symbolising, representing and switching representation, generalizing, abstracting and interpreting in mathematical contexts. Thus, for example, any tasks for which the immediate (or primary) engagement of prospective teachers is mathematical activity would provide not only mathematical experiences but also didactical experiences and a context for the broader pedagogical purposes outlined above (Watson & Bills, 2011); this kind of task is similar to what Stylianides and Stylianides (2010, pp. 163-165) called ‘pedagogy-related mathematics tasks’. Along the same lines, tasks for which the immediate (or primary) engagement of prospective teachers is pedagogical activity would be rooted in the teaching and learning of important mathematical ideas and would generate discussion around the particular mathematical content; these types of tasks are similar to what Stylianides and Stylianides (2010, p. 171) called ‘mathematics-related pedagogy tasks’. Our reason for these priorities is that generic pedagogic knowledge can be gained in the school-based components of the course, as can enculturation into professional habits of mind, whereas learning about subject-specific teaching has to include research-based knowledge. University teacher educators, who are more familiar with research findings and methods, provide a critical context for input and discussion about research-informed practices.

One of the tensions that arise in working with (prospective) teachers is that they often want something they can use to teach in their classrooms. We establish early on with our class of about 30 prospective secondary mathematics teachers that doing a mathematical task is a way to experience for themselves at their own level something of what their students might experience. This becomes a cultural norm of our classroom. However, our situation is not ‘realistic’ because the tasks and activities of our sessions are not the tasks and activities of school teaching, even if some tasks may be adaptable to the school context. The teacher education milieu is not the same as that of a school classroom; ‘reality’ has to be constructed through internalisation rather than immediate action. For this reason we are particularly interested in what prospective teachers identify as the key ideas of a session and how they envision using these ideas in their practice (e.g. possible implications of these ideas for their practice). Their direct experience in sessions has therefore to be objectified, perhaps through structuring and discussion, to make it more likely that the experience can inform their future thinking and planning. In our planning, therefore, we think about the activity that might be prompted by doing a task, and how discussion of the activity can bring new insights about learning and teaching mathematics into the public arena.

The task sequences we describe below have been carefully developed over time for particular purposes. The first sequence was designed and implemented by Stylianides and the second by Watson; each sequence was implemented during a two-hour session. We use these sequences at a time in the course when the classroom cultural norms described above are established. Consistent with what we described above about ‘key ideas’ and also to gain further insight for this paper we presented a questionnaire for immediate feedback at the end of each session on the two prompts summarised in Table 1. Prompt 1 asked for identification of three key events during the session and to explain what made them ‘key’ for our students, whilst Prompt 2 required them to anticipate how the session may influence their future planning and teaching. We shall now present the intentions, actions and outcomes of each of the two sessions undertaken with one cohort of 29 prospective teachers. Note that it is necessary to use mathematical terms and to refer to mathematical concepts in order to show the didactic detail of our work. However, we have written with general readers in mind and it should be possible to follow the discussion without needing to fully understand the mathematical terms by treating them as placeholders for concepts.

*Table 1: The two prompts to which prospective teachers responded at the end of the session.*

|  |
| --- |
| 1. Please describe briefly 2-3 key events in today’s session and explain what made them ‘key’ for you.   (Note: Interpret the term “event” in any way that makes sense to you. It could be, for example, a task, an activity, a question, a solution to a mathematical problem, a pedagogical issue, discussion about any of the aforementioned, etc.)   1. Please write down 2-3 ideas about your future planning and teaching arising from today’s session.  Explain what about the session provoked those ideas. |

## FIRST TASK SEQUENCE

In Stylianides’ session the task sequence includes a range of tasks in which prospective teachers’ primary or immediate engagement shifts several times between mathematics and pedagogy, thereby promoting discussions on the interplay of mathematics and pedagogy. In addition to a brief introduction and conclusion, the main part of the session can be divided into four major, closely connected sections. Because of space limitations, we omit here discussion of some subsidiary (to the focus of this paper) parts of the sequence.

**DESCRIPTION OF THE SESSION**

**Introduction**

At the beginning of the session, Stylianides provided an overview of the session and reminded the class about key mathematical terms to be used in the session, namely, ‘convex polygons’, ‘nonconvex or concave polygons’, and ‘interior and exterior angles of convex polygons’. Although, as we mentioned earlier, all our students are graduates in mathematics or mathematics-related subjects, we do not expect them to remember specific terms from school geometry.

**Main session part 1: Mathematical work**

The mathematical task in Figure 1 was then presented, which asked prospective teachers to work on two statements (one about the sum of the interior angles of convex polygons and the other about the sum of the exterior angles of convex polygons) and to prove their answers for each statement in two different ways. They could take as known in their work the facts at the bottom of the slide. Note that, although the mathematical content of the task (i.e., the sum of interior and exterior angles of convex polygons) is part of the secondary school curriculum, the requirement to prove each answer *in two different ways* makes the task go beyond the school level. Indeed, the requirement for two different proofs makes the task challenging even for this group of prospective teachers and was an intentional feature of the design of the task. As we explain below, our students were unable to produce two different proofs for the sum of the exterior angles (which was expected based on prior experience) and the sample of school student work in part 2 of the main session was intended to provide insight into an important proof that prospective teachers tend not to think about, thereby initiating more mathematical work on the task.

*Figure 1: Mathematical task about the sum of interior and exterior angles of convex polygons.*



The prospective teachers first worked on this mathematical task individually for a few minutes and, then, they shared ideas/strategies in their small groups of five. The small group work on the task continued until there was a whole class discussion. In the whole class discussion, different small groups presented their work on each of the two statements and the class discussed the proposed solutions.

Although the task in Figure 1 was a mathematical task, in the sharing and discussion of approaches to the task, a pedagogical perspective was also naturally raised. For example, when one small group presented a non-mathematically valid argument to one of the statements, Stylianides asked the class “What might a skeptic ask/say in relation to this argument?” and introduced to the students the following idea discussed in Mason (1982):‘convince yourself - a friend - a skeptic’. This question was aimed at encouraging prospective teachers to be more critical of the proposed argument in a way that did not feel personal. Although Stylianides’ comment was about their own mathematical work, some prospective teachers (drawing on their school experiences) quickly suggested the idea that they could use a similar strategy with their own students. Below are three excerpts from responses at the end of the session to the two prompts in Table 1:

* [Response to Prompt 1.] We saw a rule-of-thumb approach of proof: ‘convince yourself, a friend, a skeptic’ which was key because it showed to me that it is not enough to convince myself of something. As a teacher, I will have to explain my ideas to a class who do not always understand concepts after one-hearing.
* [Response to Prompt 1.] Encouraging students to challenge the teacher and each other

🡪 discussion about ‘playing skeptic’ and us all offering various arguments

* [Response to Prompt 2.] I will have to approach proving an idea as if to a skeptic, and show as much information as I can to ensure that a class is fully aware of all the ideas necessary to be familiar with a concept.

Later on in the whole class discussion of the mathematical task one small group proposed a proof that did not adequately address the general case. Specifically, the figure drawn by that group did not indicate that the argument proposed would apply for any convex polygon (which is essential for an argument to meet the standard of ‘proof’) even though the group’s description that accompanied the figure accounted for the general case. So there was a discussion in the class on how a figure could be drawn that would match the description and would also apply to the general case (see notion of ‘generic example’, Balacheff, 1988; Mason & Pimm, 1984). Similar to the previous example about convincing a skeptic, and without any prompting from the teacher educator, the prospective teachers started making connections between the mathematical issues discussed and implications of these issues in their teaching; this provides another example of the interplay between mathematical and pedagogical issues in the discussion. Below are two more excerpts from responses to Prompt 1 in the end-of-session reflection that illustrate the connections the prospective teachers made between their own mathematical work on the task and possible implications for their teaching:

* Explanation of generic/specific problem of diagrams for students. Do they understand it is to represent ANY such shape, not the specific one they see?
* The way in which we all tackled the same problems in different ways emphasised how pupils may work in different ways.

As expected, prospective teachers were not able to produce two different proofs for the second statement in the mathematical task, i.e. about the sum of the exterior angles. After the analysis of student work that follows, the mathematical task will be revisited.

**Main session part 2: Analysis of student work**

The second part of the main session of the task sequence included analysis and discussion of secondary students’ work on a related mathematical task. The student work was derived from Hadas, Hershkowitz, and Schwarz (2000). As background to the activity, Stylianides offered the following information to the class:

Some secondary students worked on the following task:

Measure with Cabri (a dynamic geometry software) the sum of the *exterior angles* of a quadrilateral. Hypothesize the sum of the exterior angles for polygons as the number of sides increases. Check your hypothesis by measuring and explain what you found.

The sample of student work that the prospective teachers were asked to try and make sense of is presented in Figure 2. The secondary students, whose work is described in the figure, have already explored the case of quadrilaterals and found out the sum of exterior angles is 3600. The figure presents the students’ exploration of what happens to the sum of the exterior angles as they move on from the case of a quadrilateral to the case of a pentagon. The discussion of the student work not only led to more mathematical work for the prospective teachers as we explain in the next section, but presumably also helped them see how their mathematical exploration could relate to engaging secondary students in a similar task.

*Figure 2: Analysis of student work on a related mathematical task.*

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**Main session part 3: More mathematical work**

As we mentioned earlier in the ‘mathematical work’ section, the prospective teachers were not able to produce two proofs for the statement about sum of the exterior angles. This was expected and the choice of the student work in Figure 2 was intended to provoke further mathematical work on the task by engaging them in a proof about the task they had not produced on their own but which used an important proof method, namely, proof by mathematical induction. In more detail, implicit in the student work is the inductive step of a possible proof by mathematical induction. The prospective teachers did not think of using mathematical induction in their initial work on the mathematical task, which was expected given (1) that this proof method causes many difficulties to students including mathematics graduates (Dubinsky, 1986, 1990; Dubinsky & Lewin, 1986; Harel, 2002; Knuth, 2002; Movshovitz-Hadar, 1993; Stylianides, Stylianides & Philippou, 2007) and (2) that most of the prospective teachers would not have ever seen this proof method used in geometry[[1]](#footnote-1). So the student work motivated revisiting the mathematical task to produce a proof by mathematical induction, first for the sum of exterior angles of convex polygons and then for the sum of interior angles. Specifically, Stylianides provided a brief reminder of proof by mathematical induction (see Figure 3) and then asked them to think about the connection between their mathematical work and the work of the student in Figure 2:

Use Proof by Mathematical Induction to prove that the sum of the exterior angles of any convex polygon is 3600. How does your proof relate to Inbal’s argument [see Figure 2]?

Below are some excerpts from prospective teachers’ responses to Prompt 1 in Table 1 in relation to their mathematical work in this section of the session:

* Proof by induction for geometry was another thing I had not done before. It was interesting to apply the mathematical induction knowledge to a different environment then I am used to.
* I found the proofs by mathematical induction interesting mainly because I can’t remember studying this myself, so it was a new concept to me.
* Correspondence between thought processes for geometrical argument and proof by induction.
* Key because: shows how inductive type arguments are common and full induction can arise from other arguments and illustrations.
* The switch from geometric diagrammatical proofs to more rigorous proofs by induction was interesting as it made me think which would be more relevant for a certain class. While the inductive proof was certainly more mathematically sound for students to understand the mechanics involved I think it would be better to use the diagrams.

*Figure 3: Proof by mathematical induction.*

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**Main session part 4: Discussion of different task formulations**

After a brief summary of the activities that were covered in the session thus far, Stylianides introduced the next activity presented in Figure 4. Note here that Stylianides did not explicitly mention the purpose served by the particular selection of the student work in terms of initiating further mathematical work on the task. It is likely that this feature of the design of the sequence went unnoticed by the majority of the prospective teachers, thinking that the analysis of student work was merely part of the normal emphasis in the course of carefully attending to student contributions during a lesson.

*Figure 4: Discussion of different task formulations.*



The new activity asked for comparison of four different formulations of the mathematical task that the class worked on earlier (Figure 1) in terms of the kind of student activity each formulation would likely support in the classroom. This activity capitalised on prospective teachers’ earlier mathematical and pedagogical work because they could use their experiences and insights in the discussion of the new activity.

As we will discuss in a following section, coding of responses to Prompt 2 in Table 1 indicates that most ideas that the prospective teachers mentioned in terms of their future planning and teaching arose from the activity in Figure 4. Below are some illustrative examples of responses to Prompt 2 in relation to this activity:

* Listening to how the statement of the task can be written in a number of different ways and how these can motivate students differently made me think that I will need to consider the best formulation of the task for my lesson objectives in each lesson.
* Careful consideration when I am designing a task, especially the purpose the task is aimed to achieve and to what extent is it going to achieve it.
* Discussion of different activities to offer students led me to think that we could also discuss these possible activities within the class – so students could have their preferences and justify their choices.
* I will think more carefully about how I phrase a question when asking pupils to prove something to get them to follow a specific path or try many different things.
* It is important to think about the students’ thinking when designing tasks. What will the task provoke them to do? Will they understand what they have to do? Will it give them an opportunity to generalise or will it be too time consuming?
* Evaluate carefully the group that you are teaching in order to decide how to set each task.
* Continuing to think about task design and the implications of the decisions (however seemingly inconsequential) that teachers make.

It seems that the discussion of different task formulations had made explicit the important connection between what the teacher says and how mathematical action is shaped by what has been said. This could be expressed as indicating a need to plan for the move from intermental to intramental understanding, so that how the task is presented becomes internalised by learners rather than interpreted merely as direct instruction. We shall return to this possibility in the analysis of the responses to the second task sequence.

**Conclusion**

The session ended with Stylianides providing a summary overview of the different activities in the session. The overview served as a reminder of the activities and did not include discussion of Stylianides’ intentions for the activities in order not to influence prospective teachers’ responses to the two prompts in the questionnaire that followed.

**CODING OF PROSPECTIVE TEACHERS’ RESPONSES TO THE TWO PROMPTS**

Table 2 summarizes the number of times each of the activities in the main part of the session was mentioned (directly or indirectly) in prospective teachers’ responses to Prompt 1. The codes were derived from the description of the session provided above:

* Mathematical work (main session parts 1 and 3);
* Analysis of student work (main session part 2); and
* Discussion of different task formulations (main session part 4).

Multiple coding of each event was possible, but this option was chosen in a restricted way.

*Table 2: Coding of prospective teachers’ responses to Prompt 1 in the first task sequence*

|  |  |
| --- | --- |
| Activities | Frequencies |
| Mathematical work | 31 |
| Analysis of student work | 12 |
| Discussion of different task formulations | 25 |

Table 3 summarizes responses to Prompt 2. The coding for Prompt 2 focused on the origin of each idea from the session, i.e. what activity / key event from the session provoked the particular idea mentioned in each prospective teacher’s response. This allowed us to use the same codes as for the coding of the responses in Prompt 1. On those occasions where it was not possible to identify the particular origin of a response or the origin related to an event other outside the main part of the session, we used the code ‘Other’. Similar to the coding in Table 2, multiple coding of each event was possible, but this option was chosen in a restricted way.

*Table 3: Coding of prospective teachers’ responses to Prompt 2 in the first task sequence*

|  |  |
| --- | --- |
| Activity from which each idea seemed to originate | Frequencies |
| Mathematical work | 19 |
| Analysis of student work | 2 |
| Discussion of different task formulations | 28 |
| Other | 10 |

**SECOND TASK SEQUENCE**

This task sequence was constructed and implemented by Watson to give teachers (including prospective teachers) experience of how particular forms of questioning can direct shifts of focus and levels of abstraction. Throughout mathematics, learning new ideas usually involves looking at existing knowledge in new ways, in particular what can be seen as a procedure at first might have to be understood as an abstract concept later on. In a Vygotskian view, without scaffolding most students might not make the ascent to a new level of thinking that takes what at first appears abstract but later becomes a concrete example for a new concept. For example, children first calculate products by multiplying; then the product itself becomes an example of a composite number, which can be factorised; then the factorisation becomes an example of a number structure; and so on. Teaching must help learners shift their perspective but new teachers often suppose this happens naturally as an outcome of completing tasks, or by being told new definitions, or by being given sophisticated methods. The aim of this session was to provide a sequence in which prospective teachers could experience and discuss how shifts of focus can be engineered.

There were nine elements to the task sequence and each had multiple purposes relating to generic pedagogy, subject specific didactics, and to specific mathematical ideas that Watson hoped the prospective teachers would encounter. The aim was to provide situations in which, at any moment, some might be focusing on mathematics, some on pedagogy and some on didactics. However, the main purpose of this sequence is to understand, through personal mathematical experience, how variation of task components and questions can shape different mathematical actions. For this reason, mathematical flow between tasks was at times interrupted for attention to be directed at the didactics. The session was not divided clearly into nine elements, but for easier reading that is how we present it. In each section the ‘general pedagogic purpose’ can give access to the ideas for the general reader, and it should also be possible to discern progression in the ‘specific didactic purpose’.

The second task sequence, contrary to the first, is not directly related to curriculum content, and is usually novel for teachers. They start by selecting a shape made by connecting four congruent squares (tetramino). They use this shape to identify and generalise relationships between four neighbouring cells on various number grids. Generalisation becomes harder with more variables as more variations of the task are introduced. The forms of questioning also change in deliberate ways to encourage abstraction to take place.

|  |  |  |
| --- | --- | --- |
| **Initial mathematical work** | | |
| 1 | *General pedagogic purpose* | Importance of personal choice; how group work can shorten time to generate materials/objects/data. |
|  | *Mathematical content* | Congruence; reflections; reasoning about completeness. |
|  | *Teacher educator action* | Introduce tetraminoes |
|  | *Teacher activity* | With others, construct all possible tetraminoes; choose a shape and its orientation to make and use for the rest of the session. |
| 2 | *General pedagogic purpose* | Share ideas about the relationships. |
|  | *Subject didactics purpose* | Variety generated through choice. |
|  | *Mathematical content* | General expressions on multiplication grids require two variables. |
|  | *Teacher educator action* | Give out multiplication grids up to 10x10; place the shape on the grid to ‘cover’ four cells. |
|  | *Teacher activity* | Look for relations between the four numbers covered; move shape somewhere else and repeat, etc.; work individually but talk in pairs about the relations between the four numbers under their shape; eventually see the relation is the similar wherever the shape is put; they will want to generalise with two variables.  *Figure 5: Example of a tetramino on a 7 x 7 counting grid* |
| 3 | *General pedagogic purpose* | Different choices about how to express generalisations |
|  | *Subject didactics purpose* | The choice of grid size might introduce irrelevant issues; choose a grid that instigates generalisation; people move from local generalisation to broader generalisation by being given varied examples. |
|  | *Mathematical content* | Generalisation with one variable; variables and parameters are different kinds of generalisation |
|  | *Teacher educator action* | Give out 7x7 and 8x8 counting grids; encourage generalisation.Discuss difference between generalising variables and generalising parameters. |
|  | *Teacher activity* | On 7x7, and then 8x8 counting grids when provided, repeat the search for relations between the four numbers covered (see Figure 5). Past experience indicates that they can be expected spontaneously to express the underlying relation symbolically for their own choice of shape and then generalise the grid size as a new variable (parameter). |
| **New forms of question** | | |
| 4 | *Subject didactic purpose* | Questions that scaffold abstract conceptualisation, i.e. treating ‘grid/cell’ combinations like a new concept. |
|  | *Mathematical content* | Algebraic generalisation |
|  | *Teacher educator action* | Ask questions that treat the ‘grid/cell’ combination as a new concept:“what tetraminoes on what grids will cover cells n+9 and n-2?” |
|  | *Teacher activity* | Reorientate their perceptions of the task to focus on ‘grid/cell’ combinations.Shift from generalisations to abstract ‘grid/cell’ combinations. |
| **Reflection on task effects** | | |
| 5 | *General pedagogic purpose* | Value of giving time to re-run a ‘memory video’ of the lesson so far for themselves**.** |
|  | *Teacher educator action* | Give a few minutes to make notes about what has happened for them so far. |
|  | *Teacher activity* | Make notes on tasks and mathematics and their own activity so far to keep track.  *Figure 6: Example of tetramino on a two-variable grid* |
| 6 | *General pedagogic purpose* | A state of expertise is expected and used. |
|  | *Subject didactic purpose* | Shift from one variable to two variables. Develop this in an unfamiliar direction. |
|  | *Mathematical content* | Functions with two independent variables seen as values on a 2-d grid. |
|  | *Teacher educator action* | Return to the multiplication grid. Give harder two-variable grids to some (see Figure 6 for an example) . |
|  | *Teacher activity* | Continue with a new kind of grid that makes harder demands about generalisation. |
| **Return to initial task with higher level perceptions** | | |
| 7 | *General pedagogic purpose* | Using questioning to scaffold thinking to a new level of abstraction. |
|  | *Subject didactic purpose* | Questions introduce new levels of abstraction. |
|  | *Mathematical content* | The difference between inductive reasoning from numerical examples and structural reasoning from visual images. |
|  | *Teacher educator action* | Pose new questions such as: “what size shape do I need to cover cells (m, n+1) and (m-7, n+2)?” |
|  | *Teacher activity* | Move from concrete work on grids and operations to positional, abstract, mental image; think about shapes and cells. |
| **Reflection on whole task sequence** | | |
| 8 | *General pedagogic purpose* | Perceptions and generalisations depend on the variation available to learners. |
|  | *Subject didactic purpose* | Identify the shifts of thinking and how they were achieved through task design.Experience the power of ‘backwards’ questioning. |
|  | *Teacher educator action* | Tell them the intentions for mathematical attention at each stage of the sequence and how the task was structured to bring them about.What did they notice about variation of questions at each stage of the abstraction process? |
|  | *Teacher activity* | Discuss the entire sequence of tasks using notes made previously and memory. Relate shifts in own thinking to changes in the task. |
| 9 | *Teacher educator action* | Give questionnaire with the two prompts in Table 1. |

All 29 prospective teachers responded to the same questionnaire as in the first task sequence (i.e., the two prompts in Table 1). Analysis was done by first collating all responses and identifying answers that appeared to be close paraphrases of each other. These were then grouped further into broader themes about mathematical tasks and didactical intentions. Here we describe what stood out for them as key events from the session (Prompt 1) and what ideas they envisioned using in their future planning and teaching (Prompt 2). A summary overview can be found in Table 4.

## ANALYSIS OF RESPONSES TO PROMPT 1: KEY EVENTS AND WHAT MADE THEM KEY

**Gradient of difficulty mentioned in a general way**

23 prospective teachers mentioned their sense of a ‘gradient of difficulty’ in the tasks. Of these, 8 mentioned abstraction as something achieved through the questioning, 3 more mentioned “variation” as a tool for supporting generalisation. 10 mentioned the choice of numbers as important, and the remaining 2 remarked on the gradient but gave no more details. 5 further responses commented that the first task had been harder and one said “engagement improved dramatically with the easier [7x7 grid with single variable] task” although others thought that seeing the harder task [two-variable multiplication grid] first had been useful as giving them something to work towards. The ordering of the tasks had prompted thought about order and difficulty as intended, but it had been hoped that more of them would be specific about exactly how variation in the grids had prompted changes in levels of generality (see below). However, this would have been a new idea for them, and probably not one that is discussed with other teachers in school, and it would be sophisticated to be able to grasp this connection and also become articulate about it immediately after the experience.

**Personal choice of examples**

11 prospective teachers mentioned the importance of *personal choice of examples* as a means of differentiation: “allowing all to get something out of the task”. They thought variety was cognitively helpful because neighbours would then have different ‘concept images’ (Tall & Vinner 1981) and insight could be gained through discussion of different cases. Such comparisons between cases highlight the critical, non-obvious, features of relations. 9 further responses also mentioned personal choice of examples but emphasised affective rather than cognitive advantages: choice gave ‘ownership’ and hence improved engagement.

The design decision had been to show how choice could aid motivation, but without it becoming a major focus of the session. The cognitive advantage of comparing different cases in element 2 had arisen through their own discussion and was not something that had been anticipated, but will be incorporated in future use of the sequence.

**Explicit connections between task design, questioning and mathematical activity**

8 prospective teachers referred to recognising that their own mathematical approaches changed during the session in response to features of the task design, rather than the more general comments reported above; 6 of them mentioned the summary of intentions and that the intended effects matched their experience of doing the tasks; one said “the pedagogical explanation was useful and related to how the lesson unfolded”. These remarks affirm the value of experiential learning for teaching. These 8 also confirmed the value of explicitness in describing design decisions.

A further 5 responses mentioned that the way in which the questions forced them to think backwards from abstract to concrete, to reflect on what they had been doing and construct particular cases that fitted abstract observations. 3 of these mentioned the adaptations in Watson’s questions; the other 2 described the ‘backwards’ questions as ‘fun and increasing engagement’. We were pleased that 13 in all had made a connection between design and their own mathematical activity. The use of the word ‘fun’ would need further investigation because it suggests that these prospective teachers may not have recognised a shift from generalising in unknown situations to constructing examples and discussing implications. This shift may have produced a feeling of ‘fun’ but the mechanism by which it was brought about was cognitive rather than merely affective.

## ANALYSIS OF RESPONSES TO PROMPT 2: FUTURE PLANNING AND TEACHING

Prompt 2 in the questionnaire asked about teaching intentions and was analysed similarly.

16 said they would think more about *gradients of difficulty* and order of tasks in future. Of these 3 did not elaborate further but 4 talked about focusing on relations and patterns; 4 talked of starting with a harder task so that ‘there is always a higher peak’. 5 students noticed that extension tasks were always more complex versions of what everyone was doing.

14 mentioned the importance of *controlling variation*, either to encourage generalization, to be clear about purpose, and to allow students to ‘use their own strategies and methods at every stage while following the structure of the lesson’. One person said that using the same resource, but slightly different, encourages a particular way of thinking.

14 prospective teachers talked about giving *choice and ownership* to generate variety in the classroom.

3 mentioned the importance of having ‘interesting’ ways to use *algebra as generalizing relationships,* and one talked about how *turning the task round* led to seeing where a ‘rule’ had come from.

*Table 4: Coding of prospective teachers’ responses to Prompts 1 and 2 in the second task sequence*

|  |  |
| --- | --- |
| Key features of sequence (Prompt 1) | Frequencies |
| Gradient of difficulty | 23 |
| Personal choice | 20 |
| Connections between task design and mathematical activity | 13 |
| Anticipated use (Prompt 2) | Frequencies |
| Gradients of difficulty | 16 |
| Controlling variation | 14 |
| Choice and ownership | 14 |
| Interesting ways to use algebra | 3 |

**Connections between questionnaire responses and task design**

A main aim in the second task sequence had been for prospective teachers to experience a form of questioning that scaffolds conceptualisation at a higher level, making the ‘grid-shape’ partnership into a new mathematical object by asking questions that focus on the relationship rather than on particular grids or shapes separately. To a great extent this had been successful, in that they reported that something had changed for them because of the task and question construction. They mainly recognised features of the work they had been asked to do and the potential power of variation and generalisation within that work. The importance of choosing examples well and thinking about sets of examples seemed to be well understood. The personal engagement they had felt by choosing a shape made a big impression and engagement is an important issue in their teaching.

It is also heartening that some of the feedback did mention the specific abstraction processes that were intended in the design. But given that most prospective teachers had indeed changed from lower levels of conceptualisation to higher levels as the task sequence progressed, and given that it had been explained to them how these shifts of level had been engineered through questioning, why were only 13 of them articulate about this in their feedback? A plausible conjecture is that this way of thinking about tasks and cognitive challenge is new for them. Such thinking is rare in both research and professional literature. To articulate this new idea at the end of the session in which it has been first introduced is a high expectation.

## CONCLUDING REMARKS

Both task sequences address the need, when doing mathematics, to generalise from mathematical experiences either by inductive reasoning or by understanding relations within a structure. In each sequence teachers learn more about the design of tasks for the classroom and also the effects of varying task presentation. Novotna and Sarrazy (2010) explain that thinking about ‘didactical variability’ needs to be explicit when teachers are planning and designing tasks. Looking at variations in tasks can reveal the relationship between how ideas are presented and how they are learnt, and can also avoid routinisation of mathematical activity. In the first task sequence the exploration of the issue of varying task presentation is an explicit activity, comparing different problem formulations and hypothesizing about the effects of these on learning. The effects of this comparison on teachers' thinking figured strongly in the questionnaire responses. In the second task sequence variability is also experienced directly but in a slightly different form: variability is used to mathematically engineer different kinds of generalisation from which abstraction is developed. The aim in the second sequence was to notice these connections, and therefore to think in future about what generalisations need to be engineered, and how this can be done. .

As mathematics teacher educators, we both have similar aims in our teaching and similar expectations of the prospective teachers’ responses, but the interplay between mathematics and pedagogy was not identical in the two task sequences even though in both cases it happened in complex and multi-faceted ways that.

In the first sequence the interplay can be summarised as described below. The starting point of the sequence was a mathematical task that, although it related to a well-known secondary school curriculum topic, the expectations in the task in terms of producing different proofs for each of the statements made the task appropriate and challenging. Theprospective teachers themselves started making such connections on their own in their contributions during the lesson and also in their responses to the questionnaire. These connections were supported by the fact that they could see the content of the task relating to the school curriculum and, also, from the fact that they were familiar with different issues of teaching and learning mathematics as a result of their school-based experience. Although the discussion of the secondary students’ work on a similar mathematical task seemed to shift the focus to pedagogy, the choice of the particular sample of student work was deliberate in order to open up more mathematical possibilities leading to a discussion of a proof by mathematical induction. The comparison of different task formulations at the end of the first task sequence capitalised on the prospective teachers’ earlier mathematical and pedagogical discussions. For example, the connection between the paths they had followed in their earlier engagement with the mathematics, and the particular formulation that had prompted these paths, served as powerful empirical experiences that informed the discussion of different formulations. At the same time, their knowledge of students and their other school-based experiences provided a complementary lens both for the discussion of the different task formulations and also for the explanations they provided in the questionnaire about the importance of this activity in their own development as learners and teachers of mathematics.

In the second sequence, the interplay between mathematics and pedagogy can be summarised below. The mathematical work is not curriculum related and was new for the prospective teachers. The emphasis remained on their own mathematical activity until attention was deliberately drawn to elements of task design and how they have shaped that activity, which served as a link to issues of pedagogy. By staying with their own experience, connections between task features and mathematical activity were made based on immediate personal evidence. In the post-session questionnaire there was the opportunity to focus more on pedagogical issues and articulate, for example, how their experience in the session might influence their own teaching. In their responses to the questionnaire, they overtly described and discussed how the deliberate introduction of new forms of questioning changed mathematical activity, and how they might use these techniques in their teaching. The sequencing of tasks is central to these effects. In element 4 of the second sequence, for example, the new forms of questioning would not have a dramatic effect on changing thinking if early stages had not first established a simpler form of thinking. A similar reorientation takes place later after the establishment of the ‘grid/cell’ generality. These shifts of attention can be replicated for classroom mathematics learners through careful task design.

The underlying principle in the reorientation described above is that learners at any stage establish successive intramental understandings that are reshaped by new forms of questioning. The direct personal experience, through the tasks above, of learning mathematics and of how learning can take place, gives prospective teachers insight into how to construct pedagogic sequences for their own students.

To conclude, we have reported on two mathematics-specific task sequences that illustrate in different ways the features offered by Watson and Mason (2008, pp. 4-6), which we summarised at the beginning of the paper. Both task sequences:

* bring aspects of mathematical knowledge to the fore in ways that promote deep understanding;
* bring aspects of mathematical thinking to the attention of teachers and provide a context for articulation of them;
* develop awareness of multiple perspectives through task-centred discussion; and
* focus on didactic decisions relevant for specific topics.

In both sequences, prospective teachers’ own learning, experience and knowledge, whether about mathematics or pedagogy, becomes the central reflective resource. The design of the task sequences shapes their immediate experience in the session, and also mediates interplay between past and present experience, school and university experience, and mathematics and pedagogy.

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1. Typically, most students would only have met this method in algebraic/number theory contexts. [↑](#footnote-ref-1)