**Exchange systematicity: interactional dynamics of variation in**

**mathematics lessons**

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**Introduction**

Variation theory is a powerful tool in thinking about the teaching and learning of mathematics (Runesson, 2005; Watson & Mason 2005a; Al-Murani 2007). This is because mathematics is often about the identification of variables and their relationships, and the control of variables can aid observations of relationships. Variation theory tells us that people notice what varies, against a background of invariance and mathematics has often been described as what Mason calls ‘the study of invariance amidst change’. Where for a learner the varied slopes of straight line graphs might come to their attention, for a mathematician the interesting observation is how, despite the varied gradients, the equations are of similar form. We illustrated this in Watson & Mason (2005a). This idea is simple enough for beginning teachers to grasp, and a class of graduate pre-service secondary mathematics teachers generated this sequence of examples for their pupils: 6 x 7; 0.6 x 7; 0.6 x 0.7; 0.06 x 7. The sequence draws attention away from the invariant digits and the numbers involved in multiplying and towards the role of the decimal point and the zeroes. This sequence therefore is more likely to lead to conjectures about decimal places through variants on the answer ‘42’.

Thus some of the task of learning mathematics can be described as identifying *dimensions of possible variation* (DoV) and extending the *ranges of permissible change* within these (RoC) (Watson & Mason 2005b). In the two illustrations given above, the dimensions of variation that jump to attention are the slope of the line and the position of the decimal point and the zeroes. The range of change in the first case is the real numbers: integers, decimals, positive and negative and zero; the range in the second is that the point can end up anywhere to the right or left of the digits ‘42’. Thus focusing on the range of change stimulates exploration of the concept being learnt.

**Exchange systematicity**

Al-Murani studied how teachers who were made aware of this way of thinking about teaching and learning enacted deliberate variation in their algebra lessons, and what effect this had on students’ learning (Al-Murani, 2006). He conducted an intervention experiment with 6 teachers, and a comparison group of 4 teachers. Teachers were not told how to teach, but were helped to see how their choice and sequencing of examples in lessons might be made according to the dimensions of variation of the aspects of mathematics they wanted students to learn (i.e. the object of learning (OOL)). Different characteristics of the mathematical object would emerge depending on the perspective from which it is viewed. The perspective could be a mathematical one, such as whether we are interested in learning about the gradient or the algebraic equation of a graph, but here we look more at the perspectives that are acted out in the pedagogic environment. Marton and Tsui and colleagues (2004) proposed the following categorization of perspectives: *intended OOL* (teacher’s viewpoint); *enacted OOL* (researcher’s viewpoint); and *lived OOL* (learner’s viewpoint).

In Al-Murani’s study students were pre-, post-, and delayed post-tested on their algebraic understanding around the teaching interventions which were based on awareness of variation. The interventions which were based on such awareness had a significant effect on both students’ algebraic and general mathematical progress between the pre-test and post-test, when compared to the groups for which the role of variation was not so clear.

Of course, as in all small studies of teaching, there were very different enactments from different teachers, and one of the non-intervention teachers was found to use dimensions of variation and ranges of change very deliberately in her teaching already, while one of the experimental teachers failed to come to crucial preparation sessions. The results of these two groups were treated as outliers and instead Al-Murani focused on those teachers whose use of variation became more deliberate and aware during the study. It was therefore part of the study to learn more about *how* variation was being used, not just *that* it was being used.

We think of this as studying how the subject matter is handled, and it happened that two of the lessons observed were about how to solve simultaneous equations, so that direct comparisons could be made that included similar content.

**Illustrative cases of how the OOL (simultaneous equations) was handled**

Al-Murani used lesson transcripts to map the flow of subject matter in the lesson through identifying the dimensions of variation being publicly worked on at any time. This process showed that, although similar dimensions were expressed by teachers and students, in some classrooms dimensions that were opened up at some stage in the lesson might never be referred to, whereas in others either the teacher or students would respond about dimensions that had been opened up, either immediately or later. Al-Murani saw this as a process of systematic exchange, and called it ‘exchange systematicity’ in that exchange, in interactions, of dimensions being attended to, was a feature of the lessons of those teachers whose students did better in the post-test. Once a dimension had opened up it was systematically pursued in dialogue. We illustrate this by comparing how variation in similar content is handled differently in two lessons by intervention and comparison teachers. Space constraints prevent us from describing the content of both lessons separately; instead we focus on the content offered in the intervention lesson and contextualise the comparison teacher’s handling of the corresponding content. In these lessons the dimensions of variation that were enacted differently were about aspects of the process of solution. Actual examples given by teachers were similarly varied.

The simultaneous equations 5*x* – *y* = 9 (i) and 4*x* + *y* = 9 (ii) were given by the teacher as introductory material for the lesson. Students had had some previous experience of solving linear equations but not a procedure for solving two together. The class was asked to suggest a method to solve them. When a student proposed ‘getting rid of y’ the intervention teacher emphasised that this was a choice (i.e. choosing which bariable to ‘get rid of’ highlighted a DoV in the lesson). Having asked the class how to proceed and following a suggestion to substitute into (i) to solve for x, she again emphasised the choice of equations in which to substitute the value of x, another DoV. In a similar situation the comparison teacher (merely) eliminated *y*, and upon receiving the suggestion ‘substitute *x* = 2 into either equation (i) or (ii) or both’ substituted into equation (i) without further comment. The comparison lesson continued with formulaic questions which elicited student queries: ‘you know you find x and put it in the equation to get y, can you find y first?’ ‘.if x and y were the other way round would you be finding y first?’ and ‘substitute into (i), (ii), both?’. The comparison teacher acknowledged these but pursued her original direction. In contrast, the intervention teacher then used the equations 3*y* + 2*x* = 7 (i) and 2*y* – *x* = 0 (ii), to demonstrate three of the principle mathematical ideas behind the concept of simultaneous equations: flexibility in eliminating variables, flexibility in substitution and flexibility in equation manipulation. The variations we observed being raised in the lesson, in this case variations in procedural options, were an explicit focus in the lesson.

These contrasts were typical of the lessons observed and analysed with intervention and comparison teachers. While teacher-offered and student-generated variation were present in all classrooms, often along the same dimensions, the intervention teachers tended to use the suggestions of students to emphasise that these were choices from a range of possible variation in methods and examples while comparison teachers tended to treat this awareness as peripheral. Test results that we cannot report here suggest that this increased the ability of students to recognise situations where it was mathematically more efficient to eliminate a particular variable first or substitute into a particular equation.

**Potential implications for teaching and learning**

The variation generated by the learner is a partial articulation of the lived OOL; it may not express everything the student is aware of, but provides a window into some awarenesses. The process of articulation moves the lived OOL from the private domain into the public domain, contributing to the potential development of the enacted OOL by making it available for all learners. The enacted OOL is therefore influenced not only by the intended OOL but also the expressed lived OOLs of learners.

|  |  |  |
| --- | --- | --- |
| Intended OOL | Enacted OOL | Lived OOL |

*Figure 1. How systematicity can inform the teacher’s understanding of the lived OOL and contribute to the convergence of the intended, enacted and lived objects of learning.*

Through public exchange of variation the teacher is able to perceive some of the lived OOL in the learner-generated part of the enacted object of learning. This allows the teacher to establish the correspondence between the intended OOL and the lived OOL and also to influence a more complex engagement with the concepts. With this knowledge the teacher is then able to decide what variation is pertinent / appropriate to offer the students subsequently. Teachers enabling exchange systematicity can then shape the variation exchange so that through a process of co-construction the enacted OOL converges with the other two objects of learning. For exchange to be worthwhile there has to be discussion and participation by students, and they have to feel able to ask questions and volunteer their ideas. In the comparison lesson above this seemed to be for the purpose of clarification for individuals, whereas in the experimental classroom it seemed to be seen as useful for everyone to explore a range of possible choices.

These two examples show how students in similar lessons have different learning opportunities because of the ways in which the teacher manages the dimensions of variation that are opened up, including those that are learner-generated. In some lessons the teacher controls the parameters of the space of learning, whereas in others the teacher and learners jointly construct the space.

**The contribution of variation theory to our understanding of classrooms**

This analysis suggests that attention to the dimensions of variation, ranges of change, and how they are handled by the teacher can distinguish between similar lessons which, in this study, were differently effective in terms of learning outcomes. In an earlier study, David and Watson looked at how apparently similar interactive practices in lessons contained different ways of handling mathematical ideas (2008). Retrospective re-analysis in the light of Al-Murani’s methods shows that one of the differences in the lessons they observed was the handling of variation. For example, In Roisin’s lessons there were several instances where she would respond to the variations opened up by learners by asking about extreme cases of such variation, such as (David & Watson, *ibid* p.40):

Roisin: Has anyone any good ways of working out percentages …?

Student: Move all the digits one place to the left

Roisin begins to follow the instruction on the board; the student interrupts:

Student: to the right

Roisin (later): … if you [just] say ‘to the right’ you could end up with them over there (pointing to far wall)

and in Susan’s lessons variation in learner-generated examples was often used explicitly to illustrate generalities underlying apparent differences (David & Watson, *ibid*, p.42):

Susan: … what I would like are a couple of your examples

Students: Me! Me! Me!

Susan then uses a selection of the offered examples to show how fraction calculations relate to a method using iconic diagrams.

David and Watson used an analysis based on theories of situated cognition to suggest that socio-cultural approaches to comparing lessons distinguish between many aspects of lessons, but do not necessarily reveal differences in the ways in which mathematics is made available to learners. They drew Gibson’s notions of affordances and constraints into the analysis to sharpen the focus on available mathematics and it is worth noting that Gibson made early reference to dimensions of variation when observing responses to nonsense symbols (1969). Al-Murani’s work adds the notion of exchange of dimensions of variation to the analysis of classroom interactions, and had evidence that this improved learning. This is helpful because it gives insight into the importance of how the variations offered by teachers and learners are taken up and developed publicly in the co-construction of the mathematical object of learning.

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