Connection Games

Variations on a Theme

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Dedicated to Merlin

For always being there... even if he
did keep dribbling on the manuscript.

3 Games as Graphs

Connection games are tightly coupled with the topology of the board surface on which they are played. Much can be learned about a game by stripping it down to its underlying graph.

Figure 3.1 shows the symbols used for this analysis. Points at which pieces can be placed are described by *vertices*, which are generally neutral until owned by either player. Connections between adjacent vertices are described by *edges*. Similarly, edges are neutral until claimed by either player. More detailed definitions can be found in Appendix A, Basic Graph Theory.

3.1 Duals

A game board can be redefined as a graph using its *dual*. This is done by creating a neutral vertex for each board point, then connecting adjacent points with neutral edges.

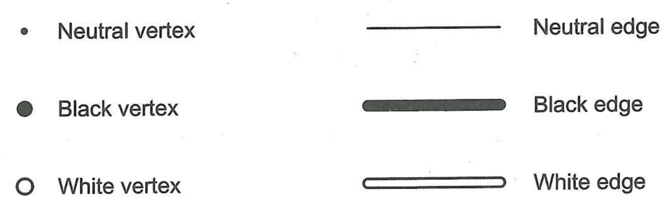

Figure 3.1. Key to the symbols used for graph analysis of games.

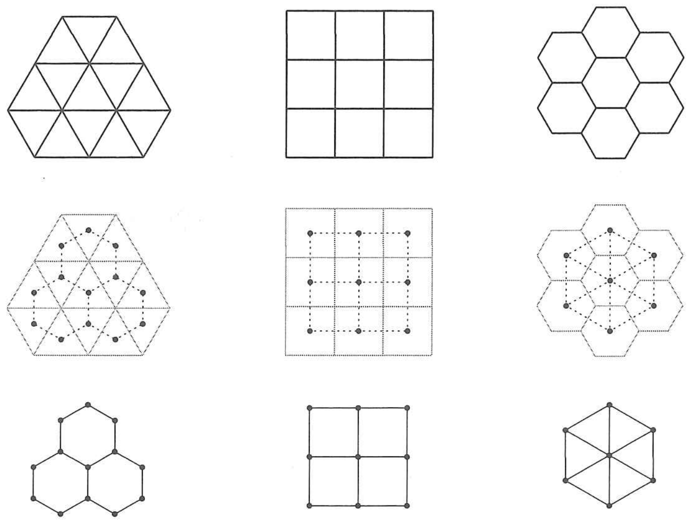

Figure 3.2. Some common grids and the construction of their duals.

This process is illustrated in Figure 3.2, which shows the duals of the three regular tessellations. The dual of the square grid is itself square, while the hexagonal and triangular grids are duals of each other. Notice that each edge of the dual graph crosses exactly one edge in the original graph. The dual representation of a game board showing the adjacency of board points is described as that game's *adjacency graph*.

For instance, Figure 3.3 shows the adjacency graph of a small Twixt board. Note that in this game bridges connect pieces a knight's move apart; physical neighbors are not necessarily adjacent in a connective sense.

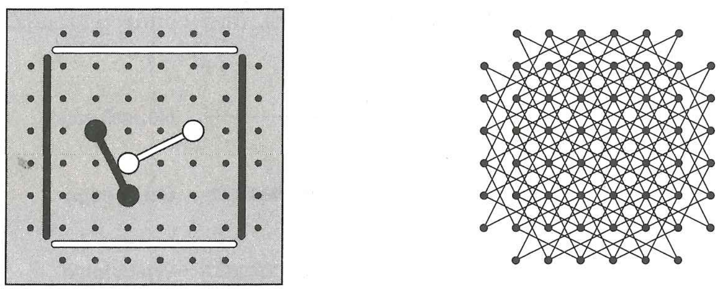

Figure 3.3. The adjacency graph of a small Twixt board.

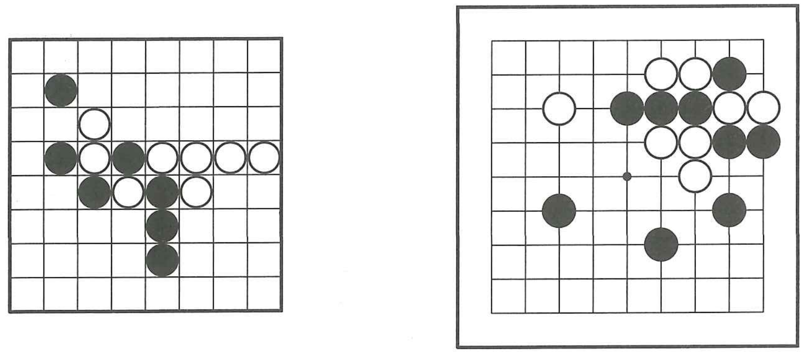

Figure 3.4. Lines of Action is played on the cells of a square board, while Go is played on the intersections (its dual).

Some games are already played on their adjacency graph. For instance, Go is played on points of intersection (Figure 3.4, right) that correspond directly to vertices in the adjacency graph. In such cases, the adjacency graph is taken directly from the board intersections, not the board cells.

3.2 The Shannon Switching Game

The *Shannon switching game*, or simply *Shannon game*, is an abstract two-person game played on a graph [Shannon 1955]. The graph is initially connected and all edges are in a neutral state. One player, *Join*, aims to permanently connect two distinguished vertices with a path of colored edges, while the other player, *Cut*, aims to permanently disconnect these vertices by cutting the graph.

Bridg-It, a board game from the 1960s, is the classic example of a Shannon game. Figure 3.5 (left) shows the interspersed grids of black and white pegs that make up a Bridg-It board. Players take turns placing a bridge between two orthogonally adjacent pegs of their color, in an effort to connect their two sides with a path of bridges.

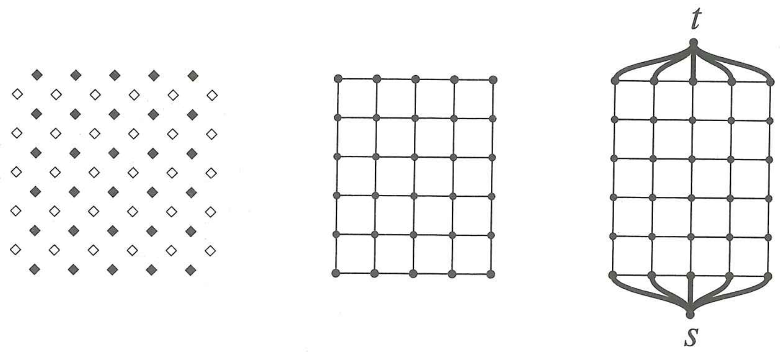

Figure 3.5. The Bridg-It board, its adjacency graph, and its game graph.

Figure 3.5 (middle) shows the adjacency graph corresponding to this board. Note that only black adjacencies are shown, because a white move implies the removal of the black connection that it crosses. Black adopts the role of Join and White adopts the role of Cut in this example.

Figure 3.5 (right) shows the adjacency graph with two distinguished terminal vertices s and t added. Each terminal is fully connected to those vertices along a Join side. The adjacency graph with terminal vertices added will be called the *game graph* throughout this book. This term is also used in some literature to describe a positional decomposition of a game; such structures will be called *game trees* to avoid confusion.

There exist two forms of the Shannon game, which are described as the *Shannon game on the edges* and the *Shannon game on the vertices*. Bridg-It is a Shannon game on the edges.

3.2.1 On the Edges

In the Shannon game on the edges, players take turns making the following moves:

- Join colors one neutral edge, and
- Cut deletes one neutral edge.

Figure 3.6 shows a game won by Black (Join) who has completed a path of bridges connecting the top and bottom board sides. It can be seen in the game graph that terminal vertices s and t have been connected.

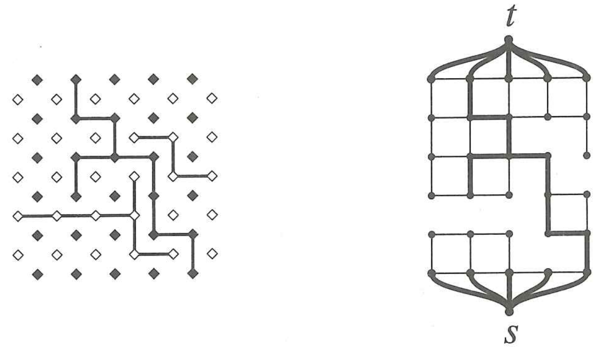

Figure 3.6. A game of Bridg-It won by Black (Join).

Figure 3.7 shows a game won by White (Cut) who has completed a path of white bridges connecting the left and right board sides. It can be seen in the game graph that this path of white bridges has cut the graph in two, permanently disconnecting terminal vertices s and t.

The Shannon game on the edges has a known solution (see Appendix B). The presence of a solution will turn some players off a game, but will intrigue another class of player who sees each game as a puzzle to be solved. In fact, Elwyn Berlekamp considers the Shannon game on the edges to be the most interesting game covered in this book, because it has an elegant solution yet still provides a worthwhile contest between players who do not know this solution.

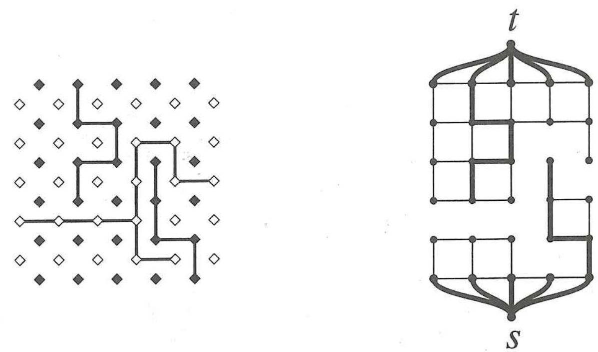

Figure 3.7. A game of Bridg-It won by White (Cut).

Figure 3.8. A 3 x 3 Hex board and its game graph.

3.2.2 On the Vertices

The Shannon game on the vertices is identical to the Shannon game on the edges except that at each turn,

- Join colors one neutral vertex and all incident neutral edges leading to another colored vertex, and
- Cut deletes one neutral vertex and all incident edges.

Hex is a Shannon game on the vertices. Figure 3.8 shows a 3 x 3 Hex board (left) overlaid with its game graph (right).

Again, Black adopts the role of Join and aims to permanently connect *s* and *t* with a chain of colored edges, while White adopts the role of Cut and aims to disconnect *s* from *t* by cutting the graph. Figure 3.9 shows the sequence of moves in a game won by Black (Join) and the modified game graph after each move.

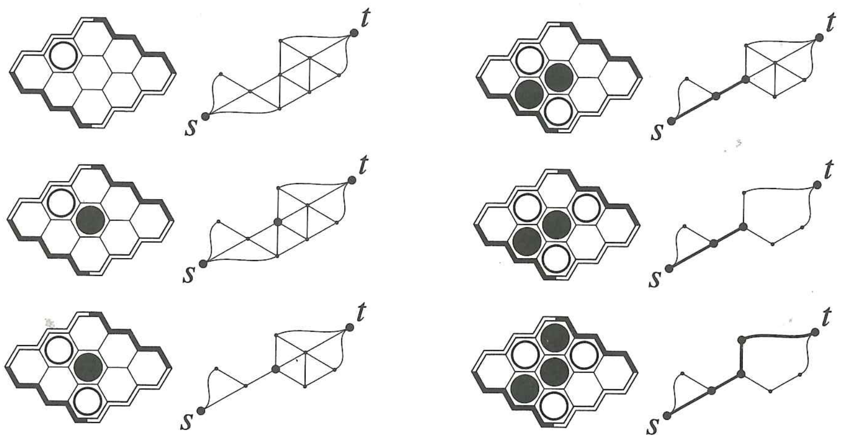

Figure 3.9. A 3 x 3 game of Hex won by Black (Join).

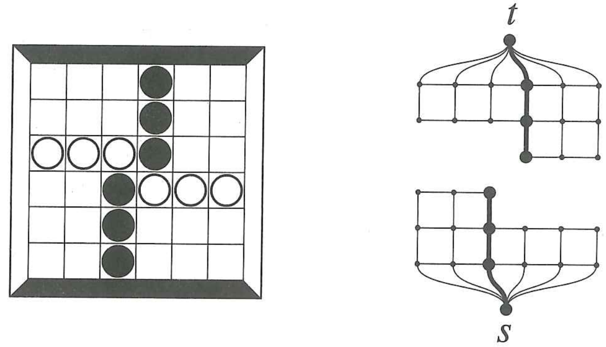

Figure 3.10. A game of Square at equilibrium ... or is it?

3.3 Deadlocks

It turns out that most connection games can be naturally described as a battle between Cut and Join. However, the serious problem of *deadlocks* must first be addressed. A deadlock occurs when opposing chains meet at a point but neither player can connect across that point.

Figure 3.10 shows a hypothetical game called Square, played by the same rules as Hex but on a square grid. The board position shown is deadlocked and can no longer be won by either player, however, its game graph (right) incorrectly suggests that White has achieved a cut and therefore won the game.

The problem can be traced to the connectivity of the underlying square grid. Consider the white piece *a* on the hexagonal grid shown in Figure 3.11 (left). The two black neighbors *b* and *c* are in consecutive clockwise order around this piece and are themselves adjacent, hence Black has a connection through *b* and *c*.

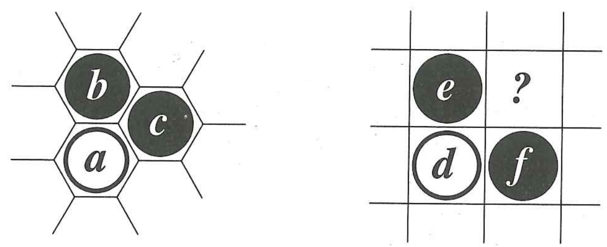

Figure 3.11. A white piece with two consecutive black neighbors on the hexagonal and square grids.

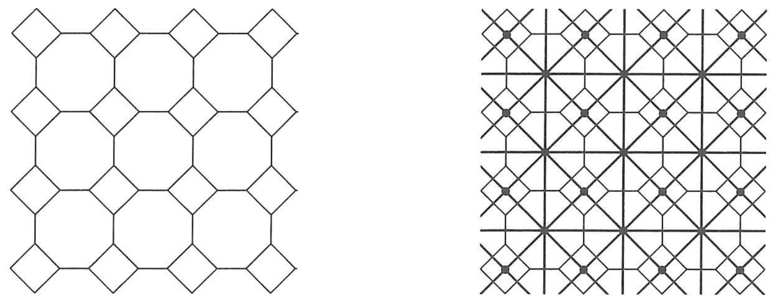

Figure 3.12. The 4.8.8 grid is not subject to deadlocks, as shown by its game graph.

On the square grid, however, consecutive neighbors around the white piece *d* are not necessarily themselves adjacent (Figure 3.11, right). If White plays at the critical point marked *?*, then a local deadlock is reached.

Games played on a planar graph can be deadlocked if the consecutive neighbors around each chain are not themselves adjacent. Luckily such potential deadlocks are easily spotted; any region of the game graph that is not *trivalent* (three-sided) is subject to deadlock. The reasons for this are explained in Appendix C, Hex, Ties, and Trivalency.

Looking back at Figure 3.2, it can be seen that the hexagonal grid is the only regular tiling guaranteed to avoid deadlocks, hence it is a popular choice for connection games.

The 4.8.8 tiling shown in Figure 3.12 is another tessellation that avoids the problem of deadlock. This tiling is used to good effect in games such as Quax and Stymie. See Appendix H for a further discussion of tessellations.

Figure 3.13 shows how the game graph can be modified to accommodate games that may be deadlocked. Note that players now have their own pair of terminals, and that White now claims edges and vertices rather than cutting them. The elegant Cut/Join simplicity of deadlock-free games has been lost; games of this type are Join/Join in nature instead.

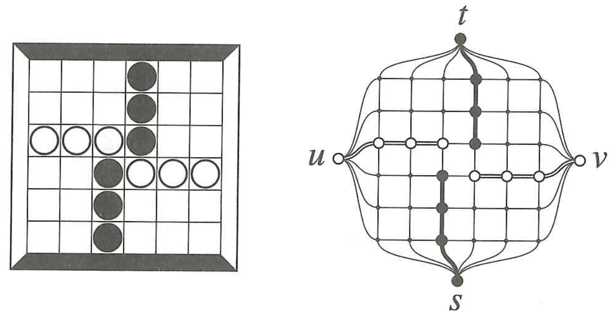

Figure 3.13. The game of Square correctly at equilibrium.

Connection games played on nontrivalent grids must involve special mechanisms to stop deadlocks spoiling the game. For instance, Gonnect allows capture to avoid global deadlocks, Trellis provides directional diagonal connections, Akron allows pieces to stack up to climb over blocking connections, and so on. Lynx provides an especially interesting solution to deadlocks on the square grid.

3.4 Example Graph Analysis

The following example shows how graph analysis can provide some insight into a given connection game. Figure 3.14 shows a game of Hex played on a map of the contiguous United States, as suggested by David Book [1998]. This idea is also included in an educational game suggested by Scott [1938].

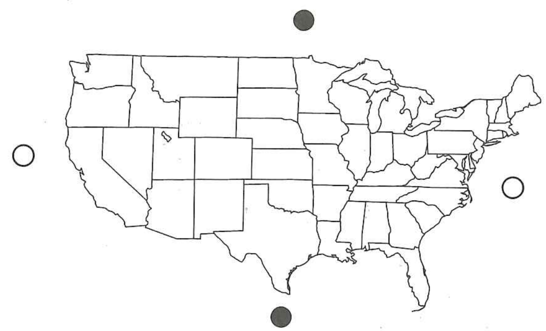

Figure 3.14. Hex played on the map of the contiguous United States.

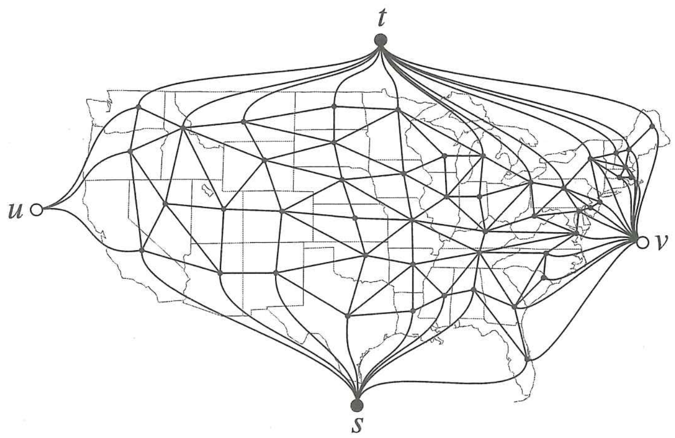

Figure 3.15. The game graph of the map.

Players take turns coloring a neutral state of their choice. Black aims to complete a black path between Canada and Mexico (including those states along the Gulf of Mexico), while White aims to complete a white path between the Pacific and Atlantic Oceans. As usual, adjacency is defined by borders shared between regions. The game graph of this map is shown in Figure 3.15.

Figure 3.16 shows the same game graph in a neater format. Terminals are shown for both players in the Join/Join style as potential deadlocks

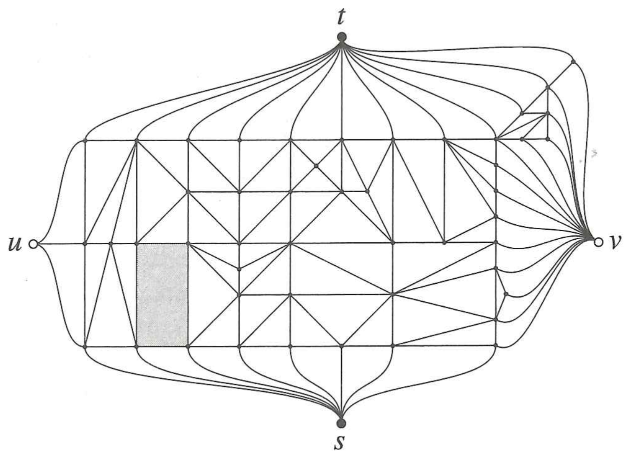

Figure 3.16. The game graph tidied up, with the potential deadlock region shaded.

may occur due to a nontrivalent region (shaded) that corresponds to the point known as the Four Corners where four states meet.

Despite the possibility of deadlock, Black can always win on this map no matter who moves first. The shortest path between Black's terminals requires only three moves, whereas the shortest path between White's terminals requires seven moves, putting White at an obvious disadvantage.

In addition, note that the six states in the top right corner are connected to the rest of the map through a single state (New York) that is adjacent to both *t* and *v*. This cluster is superfluous to the game and playing in any of these six states would be a wasted move.

This example demonstrates how the game graph can yield some useful insight into the underlying board design: whether the game can be tied, whether one player has a significant advantage, weaker regions to be avoided, and so on.