The Treatment of Elementary Geometry by a Group-Calculus
Author(s): G. Thomsen
Source: *The Mathematical Gazette*, Vol. 17, No. 225 (Oct., 1933), pp. 230-242
Published by: The Mathematical Association
Stable URL: http://www.jstor.org/stable/3607859
Accessed: 06/06/2014 10:23

THE TREATMENT OF ELEMENTARY GEOMETRY BY A GROUP-CALCULUS.

BY G. THOMSEN.

THERE is no need to recommend to teachers in this country, familiar with Dobbs' *School Course* and with later books of the same tendency, the free use of the ideas of displacement, rotation, and reflection. In the course of a paper read at the Zürich Congress last year, Prof. Thomsen, of Rostock, gave some striking examples of the effective introduction of the language and notation of the theory of groups into work of this kind, and afterwards he agreed willingly to contribute to the *Gazette* an article on the subject. For applications to their own problems we must refer readers elsewhere*, since Prof. Thomsen has paid them the compliment of explaining a point of view that would be quite unsuitable in the school. To some of our readers we must apologise for replacing the original article by a translation, but since there are no familiar symbols or recognisable formulae to facilitate the grasping of a novel theory, we think that a substantial minority will be grateful for this assistance. In making his version, Prof. Neville has taken one or two liberties with the author's notation.

In the following article the analytical geometry of Descartes is replaced by another method of dealing with elementary metrical geometry. This method does not depend on the concept of number and the laws of algebra, but is based only on the concept of a group and deals with elementary geometry by means of a pure group-calculus†.

§ 1. The group of congruent transformations in the euclidean plane falls into the two divisions of direct congruences or displacements and mirrored congruences or reversals. For particular assumptions regarding congruent transformations the notation of the theory of groups is almost instinctive. The transformation which is the result of applying the two transformations $\mathcal{A}$, $\mathcal{B}$ in succession is denoted as a product $\mathcal{B}\mathcal{A}$. If the same transformation results from applying in succession the transformations C, $\mathcal{D}$, we write $\mathcal{D}C = \mathcal{B}\mathcal{A}$. If $\mathcal{D}$ is known, we can determine C from this group relation, by "multiplying on the left" by the transformation $\mathcal{D}^{-1}$ which is the inverse of $\mathcal{D}$; we have simply $C = \mathcal{D}^{-1}\mathcal{B}\mathcal{A}$. Again, in any group relation we can always bring all the terms to one side, leaving on the other side nothing but identity, which may be denoted‡ by 1; in our example, $C^{-1}\mathcal{D}^{-1}\mathcal{B}\mathcal{A} = 1$.

Among congruent transformations, those which are involutions are of special importance. An involutionary transformation is one which coincides with its inverse ($\mathcal{A}^{-1} = \mathcal{A}$). A geometrical trans-

* G. Thomsen, "Über einen neuen Zweig geometrischer Axiomatik und eine neue Art von analytischer Geometrie," *Math. Zeitschrift*, 34, pp. 668-720 (1932). See further: G. Thomsen, "Grundlagen der Elementargeometrie", Hamburger mathematische Einzelschriften, 15 (Teubner, 1933).

† The necessary elements of this calculus are set out in the paper to which reference has already been made.

‡ In the theory of groups, identity is usually denoted by the letter *E*, reserved for the purpose.

formation is involutionary if the result of applying it twice in succession to an arbitrary figure is to reproduce the original figure ($\mathcal{A}^2 = 1$). It is well known that the only congruent involutions other than mere identity are reflections in points and reflections in lines. Reflection in a point or a line will be denoted by the same symbol as the point or line itself.

A succession of reflections which results in identity we call a circuit of reflections. Thus to say that $TdSRcba$ is a circuit means that after seven reflections, in the four given lines a, b, c, d and the three given points R, S, T, taken in the order indicated, that is, in the reverse of the written order, any figure is found in its original position.

Calculation with circuits of reflections is much simplified by the fact that there is no need to attach the negative exponent -1 to an inverse element in the process of solution. For example, suppose that for six point-reflections P, Q, R, S, T, V, the successions $PQRS$, $RSTV$ both result in identity. From the first circuit, on multiplication first by P and then by Q, it follows, since $P^2 = Q^2 = 1$, that $RS = QP$, and substituting in the second succession we find that

If PQRS *and* RSTV *are both circuits, so also is* QPTV.

The following rules evidently hold for any circuit :

(1) Any number of reflections may be moved from the beginning to the end of the circuit : for example, if $RcbQPa = 1$, then also $QPaRcb = 1$.

(2) The order may be reversed : if $RcbQPa = 1$, then also $aPQbcR = 1$.

In a phrase, *the circuit property concerns only the cyclic order of the reflections.*

If we use such natural abbreviations as $(QPba)^2$ for $QPbaQPba$ and $(cba)^3$ for $cbacbacba$, the letters inside the brackets may also be rearranged according to the rules just given : if $(QPba)^2$ and $(cba)^3$ are circuits, so are $(baQP)^2$ and $(abc)^3$.

We have one further addition to make to our notation. If a succession of reflections $QPba$ is already a circuit, so also is the succession $(QPba)^n$ formed by any number of repetitions of $QPba$. But the converse is of course not true : $(QPba)^n = 1$ does not imply $QPba = 1$, and in the simplest case, a proper involution is precisely a transformation $\mathcal{A}$, such that $\mathcal{A}^2 = 1$ while $\mathcal{A} \neq 1$. If $(QPba)^n = 1$ is assumed not to be a consequence, trivial as it would be, of any simpler relation $(QPba)^m = 1$ in which m is a factor of n, we speak of $(QPba)^n$ as a primitive circuit, or more briefly as a cycle, of order n, and we write * $QPba \in \Omega_n$. Thus $QPba \in \Omega_2$ means that $(QPba)^2 = 1$, while $QPba \neq 1$: the succession of reflections taken twice restores every figure to its original position, but taken once only is a genuine transformation of the plane.

* In the theory of equations ω_n is commonly used to denote a primitive root of $z^n = 1$, that is, a root which is not a root of a similar equation of lower degree. Two cycles of the same order are not necessarily identical, and therefore we use the logical symbol of inclusion, ϵ, rather than a symbol of equality, which might prove misleading.

If three congruent transformations $\mathcal{A}$, $\mathcal{B}$, C are such that

$$C = \mathcal{B}^{-1}\mathcal{A}\mathcal{B},$$

then C is called, in the usual terminology of the theory of groups, the transform of $\mathcal{A}$ by $\mathcal{B}$. Geometrically we obtain C *by transforming the one transformation by means of the other,* that is, by combining with $\mathcal{B}$ the correspondence which $\mathcal{A}$ establishes between points of the plane and translating this into another correspondence. For suppose that $\mathcal{A}$ carries X to Y; since $\mathcal{B}$ carries $\mathcal{B}^{-1}X$ to X, $\mathcal{A}\mathcal{B}$ carries $\mathcal{B}^{-1}X$ to Y, and $\mathcal{B}^{-1}\mathcal{A}\mathcal{B}$ carries $\mathcal{B}^{-1}X$ to $\mathcal{B}^{-1}Y$. If $\mathcal{B}$ is an involution, the transform is expressible as $\mathcal{B}\mathcal{A}\mathcal{B}$ and is the correspondence which carries $\mathcal{B}X$ to $\mathcal{B}Y$. For example, aPa is reflection in the point which is itself the reflection of the point P in the line a.

§ 2. All the simplest positional relations between points and lines in a plane can be characterised by circuits of reflections. For example, two distinct lines a, b are at right angles if and only if reflections in them are permutable, that is, if and only if $ab = ba$, or again, if and only if $(ab)^2$ is a primitive circuit. We notice that $ab = 1$ is the condition for the lines to coincide and must be excluded, since a line is not at right angles to itself. Thus the symbolical expression of perpendicularity is $(ab)^2 = 1$, $ab \neq 1$, or briefly, $ab \in \Omega_2$.

The following Table shews the characterisation of a number of elementary relations.

1. $ab \in \Omega_2$	*Perpendicular lines.*
2. $(aP)^2 = 1$	*The relation of incidence*: the line passes through the point. The relation $aP = 1$ is impossible and therefore need not be formally excluded.
3. $(abc)^2 = 1$	*The three lines belong to the same pencil,* in the sense of projective geometry, that is, are either concurrent or all parallel. Again the formal exclusion, of $abc = 1$, is unnecessary.
4. $PQRS = 1$	*The points are the vertices of a parallelogram,* taken in order.
5. $PRQR = 1$, $P \neq Q$	R is the *mid-point* of PQ.
6. $PaQa = 1$	a is the *radical axis* of PQ, that is, the equidistance locus or perpendicular bisector.
7. $acbc = 1$	c is a *midline* of ab; that is, if a and b are not parallel, c bisects one of the angles between them.
8. $PR(QR)^2 = 1$	*R divides PQ in the ratio* 2 : 1.
9. $abP = 1$	*a and b cut at right angles at P.*
10. $abZ \in \Omega_2$	a and b are *parallel,* and Z is an arbitrary auxiliary point. If it is understood that Z is arbitrary, the condition $(abZ)^2 = 1$ is adequate, since by (9) the condition $abZ = 1$ would render Z determinate.
11. $abdcZ \in \Omega_2$	*The cross* formed by the pair of lines a, b is *directly congruent with the cross* formed by the pair c, d, it being understood again that the condition is satisfied for every position of Z.

12. $A_1MA_2M\ldots A_nM=1$	M is the *centroid* of $A_1, A_2, \ldots A_n$ for equal loads at these points.
13. $\begin{cases} ahabchbc=1 \\ (abc)^2 \neq 1 \end{cases}$	h is the *altitude* line perpendicular to a in the triangle formed by a, b, c.
14. $\begin{cases} bcafacbf=1 \\ (abc)^2 \neq 1 \end{cases}$	f is the first *side of the pedal triangle* of the triangle formed by a, b, c.
15. $\begin{cases} bcafacbf=1 \\ (abc)^2 = 1 \end{cases}$	a, b, c are concurrent *bisectors of the angles of some triangle* in which f is the side opposite to the angle bisected by a.
16. $\begin{cases} a\,(bad)^2a\,(dab)^2=1 \\ (bad)^2 \neq 1 \end{cases}$	The triangle a, b, d is *isosceles*, with equal sides along b, d. If $(bad)^2=1$, the other condition is satisfied identically.
17. $\begin{cases} abCAbcABcaBC=1 \\ (abc)^2 \neq 1,\ (abZ)^2 \neq 1 \end{cases}$	The triangle whose sides are the lines a, b, c is *directly similar* to the triangle whose vertices are the points A, B, C.

The proofs of the assertions contained in this Table depend for the most part on the following considerations :

(1) If a and b intersect, the product ba is a rotation round their point of intersection through an angle which is twice the angle from a to b ; the rotation is through an angle not greater than two right angles if the direction of rotation is that in which an angle from a to b is not obtuse.

(2) If a and b are parallel, the product ba is a translation in a direction perpendicular to these lines and through a distance which is twice the distance from a to b.

(3) The product QP of two point reflections is the translation in which the vector of the displacement of every point is twice the vector PQ.

And reference can often be made with advantage to the principle of transforming a transformation, as described at the end of § 1.

As an example of the reasoning, (10) may be deduced from (1), (6), (7) as follows :

By hypothesis, abZ is a proper involution, that is, an involution that is not mere identity ; hence abZ is a single reflection. Since reflection in a line involves reversal, while reflection in a point does not, the product abZ does not involve reversal and is therefore a point reflection T. But the relation $abZ=T$ is equivalent to $ab=TZ$, and since TZ is a translation, it follows that a and b are parallel.

A few geometrical theorems can be proved by pure group theory. For example, in virtue of (4) in the Table, we have already in § 1 established a theorem of Desargues :

If PQRS *and* RSTV *are parallelograms, so also is* QPTV.

Again we can prove symbolically the result that

If $(\mathrm{abc})^2$, $(\mathrm{abd})^2$, *and* $(\mathrm{acd})^2$ *are circuits, so also is* $(\mathrm{bcd})^2$.

For $(abc)^2=1$ implies $ab=cbac$, and $(abd)^2=1$ implies $ab=dbad$, whence $cbac=dbad$, that is, $cbacdabd=1$; but the third condition, $(acd)^2=1$, implies $acd=dca$, and making this substitution in the product $cbacdabd$, we have $cbda^2cbd=1$, which on suppression of a^2 gives $(cbd)^2=1$, as required. But (3) of the Table shews that geometrically this theorem is trivial.

The great majority of theorems in elementary geometry can not however be proved by group theory alone. Consider for example the contrast between the result just proved and the theorem that

If (abc)^2 *and* (abd)^2 *are circuits and* a $\neq$ b, *so also are* (acd)^2 *and* (bcd)^2.

It is not in the least of the essence of a group that for any four involutionary elements a, b, c, d for which the relations $(abc)^2=1$, $(abd)^2=1$ are satisfied, the relations $(acd)^2=1$, $(bcd)^2=1$ also must hold; this is easily seen from examples to the contrary in a suitable group. The corresponding geometrical theorem, which is true, rests therefore on special properties of the euclidean congruence group.

§ 3. There are many ways in which we can distinguish between point reflections and line reflections, the two kinds of proper involutions in our congruence group, on the basis of group structure alone. For example, point reflections, which are direct displacements, are characterised by the fact that they can be expressed as squares of congruent transformations, namely, of rotations through a right angle; a line reflection, being a reversal, cannot be a square.

We adopt now the following point of view. Given any group whatever, finite or infinite, we construct an "artificial geometry" from the proper involutions in the group. We divide these involutions into two classes, calling a proper involution a "point" if it can be represented as the square of some element of the original group, a "line" if no such representation is possible*. Then we introduce the conditions tabulated in § 2 as *definitions* of various positional relations. For example, by two perpendicular lines we *mean* two distinct proper involutions a, b of the group, such that neither is the square of any element of the group and that their product, in either order, is an involution; by a line and a point incident in each other we *mean* proper involutions a, P of which P is the square of some element but a is not, while either product of the two is an involution.

For the time being, we shall interpret only the ideas corresponding to conditions (1)-(8) of the Table. The interesting question is, what theorems of ordinary geometry remain valid in an artificial geometry of this kind. If for example we take the group of those permutations of 8 objects which interchange the first 4 objects among themselves and the last 4 also among themselves, there are 99 of these permutations which are involutions, and according to the criterion, 15 of these are points and 84 are lines. In the geometry with these elements, there are pairs of points without a radical axis, but there are also pairs with as many as 26 radical axes, that is, pairs whose equidistance locus consists of 26 lines. Similarly there are pairs of points which cannot be joined by a line, that is, pairs P, Q such that no line a satisfies simultaneously the two conditions $(aP)^2=1$, $(aQ)^2=1$; on the other hand, there are pairs joined by 8 lines, by

* If either class is empty the geometry is uninteresting. Most geometrical theorems have assumptions which cannot be satisfied in this case, and according to the usual convention of formal logic the theorems are true theorems but there is nothing of which they are true.

20 lines, or even by 24 lines. In this geometry midpoints do not exist. Another geometry of the same kind can be constructed in which bisectors of the angles of a triangle are concurrent but perpendicular bisectors of the sides are not.

These examples shew that our artificial geometries diverge widely from ordinary geometry. To mention a few more examples, a midpoint of a pair of points need not lie either on the equidistance locus or on the line joining the pair, and these two lines, if they both exist, need not be at right angles ; again, three lines which have a common point in the sense of condition (2) need not belong to one pencil in the sense of condition (3).

§ 4. So very few of the theorems of ordinary geometry being necessarily true in the artificial geometry associated with an arbitrary group, we proceed to restrict the underlying group by postulates, choosing postulates which are in fact satisfied by the euclidean congruence group. Adding such axioms one by one, we contract the circle of available groups more and more, till finally only the euclidean congruence group is left ; more and more geometrical theorems can be proved, till the whole of elementary geometry is recovered.

The convenient course for us is to take such postulates as provide simple rules of calculation for the rearrangement of reflection circuits, rules which can be combined readily with the rules of group transformation already used. This will be made clearer by examples.

As a first postulate we choose

Ax. 1 : *For any three points,* $(\mathrm{PQR})^2$ *is a primitive circuit.*

The assumption is true in ordinary geometry, since there the product of three point reflections is a single point reflection. In the form $PQR = RQP$, the axiom gives us a simple rule of calculation :

In any succession of reflections which includes three consecutive point reflections, the two outer of these may be exchanged across the middle one.

By means of Ax. 1, we prove

Th. 1 : *If* ASBS, BTCT, *and* $\mathrm{AM(TM)}^2$ *are circuits, so also is* $\mathrm{CM(SM)}^2$.

Expressing the enunciation in the form that if S, T are the midpoints of AB, BC, and if M divides AT in the ratio 2 : 1, then M divides CS also in the same ratio, we see that essentially this is the theorem that *the medians of a triangle are concurrent.* To prove the result, we use the relations $AM(TM)^2 = 1$, $BTCT = 1$ to replace A, B in the relation $ASBS = 1$ by $M(TM)^2$ and TCT, and we have $M(TM)^2STCTS = 1$. Transferring MT to the end of the circuit, we have $M.TMS.TC.TSM.T = 1$; when the interchange allowed by the axiom is made in the two triplets TMS and TSM, the factor T^2 appears twice and can be dropped from each place, leaving $MSMCMS = 1$, the required result.

For our second assumption we take

Ax. 2 : *The total number of line reflections in a circuit must be even.*

In ordinary geometry this axiom is a theorem, since a succession with an odd number of line reflections results in a reversal and therefore cannot end by restoring an arbitrary figure to its original position.

With our two axioms we can prove

TH. 2 : *If* as, bs, *and* cs *are involutions, so is* abc,

that is to say, *three lines which are perpendicular to the same line belong to a pencil.* The argument is as follows. If a, b, c are supposed distinct from s, the assumed involutions are distinct from identity and are therefore either points or lines. But $as=t$ would imply $ast=1$, contradicting Ax. 2 ; the three products are therefore points, and from Ax. 1 we have $(asbscs)^2=1$. Since s is interchangeable with a, b, and c, each s is removable and there remains $(abc)^2=1$, as required.

Another deduction is

TH. 3 : *If for any one point* R, abcdR *is a proper involution, the succession* abcdZ *is a proper involution for an arbitrary point* Z.

For the proper involution $abcdR$ is not a line, since by Ax. 2 there cannot be a circuit of the form $abcdRt$; this involution is therefore a point, and combining this point with R and Z we have from Ax. 1 $(abcdR^2Z)^2=1$, and the result follows on removal of R^2. This theorem having been established, we can define the direct congruence of two crosses in our geometry by means of condition (11) of the Table in § 2, since we know now that this condition represents some relation independent of the auxiliary point Z. If b and c coincide, the theorem reduces to the form that $adR \in \Omega_2$ implies $adZ \in \Omega_2$, which is the analogous basis for the definition of parallelism by means of condition (9). That *congruence of crosses is a transitive relation* is the geometrical interpretation of

TH. 4 : *If* abdcZ *and* cdgfZ *are proper involutions, so also is* abgfZ.

For the proof, we have only to recall again that $abdcZ$, Z, and $cdgfZ$ are three points to which Ax. 1 can be applied, giving $abdcZ^2cdgf\,Z \in \Omega_2$, and the result follows on the removal of square factors.

We add now

AX. 3 : *If* abcRcbaR *is a circuit, then* abc *is a proper involution.*

The premiss of this axiom can be written in the form

$$(cba)^{-1}R\,(cba)=R,$$

and expresses that the transform of the point reflection R by the operation cba is the point reflection itself, or in other words that R is a fixed point of the transformation cba. Now *in ordinary geometry,* cba is a reversal and *any reversal which leaves one point fixed is a reflection* ; it follows that cba is a simple reflection, which is the result postulated in the axiom, and therefore this axiom also is satisfied in ordinary geometry.

Our three axioms are sufficient for the proof first that the perpendicular bisectors of the sides of a triangle belong to a pencil, and then that the three altitudes of the triangle belong also to a pencil.

If A, B, C are the vertices and p, q, r the perpendicular bisectors of the sides, we see from (6) of the Table in § 2 that the concurrence of the bisectors expresses

TH. 5: *If* BpCp, CqAq, ArBr *are circuits, then* pqr *is a proper involution.*

To prove this theorem, we substitute rAr for B and qAq for C in the first circuit, which becomes $rArpqQqp$, and since only cyclic order is relevant, the required result is given at once by Ax. 3.

We turn now to the altitudes. The vertices being denoted by A, B, C, the sides by a, b. c, and the altitudes by f, g, h, the symbolical expression of the result to be proved is

TH. 6 : *Six relations of the form* $(\mathrm{Ab})^2=1$, *three of the form* $(\mathrm{Af})^2=1$, *and three of the form* $\mathrm{af}\in\Omega_2$, *imply* $(\mathrm{fgh})^2=1$.

For proof, we remark that since a is permutable with both B and C, $fBC=fBa^2C=faBCa$; but the product fa is by hypothesis a point, and we can apply Ax. 1 to this point and the two which follow it, thus replacing $faBCa$ by $CBfa^2$ and so by CBf. Now the condition $(Af)^2=1$ implies identically $CBAfAfBC=1$, which is therefore equivalent to $CBA.f.ACB.f=1$, and by symmetry we have also $ACB.g.BAC.g=1$, $BAC.h.CBA.h=1$. Recalling that the products BAC, CBA, ACB are themselves points, we see that we have here a set of conditions of precisely the form required for the application of Th. 5, and the desired conclusion follows at once.

Since concurrence in a point is not the same as membership of a pencil, there is a different definition of the altitude f, as the line satisfying the two conditions $(bcf)^2=1$, $af\in\Omega_2$, and it may be objected that this simpler definition is more natural than the definition implied in Th. 6. This is true, but a further axiom seems to be required for the deduction of $(fgh)^2=1$ from the smaller set of conditions. A sufficient basis is

AX. 4 : *If* abcfghabchgf *is a circuit, either* abc *or* fgh *is a proper involution.*

This axiom, which resembles Ax. 3 but replaces the point R by a product of three lines, is valid in ordinary geometry. If now $af=fa$, we have $(bcf)^2=bcfa^2bcf=bcafabcf$, and the condition $(bcf)^2=1$ implies $bca=fabcf$; similarly from the two conditions $(abh)^2=1$, $ch\in\Omega_2$ we have $abchcabh=1$, that is, $cab=habch$. But from the two conditions $(cag)^2=1$, $bg\in\Omega_2$ we have in the same way $cabgbcag=1$, and substituting in this relation for the two products cab, bca we have $habchgfabcfg=1$, which is the premiss of Ax. 4. Thus we have

TH. 7 : *If* $(\mathrm{bcf})^2=1$, $(\mathrm{cag})^2=1$, $(\mathrm{abh})^2=1$ *and also* $\mathrm{af}\in\Omega_2$, $\mathrm{bg}\in\Omega_2$, $\mathrm{ch}\in\Omega_2$, *then either* $(\mathrm{abc})^2=1$ *or* $(\mathrm{fgh})^2=1$, which is the suggested form of the theorem regarding the altitudes, since the assumption that a, b, c are the sides of a triangle rules out the alternative $(abc)^2 = 1$. It should perhaps be added that since we have not considered concurrence of f, g, h in either Th. 6 or Th. 7, we cannot be said to have dealt in any sense with the existence of an orthocentre.

We introduce now a fifth axiom :

Ax. 5 : *If* abPQbaQP *is a circuit, either* P *and* Q *coincide or* abQ *is a proper involution.*

This axiom again is satisfied in ordinary geometry. If P and Q coincide, the premiss is satisfied automatically. If a and b coincide, the premiss is satisfied automatically and abQ reduces to the proper involution Q. If P and Q are distinct, PQ is a definite translation, and the premiss, written in the form $abPQ = PQab$, expresses that the congruent transformation ab is commutative with this translation. But if a and b intersect and do not coincide, ab is a proper rotation and is not commutative with any proper translation ; on the other hand if a and b are parallel and do not coincide, ab is a translation and is commutative with any translation. Thus the conclusion required by the axiom is that a and b should be parallel, and we see from condition (9) of the Table that this is what is expressed.

Axiom 5, with the four earlier axioms, renders possible the proof of a great mass of the theorems of elementary geometry. For example, in the geometry attached to any group for which these axioms hold :

A midpoint of a point pair lies always on a radical axis and also on a line containing the pair, and these two lines are necessarily perpendicular ;

Three lines which are parallel or which have a common point belong to a pencil

in the sense of (3) of the Table ;

*The bisectors of the angles of a triangle are concurrent.**

Two examples will suffice to illustrate the application of the last axiom. For our first example we take

Th. 8 : *If* aP, aQ, bP, bQ *are all involutions, then either* a *coincides with* b *or* P *coincides with* Q,

which is equivalent to the pair of propositions

Two distinct points have at most one line joining them ;

Two distinct lines have at most one common point.

The existence of at least one line in the one case and at least one point in the other cannot be proved from Ax. 1-5. To prove Th. 8, we argue as follows. Since a is permutable with both P and Q, $PaPQaQ = 1$, and similarly $PbPQbQ = 1$; hence $PQ = aPQa = bPQb$, and therefore $bPQbaQPa = 1$. This is precisely the premiss of Ax. 5, and implies that either P coincides with Q—the second alternative in the enunciation of the theorem to be proved—or abQ is a proper

* *Cf.* Thomsen, *Math. Zeitschrift*, 34, p. 712.

involution, which, by Ax. 2, is a point reflection T. By hypothesis, Q is permutable with both a and b, and therefore $abQ=T$ is equivalent to $Qab=T$; the two relations can be replaced by $ab=QT=TQ$, implying that QT is an involution. But if QT is a line c, we have $abc=c^2=1$, contradicting Ax. 2, and if QT is a point R, we have $QTR=R^2=1$, contradicting Ax. 1; thus QT is not a proper involution, and the only possibility that remains is that QT is identity, implying $ab=1$, that is, $a=b$, the other alternative in the enunciation.

We proceed to a second example of the application of Ax. 5, in

TH. 9. *Let six lines* f, g, p, q, u, v *be subject to the pencil relations*

$$(\mathrm{fpv})^2=(\mathrm{fqu})^2=(\mathrm{gpu})^2=(\mathrm{gqv})^2=1,\quad (\mathrm{fgu})^2\neq 1.$$

Then if the further condition $\mathrm{fugvZ}\in\Omega_2$ *is satisfied for an arbitrary position of* Z, *so also is the condition* $\mathrm{fpgqZ}\in\Omega_2$.

The pencil relations are those of the sides of a complete quadrangle, but the existence of vertices is not assumed. Taking the enunciation as referring to a quadrangle in the ordinary plane, we see from condition (11) of the Table in § 2 that the sixth condition asserts that the cross formed by f, u is congruent with the cross formed by v, g. We are, therefore, dealing with a *cyclic quadrangle*, and the result is nothing but the fundamental theorem, which plays so powerful a part throughout the geometry of the circle, that the one congruence between crosses implies the other congruences which we associate with the figure when it is cyclic. A proof of the theorem is as follows.

From the second condition, $u=qfuqf$, whence from the third condition, taken in the form $(pgu)^2=1$,

$$p\,.\,gqfu\,.\,qfp\,.\,gu=1.$$

Inserting at the points indicated the three products v^2, gvZ^2vg, vZ^2v, each of which is 1, and placing f at each end of the chain, we have

$$fpv\,.\,vgq\,.\,fugvZ\,.\,Z\,.\,vgq\,.\,fpv\,.\,Z\,.\,Zvguf=1.$$

By hypothesis, in consequence of Ax. 2, the products fpv and vgq are lines a, b, and the product $fugvZ$ is a point T; with these substitutions, the relation last written takes the form $abTZbaZT=1$, which is that of the premiss in Ax. 5. The possibility $T=Z$, implying $fugv=1$ and therefore $(fug)^2=1$, is excluded by the enunciation, and therefore $abZ\in\Omega_2$, that is, $fpgqZ\in\Omega_2$, as was to be proved.

§ 5. In our examples of the construction of a geometry from axioms, it is to be noticed that there are no assumptions as to the existence or unique determination of points and lines; in this respect the work is in contrast to Hilbert's *Grundlagen der Geometrie*, where we find as the very first axiom, *Two distinct points always completely determine a straight line*. Certainly a few assumptions as to existence and uniqueness are implicit, though unobtrusive, in our use of the axioms of group theory. There is moreover a fundamental process * by which it is possible to free these assumptions from their dependence on the group concept.

* *Cf.* Thomsen, *Math. Zeitschrift*, 34, p. 682.

Again, in our axioms there are no questions of order or continuity. The ideas of order and continuity can of course be introduced into group geometry by fresh axioms ; in particular, we can characterise the group of congruent transformations in the euclidean plane completely by adding three more axioms to those already given. But such a complete characterisation of the euclidean congruence group by intrinsic properties is not related very closely to our main investigations.

Primarily, the examples of the last section are designed to shew how a group calculus useful in the treatment of elementary geometry can be developed. *The aim is, to make the group calculus into an instrument so perfect as to take a rival place by the side of cartesian analytical geometry.* As a method, the calculus of groups is simpler than the algebra of cartesian geometry, since the former has only one operation, namely, multiplication, while the latter is built on addition and multiplication and the operations inverse to these. Further, the arbitrariness of the coordinate system at once introduces extraneous elements into the discussion of any problem by cartesian methods, and these extraneous elements have to be eliminated by a theory of invariants or by some method of vector analysis. In the treatment of geometry by the algebra of groups, no redundant elements are involved in the calculus itself, and the method appears for that reason the more suitable.

Consider for a moment the calculations necessary to prove the existence of the orthocentre by cartesian methods. We put down the coordinates of the three vertices, and we assume equations for the three sides and the three altitudes. We then write down the nine conditions of incidence and the three conditions of perpendicularity, and from our twelve equations we derive the condition which implies that the three altitudes are concurrent. As all of us are familiar with this kind of algebra from childhood, and some of us perhaps equally ready with vectorial methods, it goes without saying that we can give very elegant proofs of the theorem. But if we are to make a fair comparison between the group calculus and the cartesian method, we must take pedantic account of every step in the argument and of every application of fundamental rules like the distributive law, since the corresponding laws of the group calculus modified by our associated axioms are not yet second nature to us.

Even if in a comparison of this kind a multitude of individual examples seemed to tell in favour of the group calculus, we might still be far from entitled to speak of a serious rival to the cartesian method, which has the essential property of rendering elementary geometry "trivial". Whereas before Descartes every theorem demanded for its proof some new idea, some happy chance, as for example the drawing of some peculiar lines in the figure, analytical geometry provides once for all an infallible means of completing a proof in every case in a finite number of steps by diligence and routine alone. Even to-day we often encounter the belief that it is only on the algebraical side and not in proofs dependent on the construction and consideration of geometrical figures that we possess a method of rendering geometry in this sense trivial. This is a mistake, for with the help of Hilbert's line calculus we can easily transfer the trivialisation from the cartesian method to pure geometry. But for the majority of theorems, the proofs which have been fitted into the framework of Hilbert's calculus are essentially more complicated than familiar classical proofs. Naturally a trivialising method cannot be expected to give the "best" proof in each individual case. Nevertheless, more intimate experience of the whole nature of a method which consists in translating the cartesian process into terms of pure geometry encourages the belief that some better trivialising method must be discoverable. Perhaps the group calculus developed in this paper may point the way to the discovery. Hitherto the search for a suitable method has not prospered. Needless to say, the cartesian method could be translated into group calculus by way of pure geometry, but what is wanted is a better trivialising method natural to the group calculus itself.

§ 6. The calculus of reflections can be applied successfully to solid geometry. In the group of congruent transformations in space there are proper involutions of three kinds, reflections in points, in lines, and in planes ; these we shall denote by Latin capitals, small Latin letters, and small Greek letters (other than ϵ) respectively. Most of the positional relations can be expressed very simply. A few examples must suffice.

1. $(\lambda P)^2 = 1$. The *incidence* relation : the point is in the plane.
2. $ab \in \Omega_2$. The lines *cut and are perpendicular*.
3. $(\lambda g)^2 = 1$. The product λg cannot be identity or a line reflection, but the other two kinds of proper involution are possible ; if λg is a point reflection, *the line* g *is perpendicular to the plane* λ ; if λg is a plane reflection, *the line* g *lies in the plane* λ.
4. If abZ is a point reflection, for every position of the auxiliary point Z, *the lines* a, b *are parallel.*
5. If $(ab)^2Z$ is a point reflection but abZ is not, where Z is again arbitrary, *the lines* a, b *are perpendicular* but do not necessarily intersect.
6. If $(g\lambda)^2Z$ is a point reflection but $g\lambda$ is not, *the line* g *is parallel to the plane* λ.
7. If $(abc)^2Z$ is a point reflection, but abc is not an involution, *the three lines are parallel to one plane.*
8. If $\lambda\mu\nu$ is a point reflection, *the three planes are mutually perpendicular.*
9. If $(\lambda\mu\nu)^2Z$ is a point reflection, still for an arbitrary position of Z, but $\lambda\mu\nu$ is not, *the three planes are parallel to one line.*

The three kinds of proper involution in three dimensions can be distinguished by intrinsic properties as follows. First the line reflections can be set aside as those which are the squares of elements of the group. The point and plane reflections together compose a class in which the plane reflections can be recognised by the property that a plane reflection forms with some other element of this class a product which has period 3, that is, a proper transformation which if applied 3 times in succession to any figure restores the figure to its original place. (With any plane λ can be associated another plane μ which cuts it at an angle of $60°$, and then $(\lambda\mu)^3=1$.) It is not possible for the product of two point reflections or of a point reflection and a plane reflection to have the period 3.

If in the way just sketched we form an artificial solid geometry, instead of an artificial plane geometry, from the involutionary elements of the permutation group used for an illustration in § 3 above, we have a geometry with 36 points, 15 lines, and 48 planes.

Details of three-dimensional geometry will be found in a forthcoming paper * by H. Boldt, who gives a complete characterisation of the congruent transformations of euclidean space by intrinsic properties; the number of axioms needed for this purpose is comparatively small.

§ 7. In conclusion, let us refer again to a defect in the reflection calculus. The condition for the steps determined by two pairs of points PQ and RS to be congruent cannot be expressed as the existence of a circuit involving only the four points P, Q, R, S; the circuit must involve at least one arbitrary auxiliary point or line. The condition cannot be reduced to a simpler form than: *There exists* some line a such that $aPQaRS$ is a circuit. Similar relations have to be used in solid geometry to express the condition that two lines intersect and the condition that one pair of planes is congruent with another pair.

G. THOMSEN.

---

936. Je n'ai jamais été assez loin pour bien sentir l'application de l'algèbre à la géométrie. Je n'aimais point cette manière d'opérer sans voir ce qu'on fait, et il me semblait que résoudre un problème de géométrie par les équations, c'était jouer un air en tournant une manivelle. La première fois que je trouvai par le calcul que le carré d'une binôme était composé du carré de chacune de ses parties, et du double produit de l'une par l'autre, malgré la justesse de ma multiplication, je n'en voulus rien croire jusqu'à ce que j'eusse fait la figure. Ce n'était pas que je n'eusse un grand goût pour l'algèbre en n'y considérant que la quantité abstraite; mais appliquée à l'étendue, je voulais voir l'opération sur les lignes; autrement je n'y comprenais plus rien.—Rousseau, *Les Confessions*, livre VIe. [Per Mr. J. B. Bretherton.]

937. Quoiqu'il ne fallût pas à nos opérations une arithmétique bien transcendante, il en fallait assez pour m'embarrasser quelquefois. Pour vaincre cette difficulté, j'achetai des livres d'arithmétique, et je l'appris bien car je l'appris seul. L'arithmétique pratique s'étend plus loin qu'on ne pense quand on y veut mettre l'exacte précision. Il y a des opérations d'une longueur extrême, au milieu desquelles j'ai vu quelquefois de bons géomètres s'égarer. La réflexion jointe à l'usage donne des idées nettes, et alors on trouve des méthodes abrégées, dont l'invention flatte l'amour-propre, dont la justesse satisfait l'esprit, et qui font faire avec plaisir un travail ingrat par lui-même. Je m'y enfonçai si bien, qu'il n'y avait point de question soluble par les seuls chiffres qui m'embarrassât, et maintenant que tout ce que j'ai su s'efface journellement de ma mémoire, cet acquis y demeure encore en partie au bout de trente ans d'interruption. Il y a quelques jours que, dans un voyage que j'ai fait à Davenport, chez mon hôte, assistant à la leçon d'arithmétique de ses enfants, j'ai fait sans faute, avec un plaisir incroyable, une opération des plus composées.—Rousseau, *Les Confessions*, livre V. [Per Mr. J. B. Bretherton.]

* To appear in *Math. Zeitschrift*.