11 What does it mean to understand something and how do we know when it has happened?

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Words which appear to describe cognition, such as 'understanding' and 'knowing', are used throughout educational literature, and in teachers' shared discourse, with flexibility and fluency. When we try to use them precisely they become problematic, as they can take slightly different meanings, but we communicate effectively about them by elaborating what we mean. However, once they enter the statutory language through official documents which describe what education should be achieving they can no longer be used casually. Teachers are accountable for the ways in which they fulfil the statutory requirements, and need to have a worked-out and justifiable view of what 'understanding' means. Phrases such as 'knowledge and understanding' and 'mathematical understanding' are used in the Initial Teacher Training National Curriculum (ITTNC) (ITA, 1999), and the Mathematics National Curriculum (NC) (QCA, 1999) refers frequently to pupils' 'ability to use and understand concepts' and to assessing such ability. These requirements suggest that there is a state called 'understanding' and we can know when it exists and when it does not exist.

Understanding as a state

In this chapter I am going to argue that the idea that pupil progress in mathematics can be seen by assessing recognisable states of understanding is an over-simplification of how learning happens.

It is very common for new teachers to find themselves thinking, 'I never really understood addition of fractions (or calculus, or graph-plotting etc.) until I had to teach it!' In other words, the thinking involved in planning to teach (such as working out how to explain or exemplify and predicting

[1] Reconstructed from pages 161-175 of 'Teaching Mathematics in Secondary Schools: a Reader' edited by Linda Haggarty (2002), for the Open University. Published by Routledge, London

what pupils will find difficult) has enabled the teacher to re-examine existing knowledge and look at it in a new way that is recognised as being deeper, more connected and more secure than previous experience. Possibly the teacher has easily remembered how to add fractions, but thinking about how to teach has led to considering why it is done that way and brought new insights into the importance of equivalence, or has raised an awareness of the numerical value of the fractions. And yet the teacher has been able to add fractions, pass examinations involving this skill and be thought of as 'understanding adding fractions'. What is being recognised here is that, even when one is extremely competent in a mathematical technique, there are still ways in which understanding can grow in a new situation, when one looks at the topic differently. Understanding is not static.

Marton and Saljö (1997) classify learning as surface (learning procedures and descriptions) or deep (learning about connections and relationships with previous knowledge)[2]. This kind of distinction can be useful when planning how to teach, but fails to take account of the fact that mathematical procedures consist of strings of simpler procedures which could be described as previous knowledge. To continue the example of adding fractions, one has to multiply and to add, to identify multiples and factors, to find common multiples and common factors. all dependent on previously acquired knowledge and skills. In this sense, learning mathematical procedures inevitably involves connecting and employing previously learnt procedures. What is missing from this observation, but is implied in Marton and Saljö's distinction, is a sense of underlying meaning allowing us to explain why we add fractions this way and justify the answers we get.

Nevertheless, most mathematicians do not explain their actions when adding fractions. It is usually enough to know how to do it and to understand that the method works, but being able to reconstruct explanations, if needed, can contribute to future learning. So here we have two meanings of understanding: 'I understand that in situation X I need to do Y' and 'I understand why I need to do Y in situation X.' Ryle (1949), in describing types of knowledge, referred to these as knowing-that (factual,

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[2] I now recognise that they were talking about approaches to learning rather than the nature of what is learnt (2018).

definitional) and knowing-why. He also describes a third type, knowing-how, which is the knowledge required to carry out the chosen action.
Examples of knowing-that can be found in the NC, for example 'Understand that "percentage" means "number of parts per 100" ' (p. 59). In this case understanding appears to mean 'knowing a definition of a word', where the definition gives us some clues (but very few) about what we can do with it mathematically. Some students may be able to construct everything they need to do with percentages from this fact, others may need much more help, but all can be tested on whether they can repeat definitions and correctly use procedures in particular circumstances. There is widespread agreement that what is being tested is not 'understanding', which relates to more complex forms of knowledge, but whether pupils can act in a certain way in the very precise circumstances of the test — a very localised knowing-that.
A state of understanding would include knowing facts and procedures, but might also include a sense of underlying meaning, some connection to previous knowledge and, possibly, the ability to explain. However, as shown above in the description of previous knowledge links in adding fractions, making connections is not dependent on a sense of meaning or knowing-why. It is possible to progress in mathematics to some extent by performing increasingly complex procedures and hence displaying a kind of behavioural, fluent, automatised understanding of how to enact mathematical algorithms.

Understanding as meaning and connection

Skemp (1976) points out that knowing what it is appropriate to do, and when to do it, involves a different kind of understanding than knowing how to do it. He reports that Mellin-Olsen described two kinds of understanding: 'instrumental' as the application of rules without reasons and 'relational' as knowing what to do and why. His enthusiastic embrace of the importance of relations has influenced mathematics education hugely, but a cautious reader might well ask, 'Relate to what?' and notice that possible reasons in mathematics can range from the purely pragmatic, 'It works in these circumstances, I can check by other means' to the purely logical, 'Given these axioms and these rules of logic, this will always work.'

Repetition of a definition does not imply that the pupil attaches any meaning to what is being said. Understanding requires more than rote-learning or following procedures correctly, although these could form the basis for future work to develop understanding. A poem learnt by heart can be brought back to mind and reconsidered many times. But the development of meaning is a personal process, dependent on what the pupil makes of successive experiences of a word or concept.

> Understanding is a personal thing . . . The prize is the greater meaning that can flow from the union of isolated thoughts. All it takes is a connection but making it may not be easy. Understanding is not something that can be passed or transmitted from one person to another. No one can make the connection for someone else. Where there are connections to be made, the mental effort has to be supplied by the learner. (Newton, 2000, p. 2)

The meanings pupils develop about a concept, the relationships and reasons they attach to it, are inevitably obscure to others. Even in the education profession the nature of understanding is unclear and requires elaboration. For example, the NC contains the requirement that pupils should 'understand equivalent fractions' (p. 59). Clearly this would not be a matter of simply knowing they exist, nor is there anything to explain in this statement; it seems to be more an instruction to know-about. But what should be known? A teacher preparing to teach about them might know that they give alternative ways to represent the same numerical value, or proportion, or ratio; that, plotted as ordered pairs on a coordinate grid, they lie on a straight line; that the traditional rule 'what you do to the top, you do to the bottom' can be easily misunderstood and used to justify adding something to both the numerator and denominator, rather than only scaling. The teacher would know how to generate them, how they relate to each other and how this knowledge would contribute to later work. Given all these possible components of understanding, some of which are fortuitous, some pedagogic and some procedural, how can one assess whether a pupil understands equivalent fractions or not?

Growth of understanding

The above example suggests that understanding can change and develop, becoming more complex. Locke (1690), in his classic essay about understanding, proposed that ultimately everything is connected to everything else, hence growth of understanding relates to an increase in the number of links one makes. This a useful metaphor in mathematics because ultimately the links themselves can be named as mathematical objects (such as are expressed through abstract algebra, morphisms, networks, etc.). Since we do not know how much there is to know, there is no end to the growth of understanding.

Pirie and Kieren (1994) have developed a theory of the 'growth of mathematical understanding as a whole, dynamic, levelled but non-linear, transcendentally recursive process' (p. 62). This hierarchical model has been used to relate different levels of understanding to what can be observed in pupils' behaviour, i.e. descriptions of observable actions of mathematical understanding that express background processes. It provides a structure for considering questions such as 'What can be said about the understanding of a pupil who chooses to use symbolic forms, or manipulates familiar formats to adapt them to a new situation, or derives a new fact from some previous knowledge?'

They describe stages of primitive knowing, image-making and -having, property noticing, formalising, observing, structuring and inventising. Primitive knowing is what is known so far, making distinctions in existing knowledge and using it in new ways leads to formation of new images. Images can be manipulated and compared and lead to new properties being noticed by the learner who then abstracts something to be said about them, thus moving to a level of formalising. Reflecting on, and expressing, such formal thinking is called observing, and developing these observations as a theory is called structuring. After this the learner can create new questions and new lines of enquiry, which they call inventising. These processes, although increasingly complex, do not necessarily follow each other. In practice there is a lot of toing and froing between levels.

In secondary school mathematics it is rare for teachers to have the opportunity to observe pupils closely enough to be so precise about their understanding. The simpler models of Bruner (1960), who sees learning as a process of developing iconic and then symbolic representations of enacted experiences, with the help of interaction with others, or Floyd et al. (1981) who see learning mathematics as a process of manipulating, getting-a-sense-of and articulating, might be easier to use in the classroom. Once learners can articulate or symbolise a mathematical idea, they are ready to manipulate it further to gain more understanding, or to treat it as the raw material for abstraction or more complex manipulations.

Understanding in context

Some teachers mav interpret 'relational understanding' to be entirely about appropriateness in a context, which could be mathematical or ‘real world’, while others may look for generalised arguments or descriptions of underlying structure. To interpret ‘understand’ as ‘able to use in a real context’ implies that all mathematics can be useful outside classrooms, which is dubious, and that pupils can apply what is learnt in one place to another, dissimilar, situation. The implication is that relational understanding enables instrumental use of mathematics. But formal mathematics is rarely used outside classrooms (Nunes, 1993; Watson, 1998b), so the requirement to use it might be artificial and unrealistic. Further, Mason and Spence (1999) point out that none of the components of relational understanding (knowing that, how and why) necessarily lead to doing the most appropriate, sophisticated or efficient action in a particular situation. For a variety of ad hoc reasons the features of the situation just may not trigger a particular pupil to use the hoped-for mathematics. Cooper (in Chapter 13 of this volume), Christoforou (1999) and Watson (1999), among others, show that students' responses may be as much due to their social backgrounds and the way the mathematical question is structured as they are to understanding. Understanding appears to depend on the situation, different understandings being contingent on circumstances.

Understanding as overcoming obstacles

Sierpinska (1994), speaking of advanced mathematics, describes understanding as the overcoming of particular obstacles in mathematics.

Such obstacles include common difficulties in learning mathematics, inherent difficulties in the subject, errors, misunderstandings, overextending ideas that only work in a restricted domain, and unhelpful ways of thinking, such as generalising with too little data or failing to discriminate between opinion and fact. She sees these obstacles as arising from 'unconscious, culturally acquired schemes of thought and unquestioned beliefs about the nature of mathematics' (p. xi). In other words, some obstacles are to be expected and taken into account when teaching. It is sensible to include overcoming identifiable obstacles as a component of understanding; the ITTNC pays significant attention to this aspect but on its own this approach may do little for the development of deeper knowledge.

The example of multiplication

To illustrate that understanding is dynamic, contingent and local I shall now look at typical meanings of 'understanding multiplication'. It is possible to write $5 \times 6 = 30$ from a variety of viewpoints, each one adequate for some purpose:

- a learnt statement with no underlying number sense, from rote-learning;
- a representation of grouped counting of objects, either five lots of six or six lots of five;
- an example to show a general grasp of commutativity;
- an example of number patterns in the five-times-table;
- an example of number patterns in the six-times-table;
- a learnt statement, with underlying number sense;
- an example of multiplying two positive numbers;
- multiplication as repeated addition;
- a way to work out the answer to a problem;
- multiplication as scaling;
- multiplication by numbers greater than 1 causes increase;
- a representative of a binary operation.

In this list are hidden several potential obstacles, for example the need to understand cardinal numbers, to use numbers as objects in their own right, to have an image of what happens when one multiplies and to shift from specific examples to general properties. The list also presages future obstacles: the inadequacy of seeing multiplication as repeated addition, or grouped objects, when multiplying by negative numbers or by numbers less than 1; the successive levels of abstraction which remove the learner further and further from images of addition or scaling; multiplying vectors or matrices require a more abstract notion. For each viewpoint above apart from possibly the first, one can imagine a teacher legitimately saying the pupil understands multiplication. And yet the image of grouped objects is significantly unhelpful if one is trying to multiply matrices. Understanding, therefore, depends not only on the mathematical context but also the pedagogic situation; there is a sense in which one can understand 'enough for the moment'.

What teachers mean by understanding

If we cannot be specific about understanding, then we are unlikely to pinpoint particular moments when pupils achieve it as an attainable, definitive, stable state. When teachers say they are teaching for understanding they rarely mean that they want their students to know about formal logical systems. More often they are talking about pupils having a sense of the form and purpose of the mathematics, and the places where it is likely to be useful. They may want students to be able to 'generate' or ‘reconstruct’ an appropriate response in new situations, not just mechanically repeat back what they have learnt by heart. Some typical statements from teachers are:

I know they understand when:

- they can say it to me in their own words;
- they can tell me how they did it;
- they can use it in context without being told;
- they use it without prompting;
- they can answer a question which comes at it in a slightly different direction. (Watson, 1998a)

All of these indicate that teachers want pupils to have enough of an overview of techniques and procedures to be able to shift into another representation, generalisation or transformation which allows use in unfamiliar ways, explanations, general description and application (Dreyfus, 1991). Teachers, therefore, are recognising the abilities to generalise, represent and transform as components of understanding mathematics. Even given the temporary and local nature of understanding, it might be possible to say something about whether a pupil has generalised, represented or transformed some mathematical concept in given circumstances. But these are mental actions, so how can a teacher collect evidence of pupils' understanding?

How do we know what a pupil understands?

In order to recognise such generalisations, which are crucial in all mathematics, teachers have to rely on what they can see, hear and read. Hence there is no room for intuitive understandings in the above statements (Fischbein, 1987; Claxton, 1997) except those intuitions that might enable pupils to apply mathematics in new places. Instead there is much importance placed on verbal expressions of methods, although an essential aspect of mathematics is that structures can be expressed and manipulated in non-verbal ways. Another emphasis, which teachers might make, is that successful performance of mathematics in given contexts might indicate certain kinds of understanding. But observing pupils' actions in the classroom is difficult to manage systematically, and in the end one might only have the outcomes of written work to see.
The ITTNC says that new teachers should know:

> . . .how to use formative, diagnostic and summative methods of assessing pupils' progress in mathematics, including (ii) undertaking day-to-day and more formal assessment activities so that specific assessment of mathematical understanding can be carried out … (and) (iii) preparing oral and written questions and setting up activities and tests which check for misconceptions and errors in mathematical knowledge and understanding … and understanding of mathematical ideas and the connections between different mathematical ideas. (TTA, 1999, p.14, 9aii and iii)

In order to achieve this a teacher must have a very clear idea of what kind of understanding is being assessed: instrumental, contextual, procedural, relational within mathematics, transformable, generalised, logical or abstract, with obstacles successfully overcome. Also required is an awareness of how such understanding can be assessed.

Is it possible, for instance, to find out if a pupil's understanding is relational or instrumental? Or, if one believes all understanding to be relational, even if it is related to a fragmentary rule-performance view of mathematics, can a teacher find out to what is it related? There are problems with observations of students. Although such observations tell us something, they do not give us access to understandings that have not been expressed in accessible forms (Watson, 1997). In addition, all observations have to be interpreted by the teacher, and one may not know how such expressions were achieved. For example, a correct proof can be given because it has been learnt by heart; this may or may not mean that the student has an understanding of how the proof 'works'. The understanding could be relational or could be an instrumental response to a request to 'prove'. Neither does it indicate that the student has learnt anything about that type of proof in general, even if the student has grasped the reasoning in the learnt proof. The teacher needs to be cautious not to impute levels of understanding without evidence, and needs to probe further if more inferences have to be made.

Oral evidence, though highly valued by all the teachers, is time-consuming to organise. Language difficulties, diffidence or fear might prevent some pupils from speaking. It is rare to overhear useful remarks in a busy classroom, although such remarks often give insight into a pupil's thinking before they are able to record what they think on paper. In addition, oral evidence does not give hard evidence to support-a teacher's judgements, so that over-reliance on oral evidence may leave the teacher vulnerable to criticism when being inspected by others. Reliance on oral work ought also to be seen in the light of Bernstein's work (e.g. 1971) on how middle-class pupils are at an advantage in school because the elaborated codes of language are what they might be used to at home, where working-class pupils are expected to communicate at school in a way very unlike the codes used at home. This theory relies on a very stereotyped view of language use outside school, but it does prompt a closer look at language forms in mathematics classrooms. The request to 'explain how you did something', a common requirement in teacher— pupil discourse, is a rare form of speech outside school in any social grouping. Hence reliance on pupils' ability to demonstrate their understanding orally for assessment purposes is expecting a keen awareness of different discourses as well as mathematical ability.

Many teachers comment that written work on its own is not enough to convince them that pupils understand; they want oral evidence, or written workings and explanations as well. However, there is also wide recognition that many pupils have considerable difficulty in recording in writing what they could do mentally or practically. Assessment ofwritten work, particularly where it involves explanations or extended exploration, has to be seen in the light of research into assessing coursework. Several writers have shown that pupils can be very selective in what they write down, so that written work represents a highly-edited view of their mathematical thinking (MacNamara and Roper, 1992). Sometimes this is an attempt to produce curtailed, terse, classical mathematics, but it can also be due to a failure to appreciate what is important or an inability to find ways to represent abstract or intuitive thought on paper.

Observation in a busy classroom is difficult-to organise but can reveal that the pupil is using particular methods, such as counting instead of using number bonds. Observation of actions depends in part on the teachers' notions of how mathematical activity might be observable. Sometimes this is clear, such as when one sees a pupil use a ruler and read off a measurement correctly. At other times it has to be interpreted, such as when a pupil is trying to make a cube from six squares and may appear to us to be doing it in an obscure way, but nevertheless succeeds. Other times, there is little to intepret; the pupil who is gazing motionless at a problem may or may not be thinking about it, and the thought may or may not be productive. On the other hand, avid writing may not indicate anything useful is being done. How the teacher interprets the actions can be influenced by many factors. In the examples above, interpretation depends on what the teacher expects to see relevant to the mathematics, what the teacher expects from the particular pupil and what the teacher expects from pupils in general. It also depends on what is noted by the teacher that can be affected by previous impressions of the pupil's abilities.

In order to avoid the possible unfairness that can creep into assessment, given the warnings above, it is possible to:

- be prepared to be surprised — avoid forming firm views of a pupil's capabilities and achievements;
- use a variety of forms of assessment so that you accumulate a broad view of what a pupil can do;
- relate the way you assess, and what you record, to the purpose of the assessment;
- discuss your views and interpretations with colleagues;
- look for evidence which contradicts, as well as that which corroborates, your views;
- do not base irrevocable decisions solely on your own interpretations of what a pupil can do.

Purposes of assessment

Diagnostic assessment

This purpose of assessment assumes that you can find out something about the pupil's current state of knowledge in order to decide what and how to teach. As I have argued above, it is not possible to establish current understandings with any certainty because of the complex, dynamic and situated nature of mathematical understanding. There are commercially produced tests to help in the diagnostic process, but it is important to realise the results tell you about a pupil's response to a particular question on a particular day. Responses may indicate that common misunderstandings exist, or that the pupil was able to get the right answer using a specific method, and this is useful information when planning to teach, but general judgements about individuals made on the basis of such tests could be flawed. A more immediate way to assess the knowledge pupils bring to a mathematics lesson would be to set up an interactive situation in which pupils are somehow encouraged to reveal how they see a topic, perhaps by making up their own questions, or describing methods on the chalkboard, or telling each other what they already know. The more we can find out about how they already think about the topic, and what they know which is related to it, the more appropriately teaching can be planned and focused.

As Hawkins (1967) said, active and talkative lessons allow pupils to show us what they know, and what images they attach to concepts. When:

> children are rather passively sitting in neat rows and columns and manipulating you into believing that they're being attentive because they're not making any trouble, then you won't get much information from them. Not getting much information about them, you won't be a very good diagnostician of what they need. Not being a good diagnostician, you will be a poor teacher … When we fail in this diagnostic role we begin to worry about 'assessment'. (p23)

Since the purpose of diagnostic assessment is to find out what is not known, what is misunderstood, and to inform future teaching it is debatable whether keeping records permanently for individual pupils is of any use except to provide a base from which to assess their later progress.

Formative assessment

Teachers make judgements all the time about how pupils are responding to teaching, and what progress they are making. For progress to be observed, hierarchical criteria need to be used, such as level descriptors of the NC, or progress tests in a published scheme of work[3]. Pupils can be made aware of such criteria and possibly be involved in assessing their mathematics against them. In this way, the teacher and the pupil are both informed about how they are responding to teaching, and what topics, concepts or teaching styles are causing problems. Progress is often seen to be a one-way process, but the Pirie—Kieren model expects pupils to return to 'lower' levels of knowing from time to time, and if pupils have not worked in a particular area of mathematics for some time they may need to revisit earlier ideas. Often, formative assessment is accompanied by

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[3] On reflection in 2018 I am surprised I wrote this, but suspect it was to be realistic about what teachers were expected to do at the time; I hope the rest of the chapter provides enough to query this picture of formative assessment as having any meaning with respect to long term learning, and can be seen as more about institutionalisation.

target-setting in which pupils are given, or may suggest for themselves, some learning goals for the near future. This can be effective in helping to motivate pupils, but can work against the development of deep understandings, encouraging instead the desire to acquire more and more skills at the expense of higher levels of understanding and reflective approaches to consolidating learning.

It is clear from this description that formative assessment relates to what has been taught and what will be taught to those pupils, rather than to some overarching curriculum plan. Although the information may be of value to individual learners, it is also very important for teachers to use in order to monitor their teaching. Again, it is debatable how valid permanent record-keeping about individuals would be, and judgements made about pupils on the basis of formative assessment need to be temporary. Formative assessment does not necessarily provide useful baselines for describing progress, as it usually relates to current teaching and learning.

Summative assessment

At the end of a course, or at fixed points during the school career, pupils may br tested in a variety of ways to see how much they have learnt and identify their overall progress. These assessments are done by comparing pupil work to an overarching plan, such as a national curriculum or some course objectives, and summing up what has been achieved. This purpose of assessment is usually to categorise and grade pupils individually, thus influencing future educational choices, and to get data about a group or cohort for other purposes such as within-school or between school comparisons.

Methods of assessment

Although there are published tests for all three purposes described above, and national statutory tests for summative purposes, there are many other assessment activities which are an integral part of classroom life. In addition, teachers' judgements of what pupils have achieved are included in statutory assessments. A systematic study of methods used by teachers to find out what their pupils know (Watson, 1998a) found that, although teachers knew that their assessment findings were dependent on circumstances, they nevertheless believed that there was some ultimate state of 'understanding' about which they could say something, if only they could get enough information about pupils. More usefully for our purposes here, they described a variety of ways of finding out as much as they could about pupils' understanding because it was of central importance for their decisions about what and how to teach, who needed special support, who needed further challenges, and so on.

There are dense links between choice of assessment method and choice of teaching (see Watson, 2001) or tasks, questions and interactive strategies (see Mason, 2001) so I shall not comment fully on the pedagogy associated with every method below. Instead, I shall highlight how each method contributes to the meanings of 'understanding' given at the start of this chapter, and also how it relates to the three main purposes set out above.

1 Looking for how mathematics is used in the context of practical or investigational work, or more complex mathematics. Is a concept used where appropriate? Has it been adapted for use?

If the concept is used, this can demonstrate that the pupil has internalised it and generalised it enough to recognise where it may be useful, and to transform it for use. However, failure to use it does not mean it is not understood; it may only mean that it was not seen as appropriate, or was deemed too complicated for the context, or just failed to come to mind. How it is used can give formative information; successful use can contribute to summative assessments; a practical situation can give diagnostic information

2. Explaining to the teacher; explaining to another student

Verbal explanation can be evidence of generalisation, or of noticing and formalising properties of a procedure or concept. Some pupils may be uncomfortable about verbalising; those operating at a highly abstract level may not see how words can express the mathematics. Others can transform their understanding into words and learn more by doing so. This method can give useful diagnostic and formative information.

3. *Response to teacher-led questioning or open prompts, e g. 'Tell me about...'*

While closed questions may give information about understanding, they may also encourage instrumental, or learnt-by-heart responses. Additionally, some pupils are adept at guessing answers from the teacher's cues, while others may choose not to take part. Open prompts may reveal much more about a pupil's personally-constructed understanding, but may fail to provide enough structure to trigger the most sophisticated response possible. One prompt that appears to be effective is 'Make up the hardest example you can.' Such questions can generate useful information for formative assessment, and may incidentally allow the teacher to diagnose difficulties, but may not reveal the full extent of a pupils' knowledge and can only give summative information insofar as they reveal ways of working with mathematics.

4. Pupil expressing insight while working on an intended area of mathematics; or while working on another area of mathematics; communication pupil-to-teacher or pupil to-pupil

This is the kind of incident which is very revealing when it is spotted, but cannot be planned and hence may not be systematically incorporated into assessment practices. What is demonstrated might be intuition, or a recognition of some mathematics that has been met previously in some other form. It is more likely to take its place in the mental picture that the teacher develops for each pupil. In that sense it is formative, in that it informs the teacher that this pupil may be able to cope with particular kinds of challenge in future.

5. Response to similar, simpler, slightly different or harder examples, or examples where questions are asked in other ways

If a teacher is trying to find the extent or depth of a pupils' understanding slight alterations of a standard question-type are very useful, and can be systematically incorporated into worksheets, homework tasks and tests. These can be used to identify common misunderstandings, and show how far the pupil is able to adapt, manipulate and transform the concepts taught. Careful developments of questions can be used, therefore, to diagnose what needs to be taught, and to summarise what can be done in certain situations. The more open approach of asking pupils to make up their own hard questions, as in 3 above, can also be used.

6. Self-assessment

A formative assessment method that also motivates and informs pupils is to ask them to assess their own progress. There are several ways of doing this, but to be effective pupils must have some understanding of what it is they are supposed to achieve, otherwise the exercise can degenerate into meaninglessness. Writing journals, in which they describe what they have learnt by giving instructions or examples and recording difficulties (Waywood, 1992), is one way. Such exercises can show teachers what pupils see as the important aspects of a topic, and their sense of underlying structure.

7 Tests: teacher-written tests, impromptu questions, use of a bank of test items, test as part of published scheme, tests written by students for their class

Answering test questions is an obvious way to assess understanding, but the circumstances of the test need to be taken into account when deciding how to use the results. Timed tests consisting of closed questions assess algorithmic competence, speed, accuracy, recall, the ability to identify what is needed to answer a question and the ability to adapt what is known to fit a situation. It matters, therefore, whether the test is covering what has been actually been taught, or what is supposed to have been taught. It matters also how it has been taught, because the difference between the questions on the test and the kinds of situation the pupil is used to working with is crucial to how the pupil can engage with the questions. For this reason, teachers wishing for the best possible test results may try to 'teach to the test', in order to give their pupils the advantage of not having to adapt their understandings too much. Sometimes teachers are criticised for doing this, particularly when it leads to an instrumental approach in which pupils' responses are triggered by certain language forms in test questions. One way to teach to the test, but also to pay attention to the development of deep knowledge, would be to regard test questions as problems to be solved and develop a critical, questioning approach to the task.

Tests are commonly used for all three assessment purposes, but there are many problems with their use as summative tools. The style of question can attract some students and alienate others, questions can be ambiguous, small details of language can lead to misinterpretation of What is required (think of the difference between 'subtract' and 'subtract from'), the pressures of the test situation can lead to underachievement and so on.

8 *Analysis or discussion of errors*

For formative and diagnostic purposes pupils can be asked to explain how they did some mathematics, thus showing what sort of reasoning led to incorrect answers. This is also helpful when answers are correct! This method requires close one-to-one attention and is hence difficult to manage. However, a teacher can use similar methods with a whole class in order to become better informed about a range of understandings they might have. In addition, working on common errors can aid understanding. This is an example (as are several of the methods above) of good assessment practice merging with good teaching. Mason's article on questioning (Mason, 2001) gives further examples of this.

9 Activities which use knowledge or processes, or both, and are expressed through paper, observation, verbal, investigative or practical work

Many teachers in my research said that they would know for sure that pupils understood if they could apply their mathematics, unprompted, in a new situation. The situation might be a new mathematical context, or a practical situation such as on the sports field, or in technology lessons. Transfer mathematics from the classroom to other situations that have their own habits, ways of seeing things and ad hoc methods is complex and sophisticated, as has already been said. But application to later work in the mathematics class room would indeed inform the teacher about how pupils see the meaning, use and scope of a topic. As in all types of assessment, this is most effective if pupils are familiar with what is required of them. If a class commonly approaches new mathematics with the questions 'What do I know which is like this?' or 'How does this fit with what I already know?' then connecting and using a new topic will become a working habit. Not only will the teacher be able to see who understands the mathematics that has to be used, but how it is understood. In addition, the ground can be prepared for helping pupils see mathematics in connected way, relating one algorithm to another, and hence constructing network of knowledge.

Knowing about understanding

Although commonly used in education, the word 'understanding' is complex an open to a variety of interpretations, particularly in mathematics with its multiple layers of generalisation, abstraction and use. Understanding depends on mathematical context and on what is expected of the pupil. It can also depend on how mathematics is taught. To find out what a pupil understands is dependent on what it means to 'understand' in a context, and how the teacher identifies, collects an interprets evidence. Teaching for understanding (Watson, 2001) and assessment can be intimately related.

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