**Teaching mathematics as the contextual application of modes of mathematical enquiry**

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***Abstract***

*As teachers and educators we spent many hours in mathematics classrooms and had the privilege of being able to observe as well as participate. Our experiences, and our personal knowledge of mathematics, have led us to believe that mathematical modes of enquiry are a central part of teaching and learning mathematics. But a classroom is not a mini-mathematics laboratory in which students are apprentices. Rather, those teachers whose lessons make a significant difference to students’ understanding of mathematical ideas appear to adapt mathematical modes to the restricted frames of school mathematics.*

*We explored one of these frames, the preparation of teaching resources, to investigate our hypothesis about the central role of mathematical modes of enquiry. We set up an artificial resource preparation exercise amongst a group of knowledgeable mathematics educators and recorded their collaboration. We found that our personal mathematical modes were both transformed and transforming, and the results of this process were embedded into the resources we designed.*

*We argue that teachers’ fluency with mathematical modes is important in effective teaching. For teaching to mean anything, teachers must act in such a way that students learn something that they would not or could not have learned if the teacher had not been present. That is, the need to provide something that a textbook or annotated website cannot provide. Teachers’ fluency with mathematical modes of enquiry is the basis of their unique contribution.*

**One of Bill’s experiences**

The syllabus I was using required five lessons on 2x2 matrices for my class of 14 year olds. We had looked at arrays and been through the operations +, –, x, ÷ with other matrices and 1x2 vectors. The final section is on matrices as transformations of the unit square: reflections, stretches, sheers, rotations. We do not quite finish so I use a little of the next lesson in a tight syllabus. In response to an invitation to the students to give me a random matrix so we can look at its effect, I get a 3x3 matrix suggested. Smart kid. The class appear to have understood the 2-dimensional concept, so I extend, draw a unit cube and watch as they quickly pick up the idea and stretch and reflect it in a plane. No problem—until the same child, flush with her success asks about a 4x4 matrix, with a smile, knowing there are only three dimensions. I seize the moment to demonstrate the power of mathematics to go beyond our experience and soon hypercubes are being reflected through 3-D space using the patterns of 2x2 and 3x3 reflections. The keen students take home homework on problems in 5 or 6 dimensions. But that lesson has been used up, and half the next one, and I am dreadfully behind my schedule. Why am I not feeling concerned and why do I remember that lesson as one of my best to this day?

**One of Anne’s experiences**

One student, a good mathematics graduate training to be a teacher, told me that he had expected to shut down his intellectual engagement with mathematics in order to teach at a lower level than normal. Instead he had found thinking about mathematics as a teacher every bit as mathematical and challenging as his first degree. An example occurred when thinking about preparing a lesson on straight line graphs, when he suddenly became aware that the schoolbook use of the term linear for functions of the form y=mx+c did not equate with his university use of the term linear to refer to functions for which λf(x) = f(λx). That is, keeping within the school syllabus, f(x) = 3x–4 is not linear in the transformational sense. He used this realisation to springboard a brief discussion with his school students of other mathematical meanings that appear to vary as you learn more mathematics, such as: multiplication not always making things bigger; translation being a different kind of symmetry from reflection, rotation and enlargement because it is not a matrix transformation; division not always being represented by sharing, and so on. Another teacher said that she relied on situated definitions, so that her students have to relate a definition of proportion in one context, such as ‘proportion of a whole’, to, say, proportionality as equality of two ratios.

**The roles of mathematical modes of enquiry in teaching**

Stories like these have led us to consider mathematical modes of enquiry: a teacher sparking off a student comment to make a wider point about mathematics and extend the students’ thinking when the moment was ripe, at the later cost of syllabus squeeze; a new teacher challenging himself with elementary material by thinking about mathematical definitions, assumptions, and implications; a lecturer becoming carried away with making connections and using new perspectives to re-view familiar material; new teachers treating a curriculum topic as an arena for comparing examples, definitions, assumptions, and implications. Again and again we observe, in ourselves and in others, that some of the best teaching and learning moments occur when mathematical modes of enquiry are invoked. We have come to believe that they are central to what a teacher does, and what a student is led to do.

In this chapter we test this belief by examining more closely the teacher act of preparing teaching resources. As will be seen, we gained confirming evidence of our belief. We learn that we need to understand better how teachers come to acquire these modes, and how and when they are invoked. What makes a lecturer go off on a mathematical tangent, and how can we tell whether that is useful for students? Why will a teacher insist on a detail of mathematical argument at one moment but allow an incomplete definition or use of a technical term in another? What is it about their knowledge and awareness that links a particular mathematical mode to that teaching moment? In this chapter we cannot answer all these questions, but can begin a grounded investigation of teacher-thinking from a mathematical perspective.

To situate our investigation we comment briefly on current thinking about teachers’ mathematical knowledge and more substantially on literature about mathematical enquiry.

**Teacher knowledge**

In the late 1980’s Shulman (1986, 1987) introduced the idea of pedagogical content knowledge in contrast to subject matter knowledge. His work responded to a local trend away from the generic professional development offered in the 1980s towards programmes that recognize the importance of enabling teachers to learn how to teach particular content. As Kennedy (1999) describes, teachers need to understand the ways students hold mathematical conceptions, to know what representations and analogies will be useful in teaching, and to understand developmental stages.

Since then, mathematical knowledge for teaching has often been theorised using the idea of acquisition of types of content knowledge for teaching. While such models might be useful for adding nuance to a continuum of pedagogical content knowledge and subject matter knowledge, and may thereby inform pre-service teacher development programmes, in our view they risk missing out the most crucial aspect of what a mathematics teacher does in relation to mathematics; teachers enact mathematics. In discussing mathematical knowledge for teaching, we can easily be drawn into a curriculum of items that a teacher needs to have learned: quadratic equations, differentiation, the history of negative numbers, stages in development of number awareness, common misconceptions, and so on. What is often missed is mathematical thinking and awareness. It is not just a question of what teachers know, but how they know it, how they are aware of it, how they use it. Perhaps this can be summed up as: what mathematical habits do they have? To be effective teachers, what do they need to do mathematically? It is the assumption of this paper that these mathematical modes of enquiry need to be deeply present in teaching. Further, we are unconvinced that we can attribute what they *do* to the personal possession of certain forms of knowledge.

**Mathematical Modes of Enquiry**

In March, 2008, the ICMI Centennial conference had a Working Group on the topic Disciplinary Mathematics and School Mathematics, at which questions were asked about the relationship between research mathematics and what happens in secondary classrooms. Initially it appeared that strongly differing orientations were being expressed: one the one hand it was argued that school mathematics had to be a “shadow” of the discipline, on the other that it was fundamentally different in context (Watson, 2008). However, a consensus did emerge that “students learn through reasoning that resembles mathematical thought” (Barton & Gordeau, 2008). It was noted that a significant difference between disciplinary and school mathematical experiences was the mediation of the teacher. This begs the question of how the teacher can best undertake the mediation; “working as a mathematician” was one answer to this question.

For us, any discussion of the mathematics involved in teaching has to start from understanding of what doing mathematics entails, and then seeing how this acts out in teaching. Otherwise there is a temptation to see teaching mathematics as to do with exercising knowledge, rather than as an arena for acting mathematically.

What is acting mathematically? Krutetskii’s (1976) seminal study of gifted Soviet mathematics students identified several common features. These students all had a tendency to:

• grasp formal structure;

• think logically in spatial, numerical and symbolic relationships;

• generalize rapidly and broadly;

• curtail mental processes;

• be flexible with mental processes;

• appreciate clarity and rationality;

• switch from direct to reverse trains of thought;

• memorize mathematical objects (1976).

These tendencies have been elaborated by Cuoco, Goldenberg and Mark (1996) to attach their specific manifestations in various branches of curriculum mathematics, and also extended to include the qualities of sustained niggling that bother mathematicians. Their characterisation of ‘habits of mind’ includes: pattern-sniffing, experimenting, visualising, forming conjectures, reasoning proportionally, loving systems, embracing unifying theories, looking at variance and invariance, extending meanings, thinking generally from examples and vice versa.

Sustained niggling is also described by Hadamard (1945) and extended to include moments after one has been totally engaged with a problem for a period of time, then relaxes to do something else, when insight occurs unexpectedly. This common experience reminds us that the natural ways in which the mind works includes reflection, organising, and seeking ways to compare and generalise experience. Mason puts some structure on ‘sustained niggling’ (Mason, 1988) by focusing on stages and states of mathematical thinking. His inspiration came from Polya’s (1962) classic work on problem-solving, encapsulating Polya’s extensive list of the many strategies on which mathematicians can call. These have been sloganised as ‘specialise, generalise, conjecture, convince’, but their common use is as instructions rather than as descriptions of behaviour. This sometimes leads to an assumption that these actions happen in a given order. It is more common for mathematical thinking to roam between and within these approaches. It is also worth noting that ‘specialise’ implies a special choice of examples, rather than using examples as data for inductive purposes. We mention this here because purposeful generation and use of examples is also a major feature of being mathematical and of course one that characterises good planning and teaching. For example, in a lesson about probability the teacher offered questions in which P(r) + P(¬r) = 1 emerged as an conjecture that was obvious to many learners, followed by a question in which P(r)=1 and a following question in which P(¬r) = 1. Creating and using examples to structure generality requires that teachers see what they are teaching in terms of generalities rather than techniques. Using extreme and special examples is a common mode of enquiry to see how far a conjecture holds up. This teacher clearly understood this and tried to communicate it to her students: ‘Look’, she said ‘the mathematics is telling you something’.

Of course one cannot be mathematical without the specific intellectual toolkit and repertoire of mathematics. The ‘habits of mind’ model includes some of these, such as using understandings of probability, representation and generalisation in the example above. Simon (2006) identified key developmental understandings of mathematics not as first order knowledge, but as foundations for learning other ideas. We would see these key understandings as threads that run throughout mathematics, so that the ways in which we read mathematical situations are profoundly and lastingly influenced by them. For example, understanding number multiplicatively as a first resort, not as something to be used if additive models fail, is a key understanding in much secondary and tertiary mathematics; understanding functions as mathematical objects, rather than as algebraic representations of data sets, is key to understanding much higher mathematics. Silverman and Thompson (2008) show that merely being offered situations in which these are made apparent as useful ways to view mathematics is not guaranteed to lead to good mathematics teaching. We want to develop the reverse story: how do teachers who have, over time, developed significant ways of thinking about and interpreting mathematics, bring that experience and knowledge to bear on pedagogic tasks?

**An artificial teacher activity**

In order to explore our belief that mathematical modes of enquiry are central to effective teacher activity, we set up an artificial teacher activity, that of developing teacher resources, and asked two other experienced mathematics educators to join us. All four of us regard ourselves as mathematicians in our habits. We hoped to examine what knowledgeable mathematics educators do as they think about presenting students with mathematical situations. We took two starting-points and gave ourselves the task of using them to devise a teaching situation. We recorded the discussion and then analysed it to identify the mathematical practices and repertoire implicit and explicit in our responses. The use of the behaviour of experts to explore what is possible has strong precedents (for example, Carlson & Bloom 2005). We do not claim that what we did is what novices would do, nor that ours was the best, or only, possible behaviour; rather we are using this shared task to see what can be said about the role of mathematical expertise in typical planning discussions.

The first stimulus was a page of exercises from a school textbook chosen because it exemplified ‘dry’ problems of pure mathematics that are very familiar to any secondary mathematics teacher. The second stimulus was that day’s newspaper. The task for the team of four was to describe the possibilities they could see in these stimuli for a teacher faced with creating a lesson based on them. We acknowledge that this is a false situation. It is very rare that a teacher would have the luxury of two hours with three interested colleagues to create two lessons. Lessons are usually created within a continuous curriculum, and with certain aims. For this exercise we assumed the content aims of the author of the textbook page, and also assumed that exploring the mathematics of an ‘everyday’ issue from the newspaper could be a lesson aim in itself.

Our approach was to first identify the range of potential mathematics afforded by these artefacts, according to our mathematical knowledge. In addition we agreed to discuss, after the event, what mathematical knowledge we used and how a teacher might recognise the potential that we had recognised and bring it to realisation in a classroom.

***Stimulus 1 Problems about inverse proportion (see Appendix 1)***

This textbook page introduced inverse proportionality, expressing this relationship in a variety of ways, including two which offered interplay between data sets and algebraic representations. A range of letters as variables was used, and the independent variable was itself a function in some of the questions.

Our initial responses may be described as alerts: Pedro[[1]](#footnote-1) noted that the symbol  is not universal, and, for example, is not used in Portugal; Bill noted that the term “inverse” has multiple mathematical meanings that students of this level would know (for example, the inverse operation, and ‘invert and multiply’); Anne noted that particular letters used on the page, such as *t, L* and *v,* have common contextual meanings in mathematics and science that were not preserved in these examples; and John noted that the two ways of expressing inverse proportion algebraically  (implied in Qn.1a) and  (implied in Qn.2) are not obviously equivalent. All four of us felt that both these expressions of relationship are fundamental to a fluent understanding of, and recognition of, inverse proportionality—too important to be offered only as implications, as in Qn.1 and Qn.2.

The group then began to respond in more detail to features of the set of exercises. For example, Qn.4 essentially asked for the factors of 12, and could be completed without thinking about inverse proportion. Another feature noted was that all the numbers on the page are simple integers, or simple fractions, and have simple multiplicative relationships. The discussion quickly moved into a suggestion by Pedro that Qns. 7 & 8 would be interesting if the table did not include any matched pair. For example, instead of

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *y* | 2 | 4 |  | 1/4 |
| *z* | 8 |  | 16 |  |

one could offer

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *y* |  | 4 |  | 1/4 |
| *z* | 8 |  | 16 |  |

Such an exercise would not only make the question more open, but students working with it would be led naturally into the concept of the constant of proportionality, *k*. We wondered if the necessity to understand an expression such as *t2*to be a variable in itself, as is offered in Qn.8, detracted from the main idea of inverse proportion, or encouraged a focus on ‘inverseness’. The extensive discussion ended up with agreement that a whole series of lessons could be constructed so that students would develop for themselves the concept of inverse proportion by engaging with suitably constructed and sequenced tasks that either avoided completion by merely templating numbers, or that challenged such completion by affording cognitive conflict. However, we agreed that the page as a whole attempted to avoid the possibility that learners would get locked into simplistic assumptions about the relationship, and did offer the potential for a complex engagement with the concept approached from several different perspectives, using different symbolisations and also within composite functions.

From our initial alerts about ambiguities of language, symbolism, and possible confusion of equivalent expressions, we deduced that this page could not be used effectively without considerable discussion about the purpose and meaning of inverse proportionality. If such discussion was effective, then it is unclear what purpose the exercise would serve: the questions are so different that fluency would be unlikely to be achieved, and the conceptual understanding afforded by connecting and relating different questions would, for most students, need the teacher to mediate. While we recognised the potential of comparing answers and approaches between questions, our experience tells us that on their own learners are unlikely to make these comparisons. We decided that, in a classroom, we would probably pick out particular questions and discuss the implications of them for the meaning of proportionality and the ways that proportional relationships might appear in symbolic form. Such an approach was identified in a sample of 40 lesson videos from 13 teachers analysed by Watson and DeGeest (2008). They found that discussion of mathematical implications was always absent from the lessons taught by non-specialist teachers and frequently or always present in the lessons taught by teachers with a high level of personal mathematical knowledge.

From here the group moved away from the exercises themselves to consider what further mathematics could be developed through using such examples. Graphical plots, the effects of using different types of graph paper, and a general study of hyperbola were the first suggestions: we all regularly use graph-plotters as a mathematical tool alongside data sets and symbolic representations to increase our understanding of relationships. Next we talked about the representation of inverse proportionality as functions that vary with respect to each other (*f(x)* varying as *g(y)*, a level of complexity following that offered by the exercise), noting that such functions all go through the origin, so that the constant of proportionality is always the gradient. Finally we discussed how one might go about developing an algebra of “varies as”, such as ‘if *x* varies with *y* this way, and *y* varies with *z* that way, how does *x* vary with *z*?’. We agreed that this question could be explored by students at this mathematical level with appropriate technology.

Further discussion focused on contexts outside pure mathematics: where does inverse proportion occur, how could the concept emerge from student experiences? The Gas Law example (Qn.10) was related to the difficulty opening and closing a fridge door, and the inverse proportion arising from a fixed amount of a resource being used by differing numbers of people were suggested. We noted that many examples exist in physics and wondered whether it was wise to use letters that have conventional physical meanings in equations that do not. At a more general level it was realised that proportionality could be regarded as the invariant relationship between a pair of variables independent of scaling. Familiar examples are density, expressing the invariant relationship between volume and mass for a particular substance, and the gravitational constant, expressing the invariant relationship between vertical force and mass.

Finally we turned to the meta-question of where our ideas had come from. What mathematical learning in our own histories had led to us expressing these thoughts? There were some particular answers: Bill is interested in language and mathematics so he focused on words; Anne had a current preoccupation with notation because in her experience some notations lead too easily to manipulation without meaning (Institute of Mathematics Pedagogy, 2009); some of John’s responses were triggered by his rejection of all “copy and complete” exercises because he has found such tasks are likely to lead to over-attention to answers, and failure to grasp underlying meanings; Pedro has a habit of using “make something different” to extend a mathematical situation when he is working on his own mathematics. John’s habit leads him to reject textbook pages, where Pedro adapts a textbook task and uses it to scaffold engagement in something challenging. Such adaptation strategies can alter the nature of learning and give purpose to apparently repetitive tasks (Prestage and Perks, 2001).

We all agreed that much of how we reacted was already rehearsed in other, similar, situations. We have habits of analysing the variation, relationships, and constraints implied in collections of mathematical objects. We had all looked at the questions as a sequence of mathematical objects which were exemplifying inverse proportionality. We asked what meanings and understandings can be inferred from this collection? We brought prior experiences of many different kinds. We realised that, during the course of the discussion, we had each done some new thinking, For example, thinking about the relationship between equal ratios and constants of proportionality was not something we had done explicitly before, although it was, perhaps tacitly, embedded in our mathematical experience. What made us alert to that? In what way were we looking for new ideas?

On reflection it was clear that we had responded on several mathematical levels: we thought about symbolic representation; equivalence of notations; relevant and irrelevant features of examples; the nature and representation of relevant functions; applications and meanings in other contexts; domain of applicability as represented in these questions; the affordances of other exemplifications; and the extension of the concept of proportionality into an algebra of relations. We had both pedagogic responses (what would these exercises afford for students? how could they be changed to afford more?), and mathematical ones (what mathematics can we do in this situation? what potential mathematical meanings are suggested in the given presentation?).

***Stimulus 2 The day’s newspaper***

Bill brought a newspaper and had already identified three potential articles to use as stimuli for mathematical activity. The first was a map of UK divided into different electoral regions: three of us immediately started thinking about the Four Colour Theorem and its variants (for example, using regular-shaped areas or on different surfaces). The fourth member of the group thought of statistical questions. We asked ourselves what it was that made the stimulus different for each of us, but got only vague answers such as prior experience, familiarity with similar stimuli, disposition towards classical mathematical questions or social applications. A second article was about car finances and prompted a brief conversation about sequences and series, economic indices, and the volume of crushed scrap cars.

The item we decided to discuss in depth concerned the possibility of a new outbreak of smallpox and a discussion of plagues and epidemics. We started talking about modelling spread of disease. What was interesting, however, was that two of us began sifting our memories for the appropriate differential equations, and the best variables to use (how do we write down the probability of two people meeting and one infecting the other?), while another looked at the situation more globally as “some form of exponential growth” and thought about the reasons why or when exponential functions are the appropriate models. Of course both approaches lead ultimately to similar ideas, but it generated a discussion of how these two approaches can be balanced, which (if either) are present in the curriculum, and the importance of the interaction between the approaches.

In this discussion we all acknowledged the accessibility of our intellectual resources for modelling populations, such as knowledge of appropriate functions and prior experience, and that there are many different modelling tools that can be used (graph theory, statistical mechanics, statistical data analysis, and so on). None of us considered plotting given data to explore the situation empirically in the hope that a generality would emerge from the data, but as teachers we felt we would have to consider such an approach since some students are oriented to ‘pattern-spotting’.

We became aware of several small mathematical issues that could arise in the classroom. One issue was the way that models shift between discrete to continuous formulations, often without this being made explicit. Our facility comes from multiple situated experiences of using functions in many contexts, and this situatedness probably contributes to the difficulty in being meaningfully explicit with students. Another is the idea of big numbers and how we develop our appreciation of what big numbers “mean” in a practical sense. Our own appreciation appears to develop during adulthood, through being genuinely interested in ‘big number’ situations, such as arise in considerations of climate change and disasters.

Our discussion then turned to other links that could be made from this material. Issues of disease prevention, the experience of false positive tests, and other medical science issues that could be regarded as part of general education and for which mathematics teachers (alongside other subject teachers) have some responsibility. We reflected on our uncertainty about some of this—what are the dangers that we (or teachers at large) might not be informed sufficiently and thereby give bad practical advice? What are the problems for mathematics teachers entering these areas? This led to considering the wider role of teachers in grappling with big issues, and the extent to which the skills of mathematical analysis are important in nearly every case. We discussed the Game of Life (Gardner, 1970), John Conway’s original idea behind the mathematics of cellular automata and related simulation “games” that model the rise and fall of populations under various initial conditions. We moved on to talk about iterative models, convergence, cycling, and divergence. This led us to ideas of chaos. It also led to discussing the Game of Life as a context in which the mathematical activity of setting up rules and examining the consequences could be raised with students.

Our meta-reflection on the knowledge we had brought to our discussion focussed on our prior experience. All four of us have a wide experience of different secondary and tertiary curricula and teaching issues. We were strongly aware that the main resource we brought to the newspaper article discussion was exactly this experience of using mathematics to deal with social and economic questions. This kind of knowledge differs from the kinds we used for the proportionality discussion. It hinges on a sustained mathematical outlook on big social and political questions, rather than being a form of pedagogic knowledge. In our newspaper discussion we did not get close to the task of designing classroom tasks that would expose learners to the mathematical affordances of the issues of disease. Rather, we talked about mathematics being used as a tool to illuminate social and educational issues. The question we failed to address was whether the outcomes of engaging with such material should be the understanding of mathematical concepts, proficient use of mathematical tools, or deeper insight into the social issues, or some combination of the three. We agreed that, whatever the outcome, engaging in such mathematical teaching requires knowledge and experience of using mathematics as an applied tool so that being mathematical, in and with the world, is both implicit and explicit in our practice. The consequence of this, we agreed, was the importance of mathematics teachers continuing to think about new mathematics, new applications, and new areas of social relevance.

**Discussion**

In our responses to these two stimuli the dominant knowledge brought to bear on the pedagogic tasks of planning and teaching was the personal mathematical past experience of the protagonists. In the work on proportionality the pedagogic task was to discuss the affordances of the exercise and suggest better design, but the knowledge needed to critique the design was about representation, equivalence, exemplification, classes of functions, associated meanings, applications, tendency to treat relations as objects, and understanding of how to extend mathematical concepts beyond their obvious domains.

In the smallpox example the protagonists brought past experience of modelling, suitable functions, differential equations, and an over-arching judgement about which newspaper situations might afford what kinds of mathematical engagement. This process is the opposite of the application of mathematics to real world contexts, through which one gains the experience necessary to make these kinds of judgement.

Some mathematical modes of enquiry which arose in the situations above are:

* Interpreting mathematical statements (e.g. identifying variables, identifying relations, or constructing particular senses of structure).
* Thinking about representations (e.g. changing them, manipulating them, setting up alternatives, comparing them, identifying different potentials that arise from their use).
* Purposeful playing with an idea (e.g. instantiating it, finding particularly special or extreme examples, simplifying, asking “what if”, noticing what changes and what stays invariant). Mathematicians know, from experience, what to ask “what if” about. For example, they do *not* ask “what if I write in a different colour?” unless this might highlight new relationships, but they would ask “what if I try this with a very simple example?” or “what if I control one of the variables?”
* Conjecturing, testing, deductive reasoning, and other forms of justifying.
* Summing up by organising mathematical ideas; saying what is known or not known.
* Linking; finding similarities or isomorphisms

For teaching to mean anything, teachers must act in such a way that students learn something that they would not or could not have learned if the teacher had not been present. That is, they need to provide something that a textbook or annotated website cannot provide. As we examine the practices above, we find that mathematical modes of enquiry and past mathematical experience seem to be central to what a teacher might do. The mathematical modes of enquiry are then adapted to the restricted frames of the classroom. Teachers' own knowledge through personal experience of these modes, and their knowledge of how to draw on them and adapt them in appropriate ways at appropriate times, is important.

One aspect of what a teacher provides, that a text cannot, is a range of mathematical options offered with judgement. While a text or website may contain several options, they are not offered at a fine level in response to students’ reactions. Effective teachers will have a repertoire of, and means to construct, critical examples and a sense of when it is appropriate to present them; effective teachers understand when and why different representations are useful in a student’s learning trajectory; effective teachers make connections between mathematical ideas purposively, and focus attention on just that aspect of a relationship that is important for the task at hand. In our activity, connections were illustrated in the Smallpox example, thinking about appropriate use of representations was illustrated in the proportion example, as was our construction of new examples (or modified ones) to focus attention on particular relationships. Our argument is that a teacher for whom these are ways of being mathematical is more likely to be able to act fluently in all classroom mathematical contexts, than one who has learned a repertoire of pedagogy strategies without personal mathematical learning. Those who undertake mathematical exploration develop a pedagogic repertoire through dynamic engagement in classroom mathematics.

Another aspect that a teacher provides is prediction. It is a peculiarly educational task to need to predict the consequences of a particular experience, but in order to do this in mathematics, a teacher needs to know the mathematical consequences of particular conceptual instances. For example, at a detailed level, a teacher needs to be aware of what various letters might be used for in the future—not all will be variables, so using them that way now might lead to confusion later. Using Greek letters as general variables might be confused with their common association with angles, and so on. An example at a complex level would be the introduction of a concept within a limited frame affecting a later need to generalise to wider frames. “Knowing ahead” is a key part of mathematical learning prediction. A further kind of prediction is to look at an example or set of examples and see with new eyes what generalisations might be inferred from them. Often the cause of so-called ‘misconceptions’ is inappropriate generalisation, or paying attention to the wrong variables. Unless teachers consciously generalise from examples themselves, they may not understand how this process can go wrong even when students are thinking very carefully.

Mathematics teachers need to be able to react mathematically in the moment. They require understanding and intuition. Again, while an aspect of this is a general pedagogical skill needed by all teachers, there is a purely mathematical part. Reacting in the moment means understanding mathematics deeply enough to be aware of affordances and opportunities. An important mode of enquiry in mathematics involves being aware of all the possible connections and directions of mathematical development from any particular situation. An equation can be read this way or that, it can be factorised or expanded or divided through or several other options. It may conform to this pattern or that, it might be better to see it as a polynomial of a particular order, or make a substitution to transform it into another type of equation. It may be better to graph it or put it in matrix form. Knowing all of these (and their consequences) and choosing between them is part of both mathematics teaching and mathematical activity.

Finally, mathematics teachers must be able to “see behind”. They need to be able to interpret students’ mathematical texts and verbal explanations, to understand what the student might be meaning, or is trying to express, and then work in such a way that the mathematical thought becomes clearly expressed. Again, this is a mathematical mode of working, it is what a mathematician might do when thinking through a problem: having an idea, struggling to express it exactly, and it is through that process that justification and, ultimately, proving, are born.

A particular manifestation of this mode is in the similarity between reading a mathematics textbook and reading students’ mathematical attempts. Reading text is a mathematical mode of enquiry. Most of the available typographies of mathematical knowledge in teaching claim that knowledge of misconceptions is important, but we would say that personal experience of how misconceptions come to be constructed is a more powerful source of pedagogic knowledge. In our joint 59 years of teaching at secondary and tertiary level, we are still surprised by some students’ ways of conceptualising mathematical ideas. What enables us to work with these alternative constructions is our analysis of how they could have been arrived at, in terms of inferred relationships between variables, or inferred connections between representations. The analytical process is far more use than trying to accumulate a list of possible curriculum misconceptions to be remembered and anticipated. A route to deeper understanding about misconceptions, therefore, is to engage teachers in a new-to-them area of mathematics[[2]](#footnote-2) and to recognise what happens when their reasoning turns out to be ‘incorrect’.

As an example of what we mean, consider a first lesson in partial fractions. Students were going to find out how to express  in the form 

When we analyse this task mathematically, making some assumptions about what learners have met before, we recognise several potentially difficult features relating to previous experience of similar objects: it matters that these are ‘equivalent’ and not merely sometimes ‘equal’; we are not finding the unknown value of a variable x, we are finding the structural coefficients A and B, and we can use whatever value of x we like to help us do this; fractions here need to be seen as abstract rational structures. This topic therefore incorporates several classic difficulties in mathematics; a teacher will need to know not only that such analysis is important, but that the analysis itself has to be informed by imagining alternative meanings and interpretations, and identifying inherent ambiguity. A teacher needs to have learnt to navigate between alternative uses of letters, alternative meanings for the fraction notation, and uses of ‘=’, to recognise the problems. We are not saying that this knowledge is explicit, it may be tacit, but one cannot begin to be articulate about distinctions which one has not adopted in some earlier learning.

**Moving Forward**

What might teacher educators do with this understanding of the importance of mathematical modes of enquiry? Let us assume, for a moment, that our teachers have already developed a set of key modes. A classic situation is that teachers cannot see how to use these modes in their work because school mathematics is strongly framed by the curriculum and assessment regimes. We suggest that it is not so much a matter of learning new or more modes, it is more a matter of maintaining mathematical activity so that these modes of enquiry are active and remain part of the teacher’s way of being mathematical.

A recent experience working with teachers underscores the value of teachers maintaining their mathematical modes. A year-long research study[[3]](#footnote-3) centred around teachers who were supported in their quest for new mathematics learning in self-identified areas in which they had uncertainties. Each teacher chose their own topic, and worked with the support of mathematicians and mathematics educators from a university Department of Mathematics. They met together as a group to share their insights and experiences.

A striking outcome was the way that all these teachers, in a short period of time, adapted their teaching not only as a result of their new learning, but as a result of their mathematical modes of being. Examples were: attention to “big ideas”; reproducing learning experiences; being aware of when and what assistance was needed to understand a mathematical concept; looking for, and using, connections between different mathematical ideas; seeking mathematical structure. Another mode was the way the teachers used each other, both as support in their mathematical explorations and also as mathematical sounding boards for ideas and sources of linked knowledge.

Not only did the teachers attempt to reproduce such modes of being in their classrooms, but they all reported talking to their students about their own learning experiences—a meta-level discussion of mathematical modes. It seems that providing opportunities for teachers to be mathematicians might be a very good professional development strategy. Unlike any others we know of, it supports the development of sustained mathematical ‘being’.

**Conclusion**

Mathematical knowledge includes its own modes of enquiry, but these are not necessarily explicit, nor are they automatically drawn into action when planning to teach. We believe that mathematical modes of enquiry are learned by engaging in authentic mathematical experiences, although such learning can be enhanced both by having these modes modelled, and also by having the modes explicitly discussed at a meta-level. This applies to both teachers and students.

We know from observations that teachers are often likely, in their teaching, to focus on mathematical topics and explicit mathematical knowledge. A topic focus applies at the level of curriculum and syllabi, as well as within classroom moments. “Have you covered quadratic equations yet?”, “Do you know how to differentiate sin *x* ?”. In this teachers appear much more likely to draw on their past experiences of being taught, and the norms around them in school, than on being mathematical. Only rarely do we observe teachers applying their full repertoire of mathematical modes in a planned and explicit way.

It is with these experiences in mind that we argue that teaching mathematics is the contextual application of modes of mathematical enquiry, and that too often the modes of enquiry used in planning and teaching are drawn from a limited set, or are limited by teachers’ own experience. We hypothesise that teachers engaging in personal mathematical experiences, discussing their mathematical modes, and finding ways to organise situations involving those modes in the classroom, will result in better mathematics learning for their students. Our emphasis is on teachers’ mathematical activity as an integrating context for all aspects of mathematics teaching, in which the separate actions of doing, planning, teaching and learning mathematics connect and inform each other.

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1. We acknowledge the help of Pedro Palhares and John Mason in preparing this chapter [↑](#footnote-ref-1)
2. Anne often uses taxi-cab geometry as an arena which requires new conceptualisation, opportunities to challenge intuition, but low-risk in that nothing depends on understanding it; Bill has in-service teachers who have used a ‘modelling’ course for the same purpose [↑](#footnote-ref-2)
3. “Teachers Learning Mathematics”, funded by the Teaching & Learning Research Initiative, NZ Council for Educational Research [↑](#footnote-ref-3)