anne watson and peter sullivan

TEACHERS LEARNING ABOUT TASKS AND LESSONS

Abstract

In this chapter we suggest that mathematics teacher educators can use the obvious interest of prospective and practising teachers in planning and teaching lessons as a way of drawing their attention to key aspects of student learning. We argue that different tasks have different purposes and that consideration of those purposes can be an effective way of educating teachers. We also argue that converting tasks to lessons is complex and that the structure of lessons should ideally match the type of task. We discuss four types of task: those that foster conceptual understanding, those that develop mathematical fluency, those that create opportunities for focus on strategic competence: and those that create opportunities for use of adaptive reasoning. We also suggest templates for mathematics lessons that match each of these types of tasks. Each of the task types and lesson templates can be readily incorporated into mathematics teacher learning opportunities.

Introduction

In this chapter we examine ways in which the study of classroom tasks and mathematics lessons can be used to enhance teacher learning. We outline some of the affordances of classroom tasks that address four different aspects of mathematics, and illustrate these with four different templates that indicate how such classroom tasks can be used as the basis of lessons.

 We use ‘teachers’ to mean prospective and practising teachers involved in deliberate formal learning situations whom we assume are thinking about how to teach their students. We use ‘students’ to apply to children and adolescents in schools who are being taught mathematics by teachers. We use ‘lessons’ to refer to the units of learning and teaching used in schools for organisational purposes.

 We use ‘classroom tasks’ to refer to questions, situations and instructions that teachers might use when teaching students, and ‘tasks for teachers’ to include the mathematical prompts, many of which may be classroom tasks, that are used as part of teacher learning. We are not offering these as definitions, rather they are our attempt to narrow the use of the word ‘task’ to apply to the starting point of mathematical activity, whether it is by students in classrooms or by teachers in educational settings, for the purposes of this chapter. This means we are not going to deal explicitly with tasks for teachers such as ‘compare how students P and Q responded to this task’, although these too are tasks in mathematics teacher education and we assume that everything we say is taking place within an educational context in which such pedagogic tasks are the norm.

Using classroom tasks to prompt teacher learning

Tasks for teachers have multiple purposes in teacher education. Teachers’ engagement with such tasks can:

- inform them about the range and purpose of possible classroom tasks;

- provide opportunities to learn more about mathematics;

- provide insight into the nature of mathematical activity; and

- stimulate and inform teachers’ theorising about students’ learning.

 Teacher educators commonly combine these purposes and provide mathematical tasks, both of familiar and unfamiliar kinds, for teachers so that their engagement as learners is genuine (Watson & Mason, forthcoming). They might learn, for example, about how unfamiliar task types provide different ways of engaging with mathematics, or they might learn how different learners respond to familiar task types. Such learning may be mathematical, pedagogic, or a mixture of both depending on how task aims, discussion and reflection are structured in the teaching session, with a degree of familiarity relating to mathematical content, or task-type, or expected ways of working.

 Teachers in educative settings often want tasks or task-types that can be used in their own classrooms with minimal transformation, while teacher educators might seek to offer tasks which influence teachers’ learning. Here we combine these aims by concentrating on how teachers can be educated about classroom tasks through reflective engagement with such tasks. The starting point for our considerations is to think about how particular types of task offer the potential for students’ learning, since that is the shared concern of teachers and their educators. By starting with this as the focus, the four purposes given above can be achieved in ways which support effective teaching, and which recognise and build on existing pedagogic strengths, whatever the systemic constraints within which teachers work.

 Stein, Grover, and Henningsen (1996) offered a structure that helps to guide the following discussion. They argued that the consideration of classroom tasks by teachers goes from …

 mathematical task as presented in instructional materials,

 to …

 … mathematical task as set up by the teacher in the classroom,

 to …

 … mathematical task as experienced by students,

 which creates the potential for …

 … students’ learning.

 We focus on the first two of these steps, specifically the teacher’s choice of a task selected and adapted from available resources, and the use of the task to create a lesson which incorporates it into a sequence of events in a classroom, although clearly the other steps also inform our thinking. The ways in which these steps differ is elaborated later in this chapter.

Connecting tasks to learning

Doing tasks does not guarantee learning. It is possible for teachers and students to act together to reduce a classroom task to a sequence of things to do towards completion, and for students to trail through those ‘things to do’ without engaging with meanings, purpose or ideas. The connection between task and learning is non-deterministic and far from simple. Hiebert and Wearne (1997) propose that ‘what students learn is largely defined by the tasks they are given’ (p. 395) but, while this may be true, Christiansen and Walther (1986) report that ‘even when students work on assigned tasks supported by carefully established educational contexts and by corresponding teacher-actions, learning as intended does not follow automatically from their activity on the tasks’ (p. 262).

 Christiansen and Walther go on to distinguish between the task as set and the activity that follows, including students’ interpretations of the purpose of the task, ways of working, teacher interventions, how language and symbols are used and what is seen as valuable mathematical action. Gibson’s (1977) notions of affordance and constraint are useful to help us focus on what becomes possible because the task is set (Watson, 2004). The setting of a task opens up the potential for learning – it gives students things to hear, look at, read, think about, make sense of, relate to previous experience, and so on. These responses include rejection, confusion, misreading of symbols and other unhelpful responses. The design of a task constrains these responses so that it is more likely that students will focus on the mathematics intended by the teacher, and will draw on the knowledge, actions, and generalities which the teacher hopes will inform the ensuing activity. In addition the idea that students are attuned, through past experience, disposition and enculturation, to respond in certain ways, to transform tasks into certain kinds of activity, offers a language for talking about the non-deterministic nature of tasks.

 For teachers and their students to achieve learning goals, whatever they are and however they are described, the task must at least afford those goals to be pursued. Different classroom tasks afford different kinds of activity, and students’ experiences of different kinds of activity indicate different kinds of mathematical learning. When teachers decide to use particular tasks, and types of task, they are making choices about the nature of mathematical activity and learning that might take place; and about mathematics. In our reading of the literature we have found that these aspects which give the tasks purpose and meaning are often implicit. There is a significant body of research about observing and replicating worked examples (e.g., Atkinson et al., 2000; Moreno 2006); about the design of complex problematic situations (Freudenthal,1973; Brousseau, 1997); and about tasks which address cognitive development (e.g., Swan 2006). Results from these bodies of research can be interpreted to conflict because of a lack of explicitness about pedagogic purpose.

 In our experience new teachers, and some experienced teachers, often fail to make distinctions about the kinds of learning activity likely to be prompted by a task and may adopt attractive resources because they provide fun contexts, quiet work, material for discussion, or practical activity, without analyzing the nature of learning afforded. Any agenda associated with choice and use of classroom tasks needs to offer distinctions about learning. Teachers need to select and adapt tasks which make it more likely that particular pedagogical goals are achieved, whether these goals are about learning methods; understanding concepts, ideas, and relationships; working, thinking and reasoning mathematically; becoming a better learner; knowing more about mathematics; and so on (Mason & Johnston-Wilder, 2006). In this chapter, the distinctions we make about the nature of learning mathematics in school classrooms are developed from the five strands of mathematical learning described by Kilpatrick, Swafford, and Findell (2001). We have adapted their second strand, ‘procedural fluency’, to include the development of a repertoire of factual and conceptual knowledge which comes to mind fluently when appropriate. Definitions, names, facts, theorems and canonical examples need to be available alongside procedures. Our version of these strands is:

• conceptual understanding—comprehension of mathematical concepts, operations, and relations;

• mathematical fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately, and, in addition to these procedures, factual knowledge and concepts that come to mind readily;

• strategic competence—ability to formulate, represent, and solve mathematical problems;

• adaptive reasoning—capacity for logical thought, reflection, explanation, and justification; and

• productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

 For each of the first four strands we show how teachers can learn more about tasks, mathematics, activity and learning. In addition we show how including tasks in some standard lesson templates can raise further educative questions for teachers, and can also enrich the mathematical affordances of tasks. We do not address the fifth strand explicitly since it is connected to the tasks and lessons addressing each of the other strands, and we discuss tasks and lessons which allow for the development of a productive disposition within a pedagogic context extended over time.

From tasks to lessons

So far we have emphasised the centrality of having a learning purpose in mind when selecting, adapting and designing classroom tasks, and have pointed out that the accompanying pedagogic choices make a significant difference to whether students merely ‘do’ tasks or whether they learn mathematical ideas in the associated activity. Mathematics teachers in our countries, for a variety of reasons, tend to focus on individual lessons rather than individual classroom tasks. Planning real or imaginary lessons is seen by teachers as practical and authentic, a worthwhile way to spend time, and helps to provide a direct connection between theoretical considerations and practical imperatives. By considering the effect of stringing various tasks together to form a lesson with overall coherence, the tendency to simplify task intentions described by Stein et al. (1996) can be addressed, along with the long term aim to develop productive disposition. A classroom culture that fosters all five strands of learning over time can be developed and sustained over time through task choice and lesson design which takes into account:

- the level of implied student choice, in that student choice of focus, approach, and difficulty contributes to motivation (see Middleton, 1995);

- the potential for prompting communication, in that communication can contribute to effective learning (e.g., Wood, 2005), and is more productive if there is more to discuss than the correctness of the answer;

- the degree of risk, recognising that not all students respond well when they are uncertain on how to proceed or the risk of failure is high (see Doyle, 1986; Dweck, 2000); and

- the ‘level’ of potential student engagement: Fredericks, Blumfield, and Paris (2004) described engagement in terms of behavioural, emotional, and cognitive responses. They argued that engagement is enhanced by tasks that are authentic, meaning whether or not students can engage with tasks with the whole of their current sense of self – they do not need to leave their personalities and dispositions at the classroom door in order to participate in mathematical activity. Thus lessons must provide opportunities for students’ sense of ownership and personal meaning, fostering collaboration, drawing on diverse talent, and providing fulfillment and empowerment.

 In each of the following sections we discuss an aspect of mathematical learning from the first four strands. Each section elaborates the nature of the learning associated with the strand, offers illustrative tasks that teacher educators can use with teachers, and proposes a template of a lesson structure to which tasks within that strand can be adapted. We comment on the lesson structures in the light of the emotional and social factors outlined above. We emphasise that we are offering illustrations, and that in every strand there are other, different possibilities.

Strand 1: Conceptual understanding

It is important to work with teachers sometimes on task types which are generally familiar in schools. Traditional ‘chalk-and-talk’ methods, or whole class teacher-led discussions, can be done in ways which enable students to engage with the meaning of concepts.

 The conceptual understanding of mathematics can be problematised with teachers in at least three ways. The first is for teachers to develop greater insights into the mathematical thinking that is encapsulated in their own understandings. The second is for teachers to work with topics which are hard to teach and easy to confuse because of their conceptual nature. The third is to know that relational understanding (Skemp, 1977) is always a possibility in mathematics and that this has implications for the way they teach. To work on the first of these, teachers need to become articulate and explicit about how they came to understand concepts; for the second, recognition that mathematics is littered with inherent cognitive difficulties is needed; for the third, teachers need to recognize that their own knowledge might be limited, and what they see as procedures might be reconceptualised as expressions of generalities about relationships. These are elaborated in the following passages.

Insight into the mathematical thinking encapsulated in their existing understandings

Using concepts which are well-understood by teachers, it is possible to unpick the essential relationships and reconstruct them as ‘discovery’ tasks to show how mathematics derives from general relationships and behaviours. For example, teachers can be asked to use a calculator to demonstrate the effects of entering any number, using the ‘x2’ button , then using the ‘√’ button. Once the technicalities of using the calculator have been sorted out, school students typically make observational statements such as ‘it’s just the same’ or, if they entered a negative number, ‘it doesn’t work’. Teachers are aware of the relationship between squaring and square-rooting, but the use of the calculator helps them re-engage with this relationship as learners might. The issue to discuss with teachers is how to help students shift from these superficial observations to the notion of inverse, forms of conventional notation, and other representations which might explain what is happening, and also why negative numbers are different. These discussions help teachers appreciate the conceptual depth of topics they already ‘know’, show how simple ‘exploration’ tasks can valuably be incorporated into the curriculum, generate mathematical discussion about ‘why negatives don’t work’, and suggest that reasoning about patterns in results is something all learners can do at an appropriate level. Teachers bring their existing knowledge to the task, but may well end up with richer understandings of concepts (in this case, of ‘inverse’) and of the domains of applicability of concepts.

Topics which are hard to teach and easy to confuse because of their conceptual nature

Swan (2006), working within a cognitive perspective, devised task types which structure conceptual understanding of hard-to-teach and easy-to-confuse topics. His tasks scaffold the processes of making and refining distinctions, resolving ambiguity, and enacting the mathematical discipline which characterise the interface between conceptual learning and general human endeavour. Some of these tasks require learners to:

- classify objects using properties, definitions, language, distinctions;

- interpret multiple representations by making links between them and forming mental images;

- evaluate mathematical statements, deciding if they are always, sometimes or never true, and generating examples, counterexamples and arguments to justify such decisions;

- create mathematical problems to understand reverse and inverse processes; and

analyse other people’s reasoning and solutions by comparing methods, solutions, diagnosing errors, and identifying chains of reasoning

 His research shows that these tasks can generate talk about the mathematical ideas and meanings involved. For example, matching fraction, percentage and decimal representations of proportion can to some extent be done using instrumental knowledge, but teachers can be asked to develop such a task further by making up objects which are ‘hard’ to calculate and ‘hard’ to match (using their own definition of ‘hard’), or by being asked to place things in order and provide missing objects. Similar tasks for teachers could be set up with functions and methods of integration; or with polynomials and lists of roots. Thus the teachers explore the extent of their own mathematical understandings, while at the same time learning that the ordinary human activities of sorting, ordering, classifying, naming and so on can be harnessed through such tasks to help students learn mathematics. Often a mathematical idea is hard to teach and learn because it is easily confused with something already known, so questions which encourage learners to make distinctions for themselves are especially valuable.

Relational understanding is always a possibility

The type of relational understanding that Skemp (1977) described is about meaning, about making connections between different expressions of a mathematical idea, about connecting an idea to other known ideas. Relationally understood ideas are easier to remember than instrumentally learnt ideas, and they are more useful within a problem solving context. To illustrate this idea consider the way that Peter first learnt about standard deviation. In the days before scientific calculators, students were taught to calculate the standard deviation of a set of scores using what was termed a raw score formula. It was easy to learn to use the formula but it was not connected in any intuitive way with the concept of spread, or deviation from the mean. Later, when the calculation could be done using a scientific calculator, it was more important to explain what is variance, and why the deviation of individual scores from the mean are squared and then averaged to find the variance. This way of thinking about standard deviation then makes it clearer why variance is connected to correlation, for example. There are many similar instances in which beginning teachers realise that the way they learned to multiply a decimal number by 10 by moving the decimal point to the right, or divide by a fraction (“invert the guy and multiply”), or that negative times negative makes a positive, how to differentiate, and so on, do not make intuitive sense, and these are just rules applied to get a correct answer in the short term. For the most part, teachers have to find the meanings of these various concepts for themselves, but the task of the teacher educator is to indicate that achieving this relational understanding is worth the effort, and that using this understanding in their teaching is crucial.

An illustrative task for teachers: Comparing affordances and constraints of classroom tasks

As an example of a task that teacher educators can use with teachers to emphasise conceptual understanding, we use the notion of ‘applicability’ as a starting point to develop conceptual understanding, and we offer the task for teachers of comparing a ‘realistic’ approach to relational concept development to a more cognitive approach. This comparison draws attention to the fact that, even if the focus is on conceptual understanding, very different kinds of mental activity can be generated by different kinds of task. We offer two classroom tasks each concerned with simultaneous linear equations. Teachers can be asked to say what different kinds of activity could follow from each task, and what different sorts of learning are prompted. Teachers could be invited to consider what types of intervention might be necessary to ensure students shift from ‘doing’ to ‘conceptualising’ in each case, and come to know ways to recognise and solve simultaneous equations using these tasks as starting points.

*“Realistic” task*

Two rounds of drinks were bought, one by Anne and one by Peter. Anne bought three coffees and two fruit juices and the bill came to £6.70; Peter bought one coffee and three juices and the bill came to £4.80. Now they want to get the money back from their friends so need to know the cost of coffees and juices.

Compare your methods for working this out.

Later, they had a larger group to buy for and Anne’s bill for 13 lemonades and 6 cokes came to £26.20 while Peter’s bill for 20 lemonades and 13 cokes came to £43.70. This time, what do lemonades and cokes cost?

What methods could you use to work out similar problems in future?

*“Abstract” task*

Using a graph-plotter to help you, find, where possible, two equations in set A which go through each of the coordinate points in set B. Where it is not possible, invent new equations to make it possible.

Set A

|  |  |  |
| --- | --- | --- |
| y = x | y + x = 6 | y – x = 6 |
| y + 2x = 6 | 2x + 3y = 6 | 3x + 2y = 6 |
| y = 6x + 2 | 6y = x + 2 | x + y = 1 |

Set B

|  |  |  |
| --- | --- | --- |
| (3,3) | (0, 6) | (2,2) |
| (0, 3) | (2, 0) | (4, -3) |

Make up more sets of mathematical objects which can be matched up like this.

 The first of these classroom tasks presents a “realistic” context, while the second affords the possibility of generalising from the patterns identified by the students. Each type offers the teacher an opportunity to understand that solving simultaneous equations is not just a matter of applying methods, but involves the intersection of relationships. Relational knowledge is not only about application. Indeed, it is possible to do much of both tasks without thinking about solving equations at all. Both need purposeful interaction to shift students towards conceptual and relational understanding. Vygotsky emphasises this when he says “ neither the growth of the number of associations, nor the strengthening of attention, not the accumulation of images and representations, nor determining tendencies – none of these processes, however advanced they might be, can lead to concept formation” (1986, p.107). The central question is what kinds of experience learners need from which they can generalise and how can they be led to develop appropriate abstractions.

 The teacher development activity of comparing classroom tasks (as above) is most effective if teachers have first done the tasks themselves. This extra turn of the wheel not only brings the discussion into the arena of their professional interest (what can I do in my classroom so that these tasks promote appropriate mathematical activity?) but also enables them to talk about how tasks promote learning through mathematical activity. The pedagogic choices to be made include:

* what will they see in terms of visual impact, layout and diagrams?
* will they handle actual objects?
* how much familiarity do they need to have with notations and conventions?
* what questions can be asked about their experiences (such as what changes and what stays the same; what is the same and what is different)?
* what key words and ideas can the teacher offer to influence thinking?

 A further way to engage learners in concepts is to offer multi-stage tasks which are accessible to start with, but which need new ways of thinking about familiar objects. The investigative classroom tasks developed in the UK (e.g., Banwell, Tahta, & Saunders, 1972; ATM, 1980) provide examples of this approach. The challenge for new teachers is to realize that working on and ‘finishing’ such tasks do not in themselves guarantee learning the newly-available concepts.

An illustrative template for a lesson focusing on conceptual understanding: Active Teaching

It is also productive to discuss with teachers ways in which tasks that foster conceptual understanding can be used as the building block of a lesson. The following template, adapted from Sullivan (2007), is similar to the lesson structure that Stigler and Hiebert (1999) suggested is representative of mathematics teaching in the U.S., which in turn is similar to that described by Good, Grouws, and Ebmeier (1983) as “active teaching”. It specifies particular activities including: daily review of homework; development (including addressing prerequisite skills, lively presentations, assessment of comprehension, and controlled practice); seatwork; and homework assignment, with the teacher having an active role at each stage. The template supports teaching in which the teacher seeks to develop comprehension of a specific aspect of mathematics, or to foster conceptual understanding of a procedure or technique.

 To illustrate the elements or phases of the template, suppose the teacher wants her class to learn that it is possible to multiply any number by 99 mentally by first multiplying by 100, then subtracting the number. The left hand column of Table 1 is a description of how the template might work in this case. The right hand column in the table is intended to apply to lessons of a similar structure.

Table 1: The Active Teaching template

|  |  |
| --- | --- |
| **In this example …** | **Key lesson element** |
| The teacher poses some examples such as 600 - 6, and 1100 - 11 to check student facility with this pre-requisite skill. | The teacher revises pre-requisite content with students, and assesses current understandings. |
| Teacher poses some examples like 5 × 99, 8 × 99, and asks students to work out the answers. Some students explain what they have done. The teacher emphasises the method: 5 × 99 = 500 - 5 = 495 | Teacher uses lively methods to illustrate or model aspects of mathematics or procedures, building on suggestions from students, emphasizing relational understanding and making connections with previous learning. The teacher gives one or two carefully varied practice examples, which are reviewed.  |
| Further questions are posed, in sets of similar demand on students. The first set might be like 6 × 99, the next set like 11 × 99, and the next set like 25 × 99, and then perhaps extending to 110 × 99. | Individually or in small groups, students complete further examples, tasks or problems designed to give practice and consolidation of the content that is the focus of the lesson. Teacher monitors work of students, noticing solution strategies, adapting the questions if necessary. |
| The students’ responses to set exercises are corrected, and some further examples (e.g., 4 × 99) are posed to check both their accuracy and capacity to explain the process they used. The teacher can also draw out the key mathematical points involving the distributive law 5 × 99 = 5 (100 – 1) etc.It can also be extended to questions such as 4 × 98. | The teacher reviews the methods and answers of the students, and attends to particular problems or responses that assist in consolidating the purpose of the lesson. Students might propose questions of their own, and suggest extensions to the technique. |

 This lesson template can be used to build conceptual understanding of ways of calculating, and to illustrate a purpose for the distributive law notation. The template is characterized by limiting student choice, in that the focus and pace are determined by the teacher; there are few opportunities for prompting communication, in that it is either correctness or a particular approach that is of interest; the experience is low in risk for students in that the teacher guides the lesson; the engagement is through the energy, activity, and explanations of the teacher, or perhaps through a student’s innate desire for competence. There is some opportunity early on for students to suggest methods and later to extend the lesson to consideration of the general ways of writing such calculations.

 For all teachers, new and experienced, such a template will be familiar. What is less familiar is to take an analytical approach to specific parts of it and identify exactly where choices might make a difference to overall learning, such as choosing to extend beyond the use of 99, or choosing to use the bracket notation. Marton’s variation theories (2006) draw our attention to the significant advantages in offering variation of different aspects. In the lesson we give above, early examples stay with single digit multiplication, before moving to two digits. Further variation in the question, and even the use of formal representation, is left until the basic technique can be carried out competently. Other possible variations can be imagined. Another significant choice is in the nature of questions posed for individual work. In this lesson, students are asked to imitate or reconstruct the procedure on sets of questions, but it would also be possible to match questions to answers, to work backwards from answers to questions, to sort questions into ‘easy’ or ‘hard’, or ‘those that can/cannot be done by this method’. Thus the lesson template provides a frame for the teacher-task of imagining alternative choices, and the activity they would afford, which fit into common lesson shapes.

Strand 2: Mathematical fluency

As with Kilpatrick et al. (2001), we see the development of mathematical fluency as one of the goals of mathematics teaching. Teaching which takes mathematical fluency as the learning aim generally focuses on minimising the difficulty and effort required for learners to perform algorithms correctly. Task designers in this tradition tend to seek to reduce *cognitive load* by identifying what is intrinsic and what is extraneous (see Bransford, Jones, & Cocking, 1999)). The underlying theory is that the working memory cannot process too much input so complexity leads to inefficient learning (Baddeley & Hitch, 1974). The aim is to offer worked examples from which smooth performance can be imitated or inductively followed. Typically, writers are concerned about how many examples, of what kind, learners need to see before they can successfully do similar examples for themselves. These approaches to teaching have been challenged as not addressing the need for learners to develop conceptual understanding, nor to carry out procedures when the need for their use is not made obvious. Yet not only Kilpatrick and his colleagues, but also teachers everywhere whose students will have high-stakes assessment of a traditional kind, know that procedural fluency is one of the goals of teaching mathematics, and that teachers need to know ways to achieve this effectively. For example, it has been found that showing students a problem before demonstrating a similar example is more effective for complex ideas than offering the worked example without learners knowing why they might need to pay attention to it (Sweller, 2006). As Hewitt (1994) has pointed out, careful design of teaching sequences can ease learners’ enaction of routines and he uses repetition and pattern (aural, visual and physical) to enable learners to gain such fluency.

An illustrative task for teachers: Comparing and adapting textbook exercises

A standard way to address fluency of procedures is to provide worked examples and follow-up exercises. When teachers use sequences of worked examples and exercises they have to anticipate what learners might attend to so that examples give enough information about generality, without too much irrelevant information and without displaying common features which are unhelpful (Atkinson et al. 2000). New teachers often assume that exercises will automatically ‘provide’ learning without analysing how the whole set is structured and hence what kinds of activity will be prompted.

 Such assumptions can be challenged by asking teachers to compare the purposes of, say, three textbook exercises which purport to address the same content and to describe how the questions vary and whether such variation helps or hinders fluency; helps or hinders conceptual understanding; or has some other purpose. As with other tasks for teachers, this is more effective when teachers actually do the tasks and appreciate for themselves how the variations affect them. Teachers then find that many textbook tasks appear to be constructed within a general theme of ‘getting harder’, but the ways in which they ‘get harder’ (such as introducing negative signs or fractions) disrupt the growth of pace and automaticity and simultaneously get learners ‘bogged down’ in detail so they cannot grasp concepts either. However, it has been established that disruption after fluency can provide important insights. Anne still remembers an exercise she worked on in 1962 which entailed multiplying two binomials to get trinomials. After doing several of these, and feeling fluent, suddenly she came across a pair which only gave two terms when multiplied together. After a few attempts to get three terms she recognised that the next few questions would also only produce two terms. Fluency itself led to a sense of disruption that fed intrigue. In a more regimented classroom, with a less confident learner, it could also have led to frustration, confusion or deskilling.

 Teachers can also reorganise textbook questions into an order which (a) makes fluency more likely and (b) makes conceptual understanding more possible.

 While discussing ‘tasks which build the repertoire’, teachers often report that they find it harder to recall how to use ‘SOHCAHTOA’[[1]](#footnote-2) than to remember the ‘word’ itself. This can trigger a critical discussion about when, whether and how to introduce students to such memory aids. When working on this with teachers we find it hard to keep discussion focused on classroom tasks – often discussion becomes peppered with aids teachers ‘give’ or ‘tell’ their students which might just become ‘more things to remember’. Suitable classroom task types to address the development of repertoire include: devising personal (rather than adopting given) mnemonics, creating visual and audio representations of mathematical ideas, concept mapping, comparing similar concepts, keeping a record of typical and atypical examples and definitions, and creating their own ‘obscure’ examples of ideas.

 The word ‘practice’ is often used loosely to mean ‘doing several examples’ but it is not always clear what this ‘doing’ is supposed to achieve. Task structures that afford both fluency and memory include: chorusing mnemonics and verbalisations of procedures, working on similar examples within a time constraint, and competing with previous personal ‘best’ in terms of fluency and accuracy. Many teachers use games and investigations which require repetition of procedures to enable automatisation through subordination of repetitive tasks to the more complex task (e.g., Kirkby, 1992) – the mathematical equivalent of becoming a faster and stronger cyclist by taking on a newspaper delivery job.

An illustrative template for a lesson focusing on mathematical fluency: Purposeful Games and Puzzles

Purposeful games and puzzles (PGP) (see Sullivan, 2007) have potential to form the basis of meaningful experiences which have a focus on the development of mathematical fluency. It is suspected, though, that mathematical learning does not occur optimally merely as an incidental component of engaging in the PGP. In this case, the intent of the template is not only to emphasise the mathematical purpose of the PGP, but also to facilitate development of mathematical fluency in ways that has potential for future use.

 The following is an example of a mathematical puzzle, adapted from a suggestion by Swan (no date). The puzzle involves a set of rectangular term (or number) cards and arrow operation cards, a subset of which could be: In this case, the puzzle is to choose the two operation cards that can be placed between the two term cards to represent the connection. The point is that students have to look for the appropriate operation card to connect the terms and, by doing so, evaluate a range of possible operations simultaneously. It is also self correcting, in that there are unique operations connecting the terms.

3a

3a2b

× b

÷ b

× ab

÷ ab

 A generic lesson template for using such puzzles, and possible actions for this particular puzzle, follows. Note that fluency is not the only likely learning outcome of such a lesson.

Table 2: Lesson elements for Purposeful Games and Puzzles template

|  |  |
| --- | --- |
| In this example … | Key lesson element |
| The teacher explains that there are cards on which are mathematical terms, and arrows on which there are operations. The intention is to connect the terms using the operation cards. The teacher might model the process using different but related cards. | After explaining the rules and purpose of the PGP, the teacher demonstrates the PGP to the class. |
| After the students have worked for a while on the task, there can be a class discussion of the processes for deciding which operation card is placed where, after which the students can continue with the puzzle. Students can be invited to describe how they made decisions on operations. | Students engage in the PGP for a short while, after which there is a teacher-led class discussion of the strategies and or mathematical point of the PGP (also rules and possibilities may need to be clarified) |
| The teacher monitors the students’ work as they arrange the cards of the puzzle. It is possible to have both harder and easier sets of cards, for those who finish quickly, and those for whom the puzzle, as it is, is too difficult. Some cards sets might just involve numbers, or easier operations such as addition and subtraction. If necessary, it may be helpful to pose questions such as  3a × ? = 3a2b | The students are then offered further opportunity to engage with the PGP. There can be additional discussion and activity as needed. The teacher or the students can suggest variations, such as making the PGP more challenging for some, or less complex for others. It is possible to group students based on their success at the PGP, so that, for example, students who complete the activity quickly might be grouped together for the next implementation of the PGP.  |
| The students can be asked to complete a set of practice exercises on operations with like terms with the goal of emphasizing fluency, and/or students can be asked to create their own sets of term and operation cards. | The teacher leads a discussion of the strategies and mathematics of the PGP. Specific problems can be posed that allow either fluent practice that focuses on the mathematical point, or extension of thinking. |
| The teacher can ask for the students to suggest rules that can guide operations using algebraic terms, and to illustrate how the principles used in choosing the operation to connect 3a and 3a2b can be extended to other operations. | The teacher summarises the main mathematical ideas. The teacher has an active role to find commonalities, patterns, and principles that can form the basis of the formalisation of the intuitive insights developed during the engagement with the PGP. |

 A games format, by generating a need for efficiency and optimisation, encourages adaptation and conceptualization as well as fluency. The following is a description of the game *Race to 10* (see Brousseau, 1997). In this game, players take turns to add either 1 or 2 to the previous total. Assuming that the players start at zero, a possible sequence could be as follows:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Player One | 1 |  | 4 |  | 6 |  | 8 |  |
| *Player Two* |  | *3* |  | *5* |  | *7* |  | *10* |

 One of the mathematical points in this game is that counting on is an efficient way to add. Another aspect of the game, in this case, relates to the existence of a winning strategy. Rather than detracting from its effectiveness, the existence of the strategy enhances the search for mathematical connections. For students in the early school years, it develops the fluency with counting on.

 After playing the game for a short while, it is preferable to have a class discussion of the strategies used. This can assist by clarifying the rules and method of playing, to the process of counting on, and to the existence of a winning strategy. After more playing, possibly with winners matched with winners, there can be a review with some specific problems posed. Note that this game can be played in many ways, including for algebra, for example, by playing Race to *5x + 5y*, adding on *x*, *y*, or *x + y*. This latter type of game can be used to develop the type of fluency we are seeking when operating with like terms. This game is an example of Brousseau’s designs of *situations didactique* (1997) which offer complex explorations which have conceptualization at the core, and hence also relate to strand 1 above.

 The PGP template focuses explicitly on the importance of the development of fluency, and remembering certain facts or strategies. There is student choice in the strategy to be used, in that students choose not only the game or puzzle strategy but also the ways they solve the mathematical aspect of the game or puzzle; it is medium for prompting communication, so the teacher must take an active role in encouraging students to talk to each other about the choices they make; it is low in risk in that students have some degree of choice and the format is self correcting, and engagement is through the competition or challenge associated with the activity or game.

 As before, imagining tasks in a lesson context enables teachers to anticipate possible problems and choices. For instance, the template reminds teachers to discuss the mathematics, and not to merely let students solve the puzzle or play the game. It also offers the possibility that designing a new ‘game’ might be as valuable in conceptual terms as doing some practice exercises.

Strand 3: Strategic Competence

Strategic competence refers to the ability to formulate, represent, and solve mathematical problems. New teachers often talk of presenting mathematics as ‘relevant’ and ‘applicable’ to motivate learners, yet the ways in which people use mathematics out of school are different from those used formally, and one could argue that to learn to be mathematical outside school one has to be immersed in the relevant ‘outside’ practice. Learners have to experience a wide variety of authentic problems and discuss explicitly how to work with them, including problems within mathematics, problem solving approaches, investigative, discovery, modelling. Anderson and Schunn (2000) claim that efficient problem-solving has to be preceded by learning some prerequisites, namely the difference between declarative and analogical knowledge; knowledge about application and how to acquire new knowledge; structures of knowledge and how to use it in practice. Others from a mathematical background, such as Polya (1981) and Freudenthal, believe that one learns to solve mathematical problems by solving problems: Freudenthal (1973, p110) states ‘[t]he best way to learn an activity is to perform it’.

 Teachers can combine conceptual and strategic aims by providing:

i) situations which require thoughtful application of known procedures – for example, in the puzzle described in the previous section;

ii) situations which generate a need to know new procedures in-the-moment; and

iii) situations for which students might create methods of solution.

 Complex multistage problem-solving tasks do not always offer new mathematical insights without the teacher’s intervention to draw attention to significant features, as we saw in the ‘realistic’ task in strand 1. More often, solutions are *ad hoc*, locally relevant and robust, and learners become better at situated problem solving, rather than learn more conventional mathematics. For teachers the tension is that learners may focus on finishing the task as posed, which may include choosing to pursue it in ways not predicted by the teacher, while teachers may want some mathematical content to somehow ‘come out’ of it.

 The following comparison between tasks illustrates some of the problem. Each task engages learners in generating numbers and algebraic generalisations. Each task affords the comparison of different algebraic representations and discussion of how these representations relate to the physical properties of the situations. Each task offers extension possibilities to, in the first case, other possible designs and, in the second case, other polygons. The second case, however, also offers knowledge from the conventional mathematics repertoire, a useful fact which is likely to be called on in many future contexts. The first case may give insight into a relationship between triangle numbers and square numbers (although this is masked by the extra dot), itself a mathematical fact but not one which is drawn on much in school mathematics.

*Pattern seeking and using:*

|  |  |
| --- | --- |
| How many dots? How many ways to count them? Compare ways of counting them? Represent one dimension by n. How many ways can you express the total in terms of n? |  |

*Internal angles of polygons:*

How many triangles? Use the fact that the internal angles of a triangle add up to 180 degrees to find the sum of internal angles in this polygon. Use the same method to find internal angle sums of other polygons.

 An emphasis on problem-solving skills may confuse the issue further, in that teachers may not see that successful solving of a problem has not, by itself, led to mathematical or intellectual progress. Our experience has been that, despite the advice about argumentation and class review (Wood, 2005), the review of student work, in which important mathematical understandings are emphasised and named, is a challenge for teachers. Yet it is through serious review, in which the teacher really listens to what students are saying, that the teacher can recognise what, if any, progress has been made in mathematical knowledge or power. We argue that the attention to the mechanics of review can be an explicit focus of teacher learning sessions. This focus can include the teacher educator modelling a review of the teacher learning, and even collective critique of the teacher educator’s actions.

An illustrative task for teachers: Thinking about building on the learning of the students

Peter recently highlighted the issue of lesson review with practising teachers. The teachers were posed the task to develop advice, in the form of a mnemonic, for beginning teachers on key actions in reviewing student work after a complex, multi-stage task. One group came up with the mnemonic TEACHER:

Talk with, not at, the students

Explain any unknown concepts

Acknowledge and attribute any progress

Challenge them further by questioning

Highlight what is known by the group as a whole rather than individuals

Explore the concept further

Revisit the original idea

 We present this mnemonic not as something to be used in itself, but as an example of how teachers working explicitly on a pedagogic issue can raise each others’ awareness and make different teaching skills available.

An illustrative template for a lesson focusing on strategic competence: Imagined Representations

The *Imagined Representation* lesson template (Sullivan, 2007) is similar to a “Japanese lesson” (Stigler & Hiebert, 1999; Yoshida, this volume) and is suited to investigations that use a realistic, practical, or pseudo-practical context seeking to foster strategic competence. The template is derived from the activities suggested by Lovitt and Clarke (1988), entitled *Estimation with fractions*, and *How many people can stand in your classroom*? It provides a frame for those task-types discussed above which do not offer students something obvious to do, and supports teachers who are learning how to use less structured tasks than those they may be used to.

 The name, *Imagined Representations,* relates to the key element of problem solving associated with imagining possibilities. For example:

*The height of the Statue of Liberty is 46.05 metres. How long would you expect the arm to be?*

 Again, the right-hand column in the following is intended to apply to all tasks and lessons of this type. The left-hand column is a description of how this might work for this particular task:

Table 3: Lesson elements for Imagined Representations template

|  |  |
| --- | --- |
| In this example … | Key lesson element |
| The teacher asks the students to record an estimate of the length of the arm on the statue. The estimation engenders interest in the answer.  | After posing and clarifying the problem, the teacher asks the students to record an estimate. |
| After discussion, some groups might suggest comparing the length of the height and the arm on a photograph, others might suggest measuring a sample of people to get a common ratio of height to arm length, and others might suggest using ratios used by artists when representing people. | Students are invited to think about what strategies they might use to calculate an answer, first individually, then brainstorming in a group, and the groups report their strategies to the class. |
| The groups of students are allocated to a particular solution strategy, preferably one they have suggested. Using whatever resources are required, they implement the particular strategy and prepare a report. | Groups choose (or are allocated) a strategy, they implement the strategy to find an answer, and prepare a report. The teacher monitors the work of the groups, ensuring that all students are involved in the strategy implementation, and anticipating groups who might report at the next phase. |
| The students report on their strategy including indicating their estimate of the length of the arm. The teachers can ask questions about the most accurate method, desired levels of accuracy, and the most efficient method. | The teacher leads a review of responses, including attending to issues such as efficiency of a strategy, and appropriateness of the degree of accuracy. Ideally the teacher selects few rather than all groups to report, particularly those that are likely to contribute to the purpose of the activity.  |
| Some similar ratio tasks can be posed that allow students to practice the skills, or prompt for transfer to alternate situations. | Students complete more problems or exercises that consolidate the principles identified in the investigation or prompt transfer to a related context. |
| The teacher emphasises the process for calculating ratios, which is the purpose of posing the task in the first place, as well as the steps necessary to ensure that data collected are accurate. | The teacher summarises the main mathematical ideas addressed in the activity. One key aspect of the teacher’s role is to emphasise the “dimensions of variation” (Marton, 2006) inherent in the range of strategies and modes of communication of solutions that arise. |

 While it is difficult for the teacher to predict what will happen, this particular activity provides a context for introduction of proportional reasoning and probably for application and reinforcement of estimation and measurement of length. It includes a metacognitive element, in that it is contributing to students’ awareness not only of the possibility of multiple appropriate strategies, but also of the usefulness of planning.

 There is student choice of strategy, since they suggest which strategy to use; it is high in potential for prompting communication, in that students would be keen to explain what they found and how they found it; there would be medium risk in that, while the students have some degree of choice, what they are required to do is ambiguous; and the engagement would be through both the potential for choice, and the inherent interesting nature of the task.

 This lesson template is applicable to any practical or realistic task that requires investigation or consideration of strategy by the students, especially where there is a need for students to imagine a representation.

 As we said earlier, teachers need to work through such tasks, perhaps with others, to fully understand the affordances and constraints and hence what kinds of learning might take place. After this, it can be useful to pool teachers’ conjectures about how students might respond, and what to do with particular kinds of response. There is a difference between lessons in which teachers are listening *for* certain anticipated responses, and lessons in which teachers are genuinely listening *to* students, but a role play offering certain responses can at least prepare teachers for some of the possibilities.

Strand 4: Adaptive reasoning

Adaptive reasoning includes the capacity for logical thought, reflection, explanation, and justification. For students to develop this we need classroom tasks which focus on adaptive reasoning, rather than on methods, concepts, problems, fluency and memory. Task types which focus on reasoning could be used as isolated questions, but are probably more appropriately combined and embedded in complex tasks, as elements of overall design. Whereas a concept-focused task might use ‘sameness and variation’ as an intellectual tool to get at particular features of some objects, the same task, but with a reasoning focus, might be about developing the use of ‘sameness and variation’ as a way of acting on *any* mathematics. If the aim is to learn a concept, then the teacher will emphasise the concept; if the aim is to develop reasoning, then the teacher will talk about the purpose of identifying sameness and difference. When the focus is on developing the skills of mathematical enquiry, teachers typically are not interested so much in content knowledge or curriculum coverage, but in the development of being mathematical, a culture of mathematics, within the social world of classrooms. For new teachers, experience of working with such task structures for themselves provides a way to become more articulate about reasoning, and hence to be able to encourage reasoning through language.

 Reasoning can be included as a focus throughout teaching, so that new teachers working in placement schools can fulfil existing expectations of curriculum coverage yet also enrich the existing ways of teaching in the school. Teachers need a repertoire of the specific reasoning skills of mathematics in order to embed them throughout their teaching. such as: asking about sameness and difference (Brown & Coles, 2000); talking about how an example represents a whole class of objects (Mason & Pimm, 1984); asking students to create their own examples (Watson & Mason, 2006); asking students to conjecture about other possibilities (Brown & Walter, 1983/2005); and playing with extreme examples. Cuoco, Goldenberg, and Marks (1996) offer a further list: justifying claims, proving conjectures, seeing logical necessity; analysing answers and methods, visualising, interpreting diagrams, translating between representation, looking for invariants, generalising, seeking similarities in structure, and developing inner sayings about relationships and methods.

An illustrative task for teachers: What sorts of thinking did I do when ….. ?

A suitable task for teachers to focus on adaptive reasoning is to pose situations which will require the teachers to learn new mathematics or at least to engage with known mathematics in new ways. When teachers work together exploring some mathematics which is new to them, they can generate the kinds of language listed above for themselves, although sometimes this has to be ‘seeded’ by the teacher educator naming what they are saying. For example, a teacher might say ‘I tried a few numbers…’ and the teacher educator might name that as ‘exemplifying’; another teacher might say ‘I looked to see what would happen if…’ and the teacher educator might call this ‘controlling variables’. In our experience, the more these language forms become habitual among teachers, and between new teachers and their tutors and mentors, the more likely they are to inform planning and classroom discourse. Without this ‘habituation’ they stay as ‘more to do’ and can be lost beneath the pressures of curriculum coverage.

An illustrative template for a lesson focusing on strategic competence: The What if? template

The *What if?* template is useful for open-ended and mathematically-focused investigative tasks. Opening up tasks can engage students in productive exploration (Christiansen & Walther, 1986), enhance motivation through increasing the students’ sense of control (Middleton, 1995), and encourage pupils to investigate, make decisions, generalise, seek patterns and connections, communicate, discuss, and identify alternatives (Sullivan, 1999). The term “what if” is used to highlight the role of the teacher in prompting the dimensions of variation inherent in such tasks (Brown & Walter, 1983/2005; Marton, 2006)

 The following is an example of a task suitable for this template:

You have a box that needs 1 m of string to tie it up like this. What might be the dimensions of the box?

Assume that 30cm is needed to make the bow.

 The generic lesson template, and possible actions for this example task, are as follows:

Table 4: Lesson elements for the What If? template

|  |  |
| --- | --- |
| In this example … | Key lesson element |
| The teacher might pose the problem, clarifying terms and meanings. The students might be invited to record their answers systematically. The teacher might pose a preliminary problem such as “how might you calculate the length of the string on this box without untying it?”  | Teacher poses and clarifies the purpose and goals of the task. If necessary, the possibility of multiple responses can be discussed. |
| The teacher monitors the work of the students. For students who have difficulty answering the initial question, the teacher might prepare some boxes and loose string for students who might need to tie up a box, or a box covered in a streamer that could be cut into sections. For students who produce one or more correct responses, the teacher might ask them to find as many answers as possible, the smallest box, the nicest box, etc. | Students work individually, initially, with the possibility of some group work. Based on students’ responses to the task, the teacher poses variations. The variations may have been anticipated and planned, or they might be created during the lesson in response to a particular identified need. The variations might be a further challenge for some, with some additional scaffolding for students finding the initial task difficult. |
| Some students with simple strategies might be invited to demonstrate those to the class. Next, the teacher might choose a student who had produced an organised response to summarise their answers to the whole group. Students who have different responses can be invited to contribute their answers. | The teacher leads a discussion of the responses to the initial task. Students, chosen because of their potential to elaborate key mathematical issues, can be invited to report the outcomes of their own additional explorations. |
| Finally, the teacher can summarise the successful strategies and the collective responses. Again this is the key part of the lesson for drawing out the patterns, commonalities, and generalisations. | The teacher finally summarises the main mathematical ideas. |

 A lesson in this template focuses on adaptive reasoning although there are also possibilities for fostering conceptual understanding and strategic competence

 There is student choice in the strategy to be used, in that they can choose the degree of difficulty, and the mode of representation; it is a good task for prompting communication, in that students have the products of their own explorations to contribute; it is low in risk in that students have choice in strategies and the level at which they work; and the engagement is through their choice of strategy and the challenge of the task.

Conclusion

Effective mathematics teaching requires the alignment of a variety of complex factors, and education of the mathematics teacher should ideally prepare teachers for the challenge of addressing each of the factors. Basically we have argued in this chapter that, while there are many others aspects of teaching to learn, both prospective and practising teachers can benefit from the study of the affordances of classroom tasks, and the ways that these can be incorporated into lessons. We adapted the strands of mathematics knowledge as proposed by Kilpatrick et al. (2001) to illustrate the ways that these each contribute to mathematics learning and mathematics teacher learning; we offered illustrative suggestions of tasks for teacher educators to use in each case, and also a possible template for a lesson which used tasks within each of those domains.

 We did not elaborate the ways that such lesson templates could be used in teacher education, since we assume that the development and study of lessons is already part of all initial teacher education and many in-service programs and so take it that our suggestions can be incorporated into those existing programs. We also note that, in cultures where there has not been much attention to the study of lessons, the above suggestions can provide a useful starting point.

REFERENCES

Anderson, J. R., & Schunn, C. D. (2000). Implications of the ACT-R learning theory: No magic bullets. In R. Glaser, (Ed.), Advances in instructional psychology: Educational design and cognitive science (Volume 5), pp. 1-34. Mahwah, NJ: Lawrence Erlbaum Associates

Association of Teachers of Mathematics (1980). Points of Departure 1. Derby: ATM.

Association of Teachers of Mathematics (1982). Points of Departure 2. Derby: ATM

Atkinson, R., Derry, S., Renkl, A., & Wortham, D. (2000). Learning from examples: instructional principles from the worked examples research. Review of Educational Research. 70(2) 181-214.

Baddeley, A.D., & Hitch, G.J. (1974). Working Memory, In G.A. Bower (Ed.), The psychology of learning and motivation: advances in research and theory (Vol. 8, pp. 47-89), New York: Academic Press.

Banwell, C S, Saunders, K. D., & Tahta, D G, (1972). Starting Points.Oxford: Oxford University Press.

Bransford, J. B., Brown, A. L., & Cocking, R. R. (Eds.) (1999). How people learn: Brain, mind, experience, and school. London: Committee on Developments in the Science of Learning, National Research Council.

Brousseau, G. (1997). Theory of didactical situations in mathematics. Dordrecht: Kluwer.

Brown, L., & Coles, A. (2000). Same/different: a ‘natural’ way of learning mathematics. In T. Nakahara and M. Koyama (Eds.) Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education, p2-153-2-160, Hiroshima, Japan.

Brown, S., & Walter, M. (1983/2005). The Art of Problem Posing. Hillsdale, NJ: Lawrence Erlbaum.

Christiansen, B., & Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), Perspectives on Mathematics Education (pp. 243–307). The Netherlands: Reidel.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curriculum. The Journal of Mathematical Behavior, 15(4), 375-403.

Doyle, W. (1986). Classroom organisation and management. In M. C. Wittrock (Ed.), Handbook of research on teaching (pp. 392–431). New York: Macmillan.

Dweck, C. S. (2000). Self theories: Their role in motivation, personality, and development. Philadelphia: Psychology Press.

Fredericks, J. A., Blumfield, P. C., & Paris, A.H. (2004). School engagement: potential of the concept, state of the evidence. Review of Educational Research, 74(1), 59-110.

Freudenthal, H. (1973) Mathematics as an educational task. Dordrecht: Reidel

[Gibson](http://en.wikipedia.org/wiki/James_J._Gibson), J.J. (1977). The Theory of Affordances. In Robert Shaw and John Bransford (eds.) Perceiving, Acting, and Knowing: Towards an ecological psychology (pp. 67-82), Hillsdale , NJ : Erlbaum.

Good, T. L., Grouws, D. A., & Ebmeier, H. (1983).Active mathematics teaching. New York: Longmans.

Hewitt, D. (1994) The Principle of Economy in the Learning and Teaching of Mathematics, Centre for Mathematics Education, Ph.D. – The Open University, 1994. [[download PDF](http://www.education.bham.ac.uk/staff/Hewitt_PhD_thesis.pdf), 725 kb]

Hiebert, J., & Wearne, D. (1997). Instructional tasks, classroom discourse and student learning in second grade arithmetic. American Educational Research Journal, 30(2), 393–425.

Kirkby, D. (1992) Games in the teaching of mathematics. Cambridge: Cambridge University Press.

Lovitt, C., & Clarke, D. (1988). Mathematics curriculum and teaching program: Activity bank – Volumes 1 and 2. Canberra: Curriculum Corporation.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). Adding it up: How children learn mathematics. Washington, DC: National Research Council.

Marton, F. (2006) Sameness and Difference in Transfer. [Journal of the Learning Sciences](http://www.leaonline.com/loi/jls) 15( 4), pp. 499-535.

[Mason, J.H.](http://www.mathematics.open.ac.uk/802570F2005D1138/%28httpPeople%29/1EAAD9A9DE4625D3802570FA00444EC0?OpenDocument&subsection=mathematics), & Johnston-Wilder, S.J. (2006) Designing and Using Mathematical Tasks Tarquin Publications: St Albans.

Mason, J., & Pimm, D. (1984). Generic examples: seeing the general in the particular. Educational Studies in Mathematics, 15. 227-289.

Middleton, J. A. (1995). A study of intrinsic motivation in the mathematics classroom: A personal construct approach. Journal for Research in Mathematics Education, 26(3), 254–279.

Moreno, R. (2006). When worked examples don’t work: Is cognitive load theory at an impasse? Learning and Instruction, 16, 170-181.

Polya, G. (1981). (combined edition) Mathematical discovery. New York: Wiley.

Skemp, R. (1977). Relational understanding and instrumental understanding. Mathematics Teaching, 77, 20-26.

Stein, M.K., Grover, B., & Henningsen, M. (1996). Building students capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. American Educational Research Journal, 33(2), 455-488.

Stigler, J., & Hiebert, J. (1999). The teaching gap: Best ideas from the world’s teachers for improving education in the classroom. New York: The Free Press.

Sullivan, P. (1999). Seeking a rationale for particular classroom tasks and activities. In J.M. Truran & K.N. Truran (eds.) Making the difference. Proceedings of the 21st annual conference of the Mathematics Educational Research Group of Australasia (pp.15-29). Adelaide.

Sullivan, P. (2007). Creating mathematics lessons. In S. Close, D. Corcoran and T. Dooley (Eds) Proceedings of the Second National Conference on Research in Mathematics Education (pp. 30-43). Dublin.

Swan, M. (2006). Collaborative Learning in Mathematics: A Challenge to Our Beliefs and Practices. National Institute of Adult Continuing Education.

Swan, M. (no date). Using percent to increase quantities. Standards Unit trial materials. UK.

Sweller, J. (2006). The worked example effect and human cognition. Learning and Instruction. 16, 165-169.

Vygotsky, L. (1986) (A. Kozulin ed.) Thought and Language. MIT Press: Cambridge Mass.

Watson, A. (2004), Affordances, constraints and attunements in mathematical activity. In O. McNamara & R. Barwell (Eds), Research in Mathematics Education, 6, 23-34.

Watson, A., & Mason, J. (2006) Mathematics as a constructive activity: Learners generating examples. Mahwah, NJ: Erlbaum.

Watson, A & Mason, J.. (forthcoming) Taken-as-Shared: a review of what is known about mathematical tasks in teacher education. Journal of Mathematics Teacher Education 10.

Wood, T. (2005). From alternative epistemologies to practice in education: Rethinking what is means to teach and learn. In L. P. Steffe & J. Gale (Eds.), Constructivism in education (pp. 331-339). Hillsdale, New Jersey: Lawrence Erlbaum Associates, Publishers.

AFFILIATIONS

Anne Watson

Department of Education

Oxford University

Peter Sullivan

Faculty of Education

Monash University

1. A mnemonic, widely-used in the English-speaking world, for recalling that Sine is Opposite over Hypotenuse, and so on. [↑](#footnote-ref-2)