British Congress of Mathematics Education

BCME - 7

In collaboration with the British Society for Research into Learning Mathematics

Volume 30 Number 1

Editors:
Marie Joubert
University of Bristol
Paul Andrews
University of Cambridge

Unlike the normal volumes of informal proceedings of BSRLM day conferences, the research papers for this conference were subject to a refereeing process. We are grateful to the referees for their comments.

These proceedings consist of research reports which were written for the seventh BCME conference.

Published by the British Society for Research into Learning Mathematics.

Individual papers © contributing authors 2010

Other materials © BSRLM 2010

Proceedings of the British Congress for Mathematics Education (BCME) 2010

Volume 30 Number 1, April 2010

Analysis of classroom interaction from the combined view of self-regulating strategies and discourse analysis: What can we learn?

Mohd Faizal Nizam Lee Abdullah and Paola Iannone

University of East Anglia

The purpose of this study is to investigate the relationship between self-regulated learning (SRL) and mathematical discourse. The study involved a group of Year 9 students in the East of England engaged in mathematical tasks. Analysis on the students' interactions was carried out employing two types of analytical tools: Pintrich's (1999) model of self-regulated learning strategies, with particular attention to the rehearsal strategies, and Sfard and Kieran's (2001) discourse analysis framework. The findings show the emergence of key mathematical concepts during the engagement with SRL strategies have positive impact in producing an effective and productive discourse among the group members.

Keywords: Self-regulating strategies, rehearsal strategies, discourse analysis, secondary mathematics.

Introduction

Our study investigates students' SRL strategies while engaging with mathematical problems. Many studies have been carried out concerning mathematical problem solving processes, heuristics, and strategies but there have been few studies examining the effect of SRL strategies such as cognitive learning strategies, metacognitive and self-regulatory strategies, and resource management strategies on problem solving in mathematics (Pintrich 1999).

We also looked at students' interactions while engaging with the problems. Researchers in mathematics education agree "that mathematics can and should, at least partly, be learned through conversation" (Ryve 2004, p. 157). Communication has been observed to be an essential element for mathematics teaching and learning (NCTM, 1989; NCTM, 2000). NCTM (2000) outlines that a learner who has opportunities to engage in mathematical communication including speaking, reading, writing, and listening profits from two different aspects: communicating to learn mathematics and learning to communicate mathematically.

Our investigation linked the students' SRL strategies with their communication. In this particular study, we would like to discuss our preliminary findings on the participants' engagement with one of the component of SRL cognitive learning strategies, the rehearsal strategies and the participants' interactions in our attempt to observe mathematical learning via group problem solving. Hence our research question is formulated as follow:

What can we learn from the combined view of SRL and group discourse?

Theoretical background

SRL strategies: Drawing from the literature on SRL, Pintrich's (1999) conceptual framework allows us to characterise the various components of SRL during the

participants' engagement with a mathematical problem. The components include cognitive learning strategies, metacognitive and self-regulatory strategies, and resource management strategies. The elements of cognitive learning strategies are rehearsal strategies, elaboration strategies, and organisational strategies. The elements of metacognitive and self-regulatory strategies are planning activities, monitoring, and regulation strategies The elements in each component are usually not deployed in a given temporal order and can be used once or more throughout the problem solving process. The third component, resource management strategies are associated to the social interaction of the group which involves the commitment to work collaboratively to solve the problems. In this paper we have decided to focus on the participants' engagement with the rehearsal strategies as an exemplification of the development of the group discourse.

In the language context, rehearsal strategies, (Pintrich 1999) "involve the recitation of item to be learned or the saying of words aloud as one reads a piece of text" (460). Highlighting and underlining text informally is also considered as rehearsal strategy. Pintrich (1999) observes that via rehearsal strategies, students will be able to attend to and select important information from text.

In a group problem solving context, the rehearsal strategies include reading the problem and associate it to the relevant mathematics topic or content. The phrase 'reading the problem' refers to a member in the group reads aloud and others listen or all the members read in silent individually. This can be observed via their actions or utterances during the problem solving process. On the other hand, the phrase 'associate it to the relevant mathematics topic or content' refers to identifying the problem and categorising it to the particular topic or content of mathematics. Evoking prior knowledge that is relevant to the problem is also an element of rehearsal strategies. In addition, the rehearsal strategies can be observed via highlighting and underlining important words or phrases stated in the problem. These activities are ways for learners to take note of information or hints provided in the problem.

Discourse Analysis: Sfard and Kieran (2001) developed a theoretical and methodological framework "which aims at characterising the students' mathematical discourses while they are working in groups" (Ryve 2006 191). This framework, which is also known as communicational approach to cognition provides the platform to examine the efficiency and productivity of mathematical discourses. On the issue of effectiveness of communication, Sfard and Kieran observe that:

> The communication will not be regarded as effective unless, at any given moment, all the participants seem to know what they are talking about and feel confident that all the parties involved refer to the same things when using the same words. (Sfard and Kieran 2001, 51)

In examining the elements of effective and productive mathematical discourses, the framework offers two types of analyses: *focal* analysis and *preoccupational* analysis. On one hand, focal analysis deals with communicative successes or failures with no reasons revealed. On the other hand, preoccupational analysis offers the reasons behind the success or failure of a communication. Sfard and Kieran notice that:

> Focal analysis gives us a detailed picture of the students' conversation on the level of its immediate mathematical contents and makes it possible to assess the effectiveness of communication. This is complemented by preoccupational analysis, which is directed at meta-messages and examines participants'

engagement in the conversation, thus possibly highlighting at least some of the reasons for communication failure. (Sfard and Kieran 2001, 42)

SRL and discourse: SRL strategies are found to be one of the factors in enhancing students' academic achievement (Zimmerman and Martinez-Pons 1986). Wang et al. (1990) show that high achievement learners engaged more in self-regulative activities, such as orientation, planning, monitoring, re-adjustment of strategies, evaluation and reflection. Apart from SRL, mathematical discourse is also vital in the success of mathematical learning: "putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned" (Sfard 2001 13). Unfortunately, literature associating SRL and mathematical discourse together in mathematical learning is currently limited. Based on this, the study will focus on the combination of SRL and mathematical discourse in a problem solving process.

The study

In this paper we present some of the preliminary results of the participants' engagement with the rehearsal strategies (Pintrich 1999) and its influence towards the development of the group discourse (Sfard and Kieran 2001) during the problem solving process. Most importantly, we will observe the emergence of key mathematical concepts and how it contributes to the participants' interactions.

We employed video-recordings as our primary data in order to have a close examination of the students' interactions. As Griffee (2005) noted, video-recording provides an opportunity to reveal things that might go unnoticed. What is more important is that video-recording enables us "to re-visit the aspect of the classroom events and pursue the answers we seek" (Pirie 1996 553).

The study lasted for six months and involved a group of four Year 9 students aged between fourteen and fifteen years old at a comprehensive secondary school in the East of England. Video recordings focused on the group engaging in mathematical tasks set by the teacher (20 – 25 minutes towards the end of the one hour lesson). In addition, observational notes were kept and students' written work was taken into account to complement the video data for a more complete record of the actual situation.

A sequence of seven interacting, non-linear phases of Powell et al. (2003) model was used to analyse the video data. At the early phases, the process of viewing, listening, and describing the video data were carried out. During these processes, vignettes or episodes that were critical and significant to the study were recorded. This was followed by transcribing the critical events or episodes whereby recordings of participants' utterances and actions were fully transcribed in order to capture both what was said and what was done. This was followed by the *coding* phase whereby all critical episodes were analysed employing two different analytical tools: the Pintrich's (1999) model of SRL strategies, and Sfard and Kieran's (2001) discourse analysis framework.

All episodes were analysed in-depth to scrutinise students' engagement with the SRL strategies while working on the mathematical problem. Cognitive learning strategies, metacognitive and self-regulatory strategies were observed during the analysis process. In addition, the resource management strategies were also employed to scrutinise the social interactions among the participants. The focus of this analysis was to observe the SRL strategies used by students engaged in solving a mathematical problem. The episodes were also analysed using discourse analysis to capture the ways in which students interacted with each other. The focal analysis focused on the

coherence of the utterances involving the tripartite foci: *pronounced focus, attended focus, and intended focus*. This was followed by preoccupational analysis employing the interactivity flowchart. It focused on how students communicate between different channels of communication and different level of talks (Kieran 2001).

For the purpose of this paper, the discussion will focus on the role of mathematical discourse in the rehearsal strategies phase. We select the rehearsal strategies as an illustrative example of the data analysis carried out for this project. Finally, we selected the triangle problem as exemplification of a problem solving instance that was manageable within the scope of this paper.
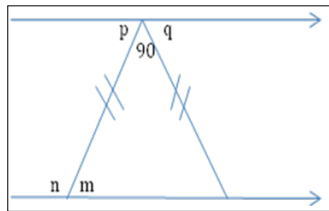
This exercise was set to the students as part of a lesson on triangles and parallel lines carried out in autumn, 2008. The students were given the diagram in Fig. 1 and asked to find the angles *p. q m* and *n*. The content of the lesson was on the properties of triangles and parallel lines including: (1) vertically opposite angles are equal, (2)

Fig 1: The *Triangle* problem

alternate angles are equal, (3) corresponding angles are equal, and (4) supplementary angles add up to 180⁰. In addition, previously, the students were taught about angles in polygon, and lines and angles. The following conversation was recorded:

[1] Megan: *n* and *m* are 90.

[2] Kathy: Are they?...no... oh yeah. (*pause for a moment*) Hah...they look like 90.

[3] Megan: So is *p*

[4] Kathy & Anne: No, they are 45 (*referring to p and q*).

[5] Kathy: Because they are same length.

[6] Megan: That is what does it... they will be the same, *m* and *p* (*pointing at m and p*).

[7] Kathy: Oh..yeah..

[8] Anne: Yes, that is not 90 (*referring to n and m*)...that means *m* is 45.

[9] Kathy: Let start again.

[10] Anne: *m* will be 45.

[11] Kathy: Do *n* first.

[12] Anne: Do this first (*pointing at p and q*). That's 45 each.

[13] Kathy: So *p* and *q* are 45.

[14] Megan: They will be equal, right?

[15] Kathy: Because they got them though (*pointing at the two small lines which mean equal length*).

[16] Anne: *n* is definitely 90 (*pointing at n*).

[17] Kathy: No, it's not.

[18] Megan & Anne: Yes, it is.

[19] Kathy: No, it is not a zig-zag (*referring to the alternate angles between n and 90*)

[20] Megan: Yeah...it has to be that one (*showing the top parallel line with the q-angle*). It has to be like that (*as though she is drawing the z*). So *n* is not 90.

[21] Anne: Alright... so *p, q* are definitely 45... Then that would be 45 (*pointing at m*) and that would be 45 (*drawing an interior angle on the left of the triangle*)... because these are the same length

[22] Kathy: Yeah...

[23] Anne: So *m* is definitely 45...and then so is *p*... and so is *q*... and then *n* is 145...

[24] Kathy: No.

[25] Megan: No. 135.

[26] Anne: Yeah.

Analysis of the data

The participants' interactions can be divided into two parts. In these two parts, we observe that the participants are engaged in the rehearsal strategy. The engagement with this strategy is seen to have a huge impact on the participants' quest to solve the problem. We observe here the emergence of two key mathematical concepts, the 'equal length' concept and the 'alternate angles' concept. At the early stage of the discussion, the participants utilise the 'equal length' concept to discuss the unknown angles, *m, p,* and *q* and later they employ the 'alternate angles' concept to focus on the value of *n*.

From the transcript, we discover that in these two parts the participants evoke their prior knowledge to justify their solution or to object to others' solution. Kathy justifies her answer to utterance [4] stating that "*Because they are the same length*" [5]. Her justification is solely based on the diagram of an isosceles triangle provided which underlines her prior knowledge of properties of lines and angles. At the later stage, Kathy demonstrates that the *n*-angle and 90-angle are not alternate angles, "*No, it is not a zig-zag*" [19] to oppose the value proposed by Anne [16]. Megan supports Kathy's ideas as she responds convincingly assuring her friends and herself that the values for *m* and *p* angles are indeed the same [6]. Based on the concept of 'equal length' and via her demonstration about alternate angles [20] Megan knows which two angles are alternate angles. On the other hand, Anne also demonstrates her prior knowledge in agreeing with the concept 'equal length' but doubts the solution proposed by Megan and proposes that the value of *m* is 45 [8], which is indeed correct. In addition, Anne is satisfied with her friends' justification and agrees upon the 'alternate angles' concept in obtaining the value of *n* [21].

We observe that during the course of finding the solution for the unknown angles, the participants' tripartite foci are centred on the 'equal length' concept and the 'alternate angles' concept. The emergence of these concepts is vital to the discourse as the participants are observed not only justifying their solution using these concepts but most importantly has inspired the group members to focus and talk about the same mathematical object as shown in Fig. 2.

Megan			Kathy		
Pronounced Focus	Attended Focus	Intended Focus	Pronounced Focus	Attended Focus	Intended Focus
[6a] That is what does it	Diagram	Solution for m and p angles	[5] Because they are the same length	Diagram	Solution for p and q angles
[6b] they will be the same	Diagram	Solution for m and p angles			
[20a] Yeah… it has to be that one	Diagram	Solution for n-angle	[19] No, it is not a zig-zag	Diagram	Solution for n-angle

[20b] It has to be like that	Diagram	Solution for n-angle			

Anne		
Pronounced Focus	Attended Focus	Intended Focus
[8] that means m is 45	Diagram	Solution for m
[21] Alright [23] … and then n is 145	Diagram Diagram	Solution for n-angle

Fig. 2: The participants' tripartite foci

Fig. 2 shows the analysis of the participants' tripartite foci. Looking through the table, we observe that the 'pronounced' focus of the participants is centred on the 'equal length' concept and the 'alternate angles' concept. Kathy proposes the concept of 'equal length' to support her solution for p and q angles stating that, *"Because they are the same length"* [5] and employing the 'alternate angles' concept [19] to rule out any possibilities of n equals 90 as proposed by Anne. Agreeing to the 'equal length' concept, Megan insists that her solutions of m and p angles are similar based on the concept ([6a] and [6b]). In another instance, Megan not only accepts the 'alternate angles' concept but also demonstrates the angles involved noting that, *"Yeah… it has to be that one"* [20a] and *"It has to be like that"* [20b]. In a similar tone, Anne uses the 'equal length' concept to determine the value of m [8] and applies the 'alternate angles' concept to confirm the value of n [23].

The second column of Fig. 2 tells us that the participants share their focus of attention as they are observed using the diagram of the triangle as a source of information. From the triangle, the participants infer the two key mathematical concepts that are relevant to the problem. Consequently, the participants' 'intended' focus is to find the value of the unknown angles as required.

To summarise, we observe that the concepts which are evoked from the participants' prior knowledge, play an important role in guiding the participants' foci. Thus, this produces an effective discourse (Sfard and Kieran 2001).
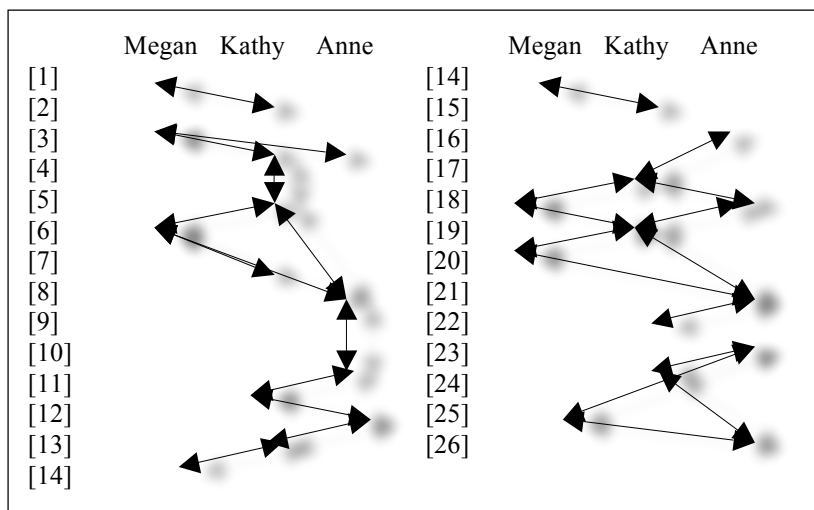
Fig. 3 Interactivity flowchart of the Triangle problem

Fig. 3 shows the participants' interaction via the interactivity flowchart in order to observe whether the discourse is mathematically productive or not. We can see that the participants' interactions are interpersonal utterances of object-level communication (Kieran 2001). This shows that the participants are interacting mathematically with each other with pro-action and reaction utterances. Nevertheless, the flowchart shows a pattern of discourse which is an interesting focus for discussion.

The interactivity flowchart demonstrates that in certain parts of the discourse a formation or a pattern is observed such as a triangular or a rectangular shape

involving the participants' interactions, for instance utterances from [5] to [8] and from [17] to [21]. A deeper investigation determines that the participants are engaged to rehearsal strategy: evoke prior knowledge that is relevant to the problem, with the emergence of two key mathematical concepts during these parts of discourse. Using these concepts in their interactions, the participants' interactions are of pro-action and re-action utterances which suggests that the participants not only propose a solution or an idea but at the same time respond to others. Looking at the pattern formed, the participants' interactions are packed (no open side) with no gaps in between which implies that at this moment the participants are interacting not only mathematically but also developing a meaningful and productive discourse.

Unlike the situation above, there are parts of the discourse that have no formation or pattern. From the flowchart, the occurrence of non-patterned discourse normally happens during the beginning and the ending of the problem solving process. This is demonstrated in three different parts: utterances from [1] to [4], from [8] to [16], and from [23] to [26]. During these periods, the participants' interactions are basically pro-action or re-action utterance and at the same time no engagement of rehearsal strategy is discovered. Thus, the interactions are observed to be loose with a lot of gaps in between which suggests that although the participants are discussing a mathematical problem no meaningful mathematical discourse took place.

Discussion and preliminary findings

In the course of solving the *Triangle* problem, the participants were engaged in the rehearsal strategy: evoke prior knowledge that is relevant to the problem, in order to justify the values for the unknown angles. Consequently, this saw the emergence of two key mathematical concepts: the 'equal length' concept and the 'alternate angles' concept. These concepts were observed to have a positive influence for the group discourse. The participants' capability to monitor their learning via the application of the key concepts had successfully developed an effective and productive discourse. The participants' interactions were focused on the employment of the concepts which encouraged the participants to focus their talk on a similar subject, in this case finding the solutions using the concepts. Thus, this was also observe to influence the participants' exchanges as the utterances were pro-action and re-action. Remarkably, these exchanges created patterns of triangular and rectangular shapes. Consequently, such formation showed that the participants were involved in a meaningful and productive discourse. We observed that utterances not associated to mathematical content such as mathematical concepts had no pattern formation. Thus, this indicates that the discourse is non-productive.

Concluding remarks

This study investigates what mathematical learning that can be achieved during the problem solving process with the participants engaged to the rehearsal strategy and discussed the problem as a group. Two different approaches were implemented in the investigation. Our approach employs the Pintrich's (1999) SRL framework for investigating the strategies used by students, and Sfard and Kieran's (2001) discourse analysis framework to investigate the effectiveness of verbal communication. Our findings suggest that during the participants' engagement with the rehearsal strategy, we observe the emergence of two key mathematical concepts that are not only significant to the problem but also crucial to the development of an effective and productive discourse. The participants' interactions involving these concepts can be

identified clearly via the interactivity flowchart as the utterances formed a pattern of closed triangular and rectangular shapes which suggests a productive discourse. Besides that, applying these concepts encourages the participants to focus and talk on a similar mathematical object. Thus, this produces an effective discourse.

To conclude, the combination of SRL and mathematical discourse offers new insights into the problem solving process. It highlights the relevance of the emergence of key mathematical concepts both as a guide for productive discourse and as an outcome of the rehearsal strategy.

References

Griffee, D. T. 2005. Research tips: Classroom observation data collection, part II. *Journal of Development Education* 29(2): 36 – 37.

Kieran, C. 2001. The mathematical discourse of 13-years-old partnered problem solving and its relation to the mathematics that emerges. *Educational Studies in Mathematics* 46: 187 – 228.

National Council of Teachers of Mathematics. 1989. *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

— — — 2000. *Principles and standards for school mathematics*. Reston, VA: NCTM.

Pintrich, P. R. 1999. The role of motivation in promoting and sustaining self-regulated learning. *International Journal of Educational Research* 31: 459 – 470.

Pirie, S.E.B. 1996. What are the data? An exploration of the use of video-recording as a data gathering tool in the mathematics classroom. In *Proceedings of the 16`th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* ed. E. Jakubowski., D. Watkins., and H. Biske, 553 – 559. Panama City: PME.

Powell, A. B., J.M. Francisco, and C. A. Maher. 2003. An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior* 22: 405 – 435.

Ryve, A. 2004. Can collaborative concept mapping create mathematically productive discourses. *Educational Studies in Mathematics* 26: 157 – 177.

— — — 2006. Making explicit the analysis of students' mathematical discourses – revisiting a newly developed methodological framework. *Educational Studies in Mathematics* 62: 191 – 209.

Sfard, A. 2001. There is more to discourse than meets the ears: looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics* 46: 13 – 57.

Sfard, A., and C. Kieran. 2001. Cognition as communication: rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity* 8(1): 42 – 76.

Wang, M. C., G. D. Haertel, and H. J. Wahlberg. 1990. What influences learning? A content analysis of review literature. *Journal of Educational Research* 84, 30 – 44.

Zimmerman, B. J., and M. Martinez-Pons. 1986. Development of a structured interview for assessing student use of self-regulated learning strategies. *American Educational Research Journal* 23(4): 614 - 628.

The Primary Mathematics Specialists – What Do They Think About Teaching and Learning Mathematics?

Barbara Allen
The Open University

Abstract

One recommendation of the Williams Review (2008) was that there should be a Primary Mathematics Specialist in every primary school within the next ten years. Research with one group of over 100 primary teachers following an Open University course suggests that the teachers who will be following the Primary Mathematics Specialists Programme will face a number of challenges including changes to their beliefs about the importance of subject knowledge and their pedagogic practices.

Introduction

In January 2010 the first cohort of primary teachers start on a programme of study that will result in them becoming Primary Mathematics Specialists. This national programme was one of the recommendations in the Williams Review (2008). These teachers are expected to emerge from the programme of study with "deep mathematical subject and pedagogical knowledge" (Williams 2008, 7). The review summarises the aims of the programme as including:

- subject knowledge in all aspects of the mathematics curriculum;
- solving problems and thinking mathematically;
- mathematics pedagogy; and
- working with colleagues to develop practice. (Williams 2008, 18).

The Williams Review criticised the National Strategies and Local Authorities for becoming more general in the approach to CPD with 'subject speciality becoming de-emphasised' (Williams 2008, para. 54). This new programme is an attempt to redress this balance by putting mathematics at the centre of teachers' professional development and is being led by universities in association with local authorities.

The work of Ernest (1989) suggests that in order for teachers to make a shift to problem solving in mathematics lessons they need to change their beliefs about mathematics, its teaching and its learning. Ernest describes three views of mathematics; problem solving, Platonist and instrumentalist. The problem solving view includes teachers who see mathematics as dynamic with content continually growing. A teacher with a Platonist view see mathematics as a static body of knowledge. The instrumentalist view of mathematics is as a collection of facts, skills and rules which often have no connections.

The significance of these views to teaching is that a teacher with a problem solving view is more likely to accept pupils' own methods of solving a problem. Whereas someone with a Platonist or instrumentalist perspective would expect pupils to find the single 'correct' answer to a problem. It is possible that the emphasis on numeracy (rather than mathematics) over the last ten years will have encouraged teachers with an instrumentalist perspective to be dominant in primary classrooms.

This paper is concerned with the beliefs of primary teachers of mathematics. In particular, it describes the way that a group of primary teachers view mathematics at the beginning of a CPD course.

Open University CPD provision

In recent years, the Open University's Centre for Mathematics Education has been running a mathematics course for primary teachers. The course, *Developing Mathematical Thinking*, which predated the Williams Review, is underpinned by a philosophy that professional development for teachers needs three interlinking strands:

- subject knowledge;
- subject related pedagogy; and
- embedding in practice.

These three strands are influenced by the work of Shulman (1986) and his notion of Pedagogical Content Knowledge (PCK) which is at the intersection of mathematics content knowledge and pedagogy. The three elements of the OU course are interwoven into the mathematical tasks that the teachers work on collaboratively.

The Open University course has proved to be highly successful in terms of supporting teachers as they improve their subject knowledge and develop their pedagogic practices. Anecdotal evidence from the teachers also suggests an increase in the attainment of their pupils.

As part of the *Developing Mathematical Thinking* course the Open University has carried out some research with these primary teachers in order to find out their beliefs about mathematics, its learning and teaching.

Method and analysis

Data were collected near the start of the course in a written questionnaire. There were two forms of items on the questionnaire: quantitative and qualitative. There were 15 quantitative items with multiple responses based on a Likert scale. For example, Item 1/9 *When I am successful in mathematics, it is because* .. had four possible responses on a five-point Likert scale: agree mostly; agree a bit; no feeling; disagree a bit; disagree mostly. These items were developed by Barbara Allen and John Mason (The Open University, 1994) as a means for OU students to profile their own beliefs about, and attitudes to, learning and teaching mathematics. The possible responses to the items relating to beliefs were designed to extract the teachers' perspectives in line with Ernest's model: problem solving, Platonist and instrumentalist. The wording of the items that were concerned with attitudes had been influenced by previous OU work with teachers and learners of mathematics (Ruffell, Mason and Allen 1998; Allen and Shiu, 1997).

The first four quantitative items asked about when the teachers felt they were successful at mathematics and how they knew when they understood a topic. The next four were the qualitative items and included statements about the teachers' perceptions of mathematics and its learning. The final eleven quantitative items were designed to highlight the teaching of mathematics including the role of the teacher, how learning happens and perceptions of themselves as effective teachers. This paper reports on four of the quantitative items and one of the qualitative statements.

The analysis of the quantitative items was a simple tally and percentage of the total responses. Not all of the teachers completed all the items. The qualitative items were sorted into general categories in line with grounded theory approaches.

The teachers

The cohort reported here is from one local authority with over 100 teachers. The vast majority of the teachers were female (84%) and most of them were in their twenties (42%) or thirties (24%). Their highest educational qualifications were mainly at first degree level with 6% having a Masters degree – not necessarily in education. The highest mathematics qualification was generally at GCSE (64%) or O level (29%) with 7% having an A level.

We do not have information about why the teachers attended the course: for example, whether they were volunteers or were selected by their Headteacher.

Quantitative Findings

Data from four quantitative items are reported here and are concerned with the teachers' own experiences of learning and teaching mathematics.

Teachers' view of success in mathematics

We were interested to find out how the teachers viewed their own experiences of learning mathematics. For some this would have been many years ago but for many it was relatively recent – within 10 years.

Item 1/9	When I am successful in mathematics, it is because …	
%	Agree mostly	Agree a bit
I persevere.	56	48
I am clever.	6	39
I follow an example.	45	63
I am fortunate	3	20

Fig 1: Teachers own experiences of learning mathematics.

The teachers were universal in their agreement that they were successful at mathematics if they persevered. But 39% thought that they were also successful because they were clever. Does this imply that some of the teachers thought that only 'clever' children could do mathematics? If so, how are these clever children identified? How do the teachers work with children who are not clever and do they assume they can not be expected to be successful at mathematics? And what do the teachers mean by clever? Are the clever children the ones who get the answers correct?

Most of the teachers also thought they were successful if they followed an example. This suggests that their experiences of learning mathematics were in a 'traditional' setting where the teacher demonstrates a mathematics technique to the class who then complete a series of similar questions. Does this mean that these teachers teach in a similar way? If this has been a successful learning experience for them then one might assume that they believe that this traditional form of teaching would be effective in current primary classrooms. This suggests that a majority of the teachers may have an instrumentalist perspective on mathematics.

Some additional information came from item 8/11 when the teachers were asked to what they attributed their pupils' success.

Item 8/11	When teaching I assume that pupils' success is primarily due to ...	
%	Agree mostly	Agree a bit
their hard work.	35	61
luck	0	9
their natural ability.	14	73
them being set suitable work.	74	27
good teaching.	75	29

Fig 2: Teachers' attributions of pupils' success

Most of the teachers appeared to attribute some of their pupils' success to hard work and natural ability. But they were more strongly in agreement that success was attributable to them being given the appropriate work with quality teaching. It appeared, then, that the teachers felt that their pupils' success depended mainly on their skill as an effective teacher.

There seemed to be conflicting information here. Whereas the teachers attributed their own success in mathematics to perseverance (a personal quality) and following an example (a personal strategy) it is not these personal qualities or strategies they felt made pupils successful. This appears to be evidence for an emphasis on teaching rather than learning. The Primary Mathematics Specialist Programme includes an understanding of the ways people learn mathematics which suggests it is possible that this shift from teaching to learning could be a challenge for some teachers.

Teachers' view of teaching mathematics

The teachers were also asked about their expertise as a teacher and how they approached their teaching. These two items produced responses that contradicted those described above. The first concerned the teachers' beliefs about learning mathematics and how this impacted on their teaching.

Item 10/11	My approach to teaching is mainly based on a firm belief that ….	
%	Agree mostly	Agree a bit
everyone can learn maths if only they'd work at it.	15	47
some people are naturally good at maths and some are not.	25	57
I can accelerate learning by stimulating and challenging pupils.	76	25
my job is to get pupils through the assessments and SATs.	7	33
everyone can learn maths if I support and encourage them.	71	31

Fig 3: Approach to teaching

Item 8/11 the teachers attributed their pupils' success in part to hard work (35% and 61%) and natural ability (14% and 73%). Item 10/11 gave a different perspective with the majority of teachers feeling that some people are naturally good at mathematics (25% and 57%) but also that anyone can learn if they have the support

of a teacher (71% and 31%). This seems to suggest that the teachers feel that although some pupils have natural ability in mathematics anyone can learn mathematics provided they work hard and have effective teaching.

It is, of course, possible that the teachers were giving the answers that they imagined the researchers wanted to see. It is also possible that the teachers have never before been asked for their perspective on teaching and learning mathematics and therefore have not thought through possible answers.

We also do not know how the teachers interpreted the word 'maths'. Do they see maths as just number or numeracy? Or do they see maths as a larger subject?

If these teachers believe that their teaching is a major influence on whether or not pupils learn then it suggests that they will be supportive of the Primary Specialist Programme which aims to improve pedagogical subject knowledge.

An effective teacher of mathematics

It seems from previous answers that the teachers felt that one of the keys to success was quality teaching but what did they think made an effective teacher? Item 7/11 was designed to establish their opinions about how they could be more effective as a teacher of mathematics.

Item 7/11	I think I would be a more effective teacher if only ….	
%	Agree mostly	Agree a bit
I had more preparation time.	57	40
pupils were motivated to learn.	26	33
I had more support.	23	42
I could spend more time on professional development.	23	51
I knew more mathematics.	12	46

Fig 5: More effective teacher

It appears that the teachers believe that they would be more effective teachers if they had more preparation time. We do not have any information on how much preparation time they currently get or would want. One feature of the Primary Specialist Programme is Lesson Study which will involve teachers planning, teaching and reflecting on lessons together. This is going to involve a change in the way many teachers plan a programme of work for their pupils.

Time spent on professional development appears to be important to the teachers which suggests that they might be supportive of the Primary Mathematics Specialist programme.

The teachers do not see increased subject knowledge as the highest priority. This is of some concern since one of the main thrusts of the Primary Specialist Programme is the raising of teachers mathematics knowledge. This could be a concern to the university providers who will be required to work on mathematics with the teachers and increase their subject knowledge.

Although the quantitative date gave some information about the way the teachers viewed the teaching and learning of mathematics, it was the qualitative data that added some detail.

Qualitative findings

The open-ended item being reported here asked the teachers to complete the following sentence.

Mathematics is ... ...

It was intended that the responses be analysed in terms of Ernest's three views of mathematics; problem solving, Platonist and instrumentalist. On reading the responses it was clear that they could not be sorted into these three groups, instead they fell into 4 different categories that were named: affective (38%); real world (26%); subject content (24%); and mathematical relationships (12%). Clearly some of these categories have links with Ernest's model and these will be discussed later.

Affective responses

A large number of the affective responses were single words like; creative, challenging, hard, great, enjoyable, daunting. A few expanded on the single word with

Mathematics is fun when I get it!

Mathematics is fun when it is understood.

Mathematics is good when you understand it.

Mathematics is difficult when I am expected to provide answers to others.

The longer responses in this category gave more information.

Mathematics is finding a way which works for you and working with it to succeed.

Mathematics is fun when I have success and frustrating when I fail.

Mathematics is taxing because it makes me think and takes me out of my comfort zone.

Mathematics is very challenging for me. I have struggled with maths since being in school and that puts up barriers against extending my mathematical understanding.

Mathematics is motivating because there are answers to be found out and the possibility of getting it right which feels good.

Some of the teachers seemed to feel that understanding mathematics was important while others linked feelings of success with getting the answers right. These affective responses were not part of Ernest's model but it is possible that getting the answers right indicates an instrumentalist view. These responses do not really give any indication of how the teachers view mathematics. It seems that the word mathematics engendered only an emotional response for 38% of the teachers.

Real world

The 26% of responses in this category split into three different sub-groups: those that referred to mathematics being everywhere; those that related to needing mathematics to understand the real world; and some that regarded mathematics as a skill.

The responses that were concerned with mathematics being everywhere generally consisted of single word responses; essential, everything, everywhere, a subject.

A number of teachers made comment about needing mathematics in order to make sense of the real world.

Mathematics is making sense of the world through generalising patterns to solve problems.

Mathematics is using numbers to explain the world.

Mathematics is learning skills, knowledge and concepts to help you undertake tasks in the real world.

Mathematics is an important life skill.

These responses suggest an awareness of the uses of mathematics outside the classroom. They do not however give any indication that the teachers can see connections between the topics in mathematics. The teachers giving real life responses did not seem to be displaying an instrumentalist view of mathematics.

Subject content

The responses in this category mentioned the content of mathematics.

Mathematics is working with numbers and patterns.

Mathematics is the knowledge, understanding and application of number, shape and concepts to do with these.

Mathematics is a combination of facts and figures.

Mathematics is an understanding of number.

Mathematics is the ability to calculate values.

Mathematics is problem solving, number concepts, patterns, collecting data, understanding measures.

Some of the statements overlapped with those in previous categories. These were placed in the subject content group because that was the overriding issue.

Mathematics is learning rules, patterns and examples about numbers, shape, space, measures and using them to solve problems that will help in our everyday lives.

The responses in this category suggest that these teachers see mathematics consisting only of school mathematics. Only a few made links to the use of school mathematics in everyday life. Schoenfeld (1988) criticised the school curriculum for only introducing learners to a subset of mathematics, namely school mathematics. It seems from these findings that some of the teachers were only aware of this subset of mathematics. The teachers giving a subject content response appear to have an instrumentalist view of mathematics but it is not known if they also display signs of a Platonist view.

Mathematical relationships

The 12% of responses in this category talked about making links between mathematical concepts and recognising relationships. Some of the responses recognised links between areas of school mathematics while others were concerned with the big picture of mathematics.

Mathematics is making connections between different elements of number, shape and space, algebra etc. to work out a given problem.

Mathematics is making links between many ideas of number and shape.

Mathematics is spotting patterns and solving problems.

Mathematics is understanding number patterns and their relationships.

> **Mathematics is** making connections.
>
> **Mathematics is** understanding concepts and applying them to different contexts.

For many mathematics educators, mathematical relationships are at the core of the subject. Although most of the teachers' comments in this category contained the word 'link' or 'relationship', a number of them still only cited these as existing within the subset of school mathematics. The teachers giving these responses appear to have more of a problem solving view of mathematics.

What next?

Ernest's (1989) model appears to suggest that the teachers with a problem solving view of mathematics are the most effective. There is little evidence from this cohort of primary teachers that they view mathematics in terms of problem solving. It seems that in order for them to become effective teachers of mathematics they will need to shift their view of mathematics from Platonist or instrumentalist to one of problem solving.

By the time that BMCE starts in April 2010, the first cohort of teachers will have embarked on this programme. We will have greater information about who the teachers are and their own knowledge of mathematics. The findings in this paper suggest that for primary teachers, mathematics engenders an affective response rather than a clear view of how they see mathematics.

The challenge to the universities and local authorities involved in this programme is to facilitate teachers to change the way they view both the teaching and learning of mathematics in order to raise the attainment of their pupils. This paper suggests that the providers of the programme will also need to challenge the way the teachers view mathematics as a subject.

References

The Open University. 1994 ME825 *Researching Mathematics Learning*, Milton Keynes, The Open University.

Allen, B. and C. Shiu. 1997 Learning Mathematics is Like … Views of tutors and students beginning a distance-taught undergraduate course, *British Society for Research into Learning Mathematics, Conference Proceedings*, University of Nottingham, 1997: 8-11.

Ernest, P. 1989. The Impact of .Beliefs on the Teaching of Mathematics, in *Mathematics Teaching: The State of the Art* , ed. P. Ernest. London, Falmer Press, 249-254.

Ruffell, M., J.H Mason, and B.M. Allen 1998 Studying Attitude to Mathematics**,.** *Educational Studies in Mathematics* 35(1): 1-18.

Schoenfeld, A.H. 1988. When Good Teaching Leads to Bad Results: The Disasters of "Well-Taught"' Mathematics Courses. *Education Psychologist*, 23(2): 145-166.

Shulman, L.S. 1986. Knowledge and Teaching: Foundations of the New Reform. *Harvard Educational Review*, 57 (1): 1-21.

Williams, P. 2008. *Independent Review of Mathematics Teaching in Early Years Settings and Primary Schools*, London: DCSF.

Acknowledging the cultural dimension in research into mathematics teaching and learning

Paul Andrews

University of Cambridge

In this review paper I make a plea for those involved in research into the teaching and learning of mathematics to acknowledge that however their work is framed, it will be located in a culture, not always visible to their readers, that needs making explicit. In order to do this I first examine three key models of culture and their significance for education. Secondly I consider various models of curriculum and ways in which school mathematics is presented. Finally, I summarise some recent comparative research in mathematics teaching showing substantial variation in the ways in which teachers manage the presentation of mathematics in their classrooms. In so doing a plea to researchers is framed: Culture permeates all aspects of educational endeavour and should be acknowledged more explicitly than it is.

Introduction

The purpose of this review paper is to invite researchers in mathematics education to acknowledge, in the theoretical framing and reporting of their work, the manner in which culture pervades all aspects of education in general and mathematics education in particular. Consequently, without wishing to over-assert what I feel is an important point, educational research should not be conducted independently of an acknowledgement of such influences. Having said that, culture is a difficult construct to define, although the consensus seems to be that it is, essentially, a collective manifestation of psychological conditioning and can be construed as a societal analogy to individual memory (Triandis and Suh 2002). Culture embodies the "implicitly or explicitly shared abstract ideas about what is good, right, and desirable in a society" (Schwartz 1999: 25). Culture includes those beliefs, artefacts and practices that history has shown to be effective for the maintenance of a society and its future generations (Hofstede 1980, Triandis and Suh 2002). In summary, cultures are social, historical and behavioural constructions that reflect the "collective mental programming" of their people (Hofstede 1980: 43). Through the educational transmission of their embedded traditions, values, beliefs, knowledge and skills, they ensure their continuing replication. In the following I consider three features of culture and show, in various ways, how they impact on the opportunities to learn students receive. These are various models of culture and how they shape educational systems, models of curricula and their culturally derived expectations, and a selection from the mathematics education literature highlighting the extent to which culture influences classroom practice and related constructs.

Models of culture

A number of researchers have attempted to categorise the ways in which cultures differ, and it is to them that we first turn. Hofstede's well known study initially identified four dimensions of culture. The first, *power distance*, concerns the extent to

which followers accept being led. "A society's power distance level is bred in its families through the extent to which its children are socialised toward obedience or toward initiative" (Hofstede and McCrae (2004: 62). The second, *uncertainty avoidance*, relates to the extent to which "a culture programs its members to feel either uncomfortable or comfortable in unstructured situations" (62). The third is *individualism*, or the "degree to which individuals are integrated into groups". Lastly there are *masculine* as opposed to *feminine* cultures. *Masculine* cultures "strive for maximal distinction between what men are expected to do and what women are expected to do". With respect to education, Hofstede's has shown how his four dimensions of culture can predictably explain variation in educational practice, not least through their impact on differences in the social positions of teachers and students, curriculum relevance, cognitive expectations, and differences in the patterns of participant interactions (Hofstede 1986). Others have exploited Hofstede's dimensions in the framing and interpretation of their work. In a study of 43 countries concerning class size, Cheung and Chan (2008), for example, found that cultural norms associated with power distance and collectivism had more significant impact on decisions concerning class size than economic factors.

Triandis (2001) has proposed an eleven dimensional model of culture, which, by way of brevity, is presented without discussion. Cultures differ in their *complexity*; for example, cities are more complex than villages. Cultures differ in their tightness; *loose* societies tend to be more tolerant of diversity and non-conformist than *tight* cultures. Cultures differ in their perspectives on the individual and collective. For example, tight and simple cultures tend to the collectivist. They differ in their *verticality* and *horizontality*, where the former are more accepting of hierarchical differences than the latter. There are *active* and *passive* cultures. In the former people try to mould their environment to suit them and tend to be more competitive. *Universalist* cultures differ from *particularist* cultures in that any suitably qualified person is eligible for a job. In *diffuse* cultures people are judged holistically while in *specific* cultures they are not. *Ascriptive* cultures judge according to an individual's attributes while *achievement* cultures judge them on their achievement. In *instrumentalist* cultures social activity is subordinated to work, while the opposite is found in *expressive* cultures. Cultures may be *emotionally expression* or *emotionally suppressive*. Lastly, the extent to which norms concerning *physical contact* varies. As with Hofstede's work, such characterisations help us understand the role of culture in framing humans' day-to-day actions and decision making in general and the conduct of education in particular.

With particular regard to education, Schwartz (1999), examined elementary teachers' and undergraduate students' education-related values and, drawing on data from more than 40 countries, identified seven dimensions of culture including conservatism (embeddedness), hierarchy, mastery, autonomy, egalitarian commitment (egalitarianism), and harmony. Thus, with respect to culture and education, for example, a conservative culture which emphasises the "maintenance of the status quo, propriety, and restraint of actions or inclinations that might disrupt the solidary group or the traditional order" would structure educational opportunities very differently from an autonomous culture in which an individual finds "meaning in his or her own uniqueness, who seeks to express his or her own internal attributes (preferences, traits, feelings, motives) and is encouraged to do so" (Schwartz 1999: 24). Importantly, not only for justifying his choice of research subjects but also for highlighting the significance of culture in this discussion, Schwartz (1999:) writes that teachers were chosen for his study because they "play an explicit role in value socialisation", are likely to be "key carriers of culture, and... reflect the mid-range of prevailing value

priorities in most societies".

Culture and curricula

The curriculum, according to the second international mathematics study, comprises intended, implemented and attained forms. However, what is frequently missing from such discussions is the extent to which historical and cultural forces have shaped the development of the curricula, in all its forms, of different countries. A number of researchers have attempted to categorise the manner in which curricula come into being. One example can be found in the work of Holmes and McLean (1989), who summarise four curricular traditions. The first, Essentialism, is linked primarily to English education and refers to the liberal arts curriculum of the late mediaeval English public[1] school, its replication in early models of public education, and its rejection of science and engineering as subjects studied by gentlemen. Cummings (1999: 423) describes the public school as "a boarding school with many rules, mandatory chapel, an emphasis on mind and body, which included daily athletics,... a strong classical thrust in the curriculum, and so forth". The second, Encyclopaedism, is generally linked to post-revolutionary France and draws on Enlightenment principles that education should include all human knowledge, be accessible to all and free from superstition (Cummings 1999). The third, Polytechnicalism, is linked to the Soviet Union and its satellites and incorporates notions of vocationalism into an essentially encyclopaedic model of knowledge located in a socialist moral philosophy (cummings 1999). Lastly, Pragmatism, is linked to the United States and addresses the acquisition of the knowledge and skills necessary for tackling real world problems of a participative democratic. Inevitably, despite their crudity and blurred edges, such models help explain why, for example, why French teachers present an intellectually sophisticated mathematics in comparison to the intellectually simplified mathematics presented by their English colleagues (Jennings and Dunne 1996).

In similar vein, Cummings (1999) writes that all curricula are based on cultural constructions of the ideal person, which, in the light of the discussions above, vary considerably from one culture to another. His analyses led him to conclude that most modern educational systems are derivatives of the traditions of six *core* curricula – the Prussian, Russian, French, English, Japanese and United States. Drawing on a range of factors, he offers a compelling account of not only the development of these six traditions but also the ways in which they have influenced curricula world-wide. In respect of three of these six core nations, Pepin (1999) has argued that the English tradition, with an emphasis on personal morality, has had an anti-rational humanist core privileging intuitive knowledge over the systematic construction of knowledge. With regard to the French tradition she writes that the post-revolutionary principles of *égalité* and *laïcité* facilitated the removal of social inequalities through a common broad curriculum and the expectation, unlike in England, that pastoral and moral issues will be addressed by the family. Lastly, the German humanistic tradition,

---

[1] As BCME readers will know, the English public school is an elite, usually very old, independent school. They employ competitive entrance examinations and are frequently so popular among the wealthy classes that parents may have to register their desire for their child to attend shortly after the child's birth. They are steeped in tradition and continue, as they have for many centuries, to furnish the higher ranks of the English civil service, the clergy, the judiciary, around half of all students at the leading universities of Cambridge and Oxford, and, of course, a disproportionately high number of members of parliament. Despite the name, they should not be confused with the public school systems of other countries, which, in England, is known as the maintained sector.

drawing on notions of *Bildung*[1], incorporates both encyclopaedic rationalism as well as moralism in its unified promotion of academic and moral education. Such views lead to the equal valuing of academic and practical knowledge as found in the tripartite structure of German schools.

Of course, the extent to which such underlying principles and traditions are manifested in the written, or intended, presentation of curriculum mathematics may vary from one country to another. By way of illustration, three European perspectives on the teaching of linear equations at the lower secondary level are presented below. Choice, in this respect, was constrained by the availability of curricula in English, while the topic was determined by other work on which I am currently engaged. They are presented alphabetically and, in respect of linear equations, verbatim.

The Finnish national curriculum[2] for grades 6-9 asserts that students, by the end of grade 8, "will know how to… solve a first degree equation".

The Flemish mathematics curriculum[3], expects students in the first grade of secondary education to "solve equations of the first grade with one unknown and simple problems which can be converted to such an equations". During the second grade they will "solve equations of the first and second degree in one unknown and problems which can be converted into such equations".

The Hungarian curriculum[4] for grades 5-8 (upper primary) writes that in year 5 students should "solve simple equations of the first degree by deduction, breaking down, checking by substitution along with simple problems expressed verbally". In year 6 they should "solve simple equations of the first degree and one variable with freely selected method". By year 7 they should "solve simple equations of the first degree by deduction and the balance principle. Interpret texts and solve verbally expressed problems. Solve equations of the first degree and one variable by the graphical method". Lastly, by year 8 students should "solve deductively equations of the first degree in relation to the base set and solution set. Analyse texts and translate them into the language of mathematics. Solve verbally expressed mathematical problems"

In these three examples can be seen very different perspectives on a core topic of the lower secondary curriculum. The Finnish specification is loose and offers little in terms of how content should be interpreted and presented at any specified moment in the four year cycle. The Flemish appears more tightly specified although the shift from one year to the next, in respect of linear equations, seems vague. The main difference lies in the explicit expectation of problems to be translated into equations for solving. Lastly, the Hungarian curriculum offers a tightly specified progression over a four year period with methods and problem solving, including word problems, increasingly exploited. While not wishing to over-speculate, it is interesting to compare such specifications with three of Hofstede's (1986) dimensions. When compared with other developed European nations, he presents Finland as low power distance, low uncertainty avoidance and a strong tendency towards the feminine. This contrasts with Belgium (not just Flanders) as not only substantially higher than Finland on power distance and uncertainty avoidance but also among the most

---

[1] According to Prange (2004) Bildung is a broad concept that does not translate easily, but is "something noble and undeniably good... Bildung is much better than mere education, or Erziehung, to give the German word. It is associated with liberty and human dignity, whereas education is associated with teaching skills and morals" (Prange 2004: 502).

[2] See http://www.oph.fi/english/publications/2009/national_core_curricula
[3] See http://www.ond.vlaanderen.be/dvo/english/
[4] See http://www.okm.gov.hu/letolt/nemzet/kerettanterv36.doc

masculine of the developed European nations. Unfortunately, Hungary is not represented in Hofstede's analyses as they derived from data collected before the countries of Eastern and Central Europe joined the capitalist world. However, there is some evidence here of the ways in which cultural norms and characteristic patterns of social behaviour find a voice in the curricular presentation of mathematics.

Culture and mathematics teaching

As indicated above, teachers are proxies for an educational system's values and there is growing evidence that mathematics teachers in one country behave in ways that identify them more closely with teachers in their own country than teachers elsewhere. Much of this research draws on the perception that "teaching and learning are cultural activities (which)... often have a routineness about them that ensures a degree of consistency and predictability. Lessons are the daily routine of teaching and learning and are often organised in a certain way that is commonly accepted in each culture" (Kawanaka, 1999: 91). This sense of routine predictability has been variously described as the *traditions of classroom mathematics* (Cobb et al, 1992), the *cultural script* (Stigler and Hiebert, 1999), *lesson signatures* (Hiebert et al, 2003) or the *characteristic pedagogical flow* of a lesson (Schmidt et al, 1996). The latter embodies the pedagogical strategies which, through repeated enactment, are typical of a country's lessons, routine, and beneath the consciousness of most teachers (Cogan and Schmidt 1999). Explanations for such patterns draw on beliefs that cultures "shape the classroom processes and teaching practices within countries, as well as how students, parents and teachers perceive them" (Knipping 2003: 282), to the extent that many of the processes of teaching are so "deep in the background of the schooling process ... so taken-for-granted… as to be beneath mention" (Hufton and Elliott, 2000: 117).

In this regard, there is a growing body of research highlighting substantial national variation not only in the ways in which teachers act out their roles but also in the resources available to them. For example Kaiser et al. (2006) have offered persuasive summaries of the distinguishing characteristics of English, French, German and Japanese mathematics teaching, particularly in respect of proof and the structural properties of mathematics. The two TIMSS video studies have examined a range of teacher practices in Australia, the Czech Republic, Hong Kong SAR, the Netherlands, Switzerland, the United States and Japan. Huegener et al (2009) have focused on differences in the ways teachers present the theorem of Pythagoras in Germany and Switzerland, while Santagata (2005) has highlighted substantial differences in the ways in teachers handle students' mathematical errors in Italy and the US. Campbell and Kyriakides (2000) and Haggarty and Pepin (2002) have shown how school texts reflect differences in systemic expectations and traditions. An et al. 2004) have identified culturally located differences in teachers' mathematical content knowledge, while Correa et al. (2008) have done the same for teachers' mathematics-related beliefs. Finally, although it is acknowledged that the scope of this paper presents an extensive review, it is important to acknowledge the Learner's Perspective Study. In this respect, Clarke and his many colleagues have contributed much to our collective understanding of differences in the ways in which mathematics teachers around the world construe their roles (Clarke et al 2006a, 2006b).

With regard to my own and my colleagues' work, we have examined the ways in which teachers present mathematics to students in the age range 10-14 in England, Flanders, Hungary and Spain. The episodes of videotaped lessons were coded against a framework developed iteratively and collaboratively over the course of a year

(Andrews 2007a), where an episode was that part of a lesson in which the teacher's observable didactic intention remained constant. In terms of teachers' observable learning objectives, the project found (Andrews 2009a) that teachers in all four countries privileged the development of both conceptual procedural knowledge in high and comparable proportions. The major variation lay in the other outcomes. For example, Hungarian teachers placed a higher emphasis on the structural properties (links within and between topics), mathematical efficiency (comparing solutions strategies for elegance and efficiency), problem solving and reasoning than elsewhere. In similar vein, English teachers were rarely seen to encourage structural links or efficiency. Such differences reflect not only differing curricular expectations but long standing mathematics teaching norms. The same study (Andrews 2009b) found, in relation to the observable didactic strategies employed by teachers, that teachers explain regularly and in comparable proportions irrespective of country. However, all other strategies distinguished between the didactical practices of project teachers. For example, Flemish teachers were exploited explicit motivational strategies in smaller proportions than elsewhere, while Spanish teachers employed them in over half of all observed episodes. English teachers very rarely questioned (used higher order questions) while Hungarian teachers did so constantly. Spanish teacher coached - offered hints and suggestions to facilitate their students' successful completion of given tasks - in more than three quarters of observed episodes, while teachers elsewhere did so in equal and significantly smaller proportions. Hungarian teachers invited students to share publicly their solution strategies in almost every episode while teachers elsewhere did so consistently at around the 60 per cent level.

Of course, cultural emphases do not end with learning outcomes and didactic strategies. In a comparison of Hungarian and English mathematics teachers' beliefs Andrews and Hatch (2000) found that while English teachers valued the systematic decorating of their classrooms with examples of students' work or mathematical posters, such practices were alien to Hungarian teachers who tended to work within classrooms with, essentially, bare walls. In a second study Andrews (2007b) found English teachers espousing collective beliefs about the value of school mathematics lying in its applicability to a world beyond school, while Hungarian teachers articulated a collective belief whereby the value of mathematics lay within mathematics itself. In short, the ways in which teachers conceptualise and present mathematics to their learners, the environments they create and their beliefs as to the nature and importance of the subject highlight well that mathematics teaching, in all aspects, is a cultural activity that differs significantly from one country to another.

Final appeal

In conclusion, my appeal to colleagues researching any aspect of mathematics teaching and learning is that they make explicit in the reporting of their work the cultural context in which it was undertaken. Too frequently research is reported with no indication, other than the authors' designation, of a study's location. Moreover, and I know I am equally to blame in this regard, writers assume when synthesising the literature, that generalities derived from a study undertaken in one cultural context are generalisable to another. That is, when constructing the theoretical frames within which we conduct our research, too frequently we ignore the cultural implications embedded in the studies we analyse. Such assumptions, that literature can be synthesised independently of context, may lead to poorly designed and reported studies.

References

An, S., G. Kulm, and J. Wu. 2004. The Pedagogical Content Knowledge of Middle School Mathematics Teachers in China and the U.S. *Journal of Mathematics Teacher Education* 7 (2): 145-72.

Andrews, P. 2007a. Negotiating Meaning in Cross-National Studies of Mathematics Teaching: Kissing Frogs to Find Princes. *Comparative Education* 43 (4): 489-509.

———2007b. The Curricular Importance of Mathematics: A Comparison of English and Hungarian Teachers' Espoused Beliefs. *Journal of Curriculum Studies* 39 (3): 317-338.

———2009a. Comparative Studies of Mathematics Teachers' Observable Learning Objectives: Validating Low Inference Codes. *Educational Studies in Mathematics* 71, (2): 97-122.

———2009b: Mathematics Teachers' Didactic Strategies: Examining the Comparative Potential of Low Inference Generic Descriptors *Comparative Education Review* 53 (4): 559-581.

Andrews, P., and G. Hatch. 2000. A Comparison of Hungarian and English Teachers' Conceptions of Mathematics and Its Teaching. *Educational Studies in Mathematics* 43 (1): 31-64.

Campbell, R. and L. Kyriakides. 2000. The National Curriculum and Standards in Primary Schools: A Comparative Perspective. *Comparative Education* 36 (4): 383-395.

Cheung, H., and A. Chan. 2008. Relationships Amongst Cultural Dimensions, Educational Expenditure and Class Size of Different Nations. *International Journal of Educational Development* 28 (6): 698–707.

Clarke, D., C. Keitel, and Y. Shimizu, eds. 2006a. *Mathematics Classrooms in Twelve Countries: The Insider's Perspective*. Rotterdam: Sense Publishers.

Clarke, D., J. Emanuelsson, E. Jablonka, and I. Mok, eds. 2006b. *Making Connections: Comparing Mathematics Classrooms around the World*. Rotterdam: Sense Publishers.

Cobb, P., T. Wood, E. Yackel, and B. McNeal. 1992. Characteristics of Classroom Mathematics Traditions: An Interactional Analysis. *American Educational Research Journal* 29 (3): 573-604.

Cogan, L., and W. Schmidt. 1999. An Examination of Instructional Practices in Six Countries. In *International Comparisons in Mathematics Education*, ed. G. Kaiser, E. Luna and I. Huntley, 68-85. London: Falmer.

Correa, C., M.. Perry, L. Sims, K. Miller, and G. Fang. 2008. Connected and Culturally Embedded Beliefs: Chinese and Us Teachers Talk About How Their Students Best Learn Mathematics. *Teaching and Teacher Education* 24 (1): 140–153.

Cummings, W. 1999. The Institutions of Education: Compare, Compare, Compare! *Comparative Education Review* 43 (4): 413-437.

Haggarty, L., and B. Pepin. 2002. An Investigation of Mathematics Textbooks and Their Use in English, French and German Classrooms: Who Gets an Opportunity to Learn What? *British Educational Research Journal* 28 (4): 567-590.

Hiebert, J., R. Gallimore, H. Garnier, K. Bogard Givvin, H. Hollingsworth, J. et al. 2003. *Teaching Mathematics in Seven Countries: Results from the Timss 1999 Video Study*. Washington: National Center for Educational Statistics.

Hofstede, G. 1980. Motivation, Leadership and Organization: Do American Theories Apply Abroad? *Organizational Dynamics*, Summer: 42-63.

———— 1986. Cultural Differences in Teaching and Learning *International Journal of Intercultural Relations* 10: 301-120.

Hofstede, G.and R. McCrae. 2004. Personality and Culture Revisited: Linking Traits and Dimensions of Culture *Cross-Cultural Research* 38 (1): 52-88.

Holmes, B. and M. McLean. 1989. *The Curriculum: A Comparative Perspective*. London: Unwin Hyman.

Hufton, N., and J. Elliott. 2000. Motivation to Learn: The Pedagogical Nexus in the Russian School: Some Implications for Transnational Research and Policy Borrowing. *Educational Studies* 26 (1): 115-136.

Hugener, I., C. Pauli, K. Reusser, F. Lipowsky, K. Rakoczy, et al.2009. Teaching Patterns and Learning Quality in Swiss and German Mathematics Lessons. *Learning and Instruction* 19 (1): 66-78.

Jaworski, B. and D. Phillips, eds. 1999. *Comparing Standards Internationally: Research and Practice in Mathematics and Beyond*. Oxford: Symposium Books.

Jennings, S., and R. Dunne. 1996. A Critical Appraisal of the National Curriculum by Comparison with the French Experience. *Teaching Mathematics and its Applications* 15 (2): 49-55.

Kaiser, G., K. Hino, and C. Knipping. 2006. Proposal for a Framework to Analyse Mathematics Education Traditions in Eastern and Western Traditions. In *Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West, the 13th ICMI Study*, ed. F. Leung, K.-D. Graf and F. Lopez-Real, 319-51. New York: Springer.

Kawanaka, T., J. Stigler, and J. Hiebert. 1999. Studying Mathematics Classrooms in Germany, Japan and the United States: Lessons from the Timss Videotape Study. In *International Comparisons in Mathematics Education*, ed. G. Kaiser, E. Luna and I. Huntley, 86-103. London: Falmer.

Knipping, C. 2003. Learning from Comparing: A Review and Reflection on Qualitative Oriented Comparisons of Teaching and Learning Mathematics in Different Countries. *Zentralblatt fur Didaktik der Mathematik* 35 (6): 282-293.

Pepin, B. 1999. The Influence of National Cultural Traditions on Pedagogy: Classroom Practices in England, France and Germany. In *Learners and Pedagogy*, ed. J. Leach and B. Moon, 122-135. London: Paul Chapman.

Prange, K. 2004. Bildung: A Paradigm Regained? *European Educational Research Journal* 3 (2): 501-509.

Santagata, R., and A. Barbieri. 2004. Mathematics Teaching in Italy: A Cross-Cultural Video Analysis. *Mathematical Thinking and Learning* 7 (4): 291-312.

Schmidt, W., D. Jorde, L. Cogan, E. Barrier, I. Gonzalo, et al. 1996. *Characterizing Pedagogical Flow: An Investigation of Mathematics and Science Teaching in Six Countries*. Dordrecht: Kluwer..

Schwartz, S. 1999. A Theory of Cultural Values and Some Implications for Work. *Applied Psychology* 48 (1): 23-47.

Stigler, J. and J. Hiebert. 1999. *The Teaching Gap*. New York: The Free Press.

Triandis, H. 2001. Modern Education Needs Cross-Cultural Psychology. In *Research on Sociocultural Influences on Motivation and Learning*, ed. D. McInerney and S. van Etten, 1-13. Greenwich CT: Information Age Publishing.

Triandis, H. and E. Suh. 2002. Cultural Influences on Personality. *Annual Review of Psychology* 53: 133-60.

Can Australians Mark KS3 Mathematics Exams? A Study in Cultural Differences

Paul Ayres

University of New South Wales, Sydney, Australia. (p.ayres@unsw.edu.au)

Highly experienced Australian teachers (N = 38) marked a sample of the 2006 KS3 mathematics exams, following similar training to their counterparts in the United Kingdom. Results indicated that were able to mark at a very high standard, but experienced a number of difficulties in doing so. Marking diaries revealed that a number of cultural differences existed concerning quality control, the number of questions marked, the different emphases attached to calculation accuracy, mathematical processes and conceptual understanding.

Introduction

High stakes examinations have for a long time been controversial. A major criticism of such examinations is that they lead to restricted teaching practices, as teachers are often found to teach-to-the test (Barksdale-Ladd & Thomas, 2000). In more recent times the marking of examinations has also become highly controversial. No more so than in the United Kingdom where the delayed marking of Key Stage 3 (KS3) examinations led to a Parliamentary inquiry and ultimately the closure of the National Assessment Agency (NAA). Nevertheless, key government agencies, such as the Qualifications and Curriculum Authority (QCA) have recognised the importance of examination marking and commissioned a number of studies to investigate marking accuracy across a number of national curriculum tests (see QCA, 2009). This paper reports on one such study on the marking of KS3 mathematics (Baker et al., 2006). The main purpose of this paper is to report on cultural differences in mathematics test marking. It also reports on marking accuracy from a cross-cultural perspective (Australian v English markers) and the use of different marking locations (Home v Centre marking).

There has been a multitude of international studies investigating cross-cultural differences in mathematics performance and the underlying factors. Of great significance has been the ongoing international comparative research conducted by the various TIMSS and the PISA studies, which have investigated a wide range of issues related to the teaching and learning of mathematics (see Clarke, 2003). In these studies marking accuracy has not been a focus of investigation but a major technical concern. In the original TIMMS study, the designers recognised the need for marking (scoring) validity on free response items (Beaton et al., 1996). Hence scoring rubrics were developed and training sessions conducted to achieve a high rate of reliability. As part of this process studies were completed that compared markers on common scripts both within and between countries. For later TIMSS studies as well as PISA 2000 and 2003 similar strategies were used (see OECD, 2005). It was found that high rates of consistency could be achieved within and across the different countries that participated provided extensive training was provided. However, there were some variations found particularly on specific items. Although these studies have reported these findings, the underlying reasons behind these variations have not been investigated because the researchers' main aim was to achieve marking consistency.

This present study aimed to explore potential cultural reasons why there might differences in marking accuracy other than human error. To achieve this, Australian mathematics markers, chosen because of similarities between the two countries, were required to mark a sample of the 2006 KS3 mathematics exam (already marked and graded) under the same training conditions as their English counterparts. Their marking accuracy was then compared with the English markers. Furthermore as a secondary comparison, the markers were divided into two groups where one group marked at home and a second group marked at a common centre. To collect information on why marking differences might occur, the Australian markers kept diaries that documented their experiences and perceptions about the marking process.

Method

Participants

The study included 38 participants (F=21 and M=17) with an average age of 51.3 years from Sydney (Australia) or neighbouring towns. Thirty-two were currently teaching mathematics in a high school, three were recently retired from school teaching, and three taught mathematics at the tertiary level. Overall teaching experience in secondary schools was 26.5 years, with a mean marking experience of state examinations of 10.3 years. The participants were recruited because they were highly experienced and were paid market rates for state examination marking. Markers were randomly assigned to either the Home or Centre group.

Instruments and study procedures

The examination papers marked were copies of the 2006 KS3 mathematics tests, which had already been marked in England. All examination answer booklets were "cleaned" to ensure that no English markers' comments or marks were visible to Australian markers. A survey instrument collecting marker background information, such as demographic data and prior related marking experience, was developed.

Pre-training day materials

Prior to the training day, participants were mailed the Teacher Pack (QCA, 2006) that was identical to the pre-training package sent to the mathematics markers in England for the 2006 KS3 mathematics exam. The Australian markers were required to mark the five training scripts using the marking schemes prior to the training day.

The training day

Both groups undertook a 4-hour training session. During this time, an English Marking Trainer worked through the marking schedule using the 5 training scripts. On specific answers the trainer would emphasise certain aspects of the marking scheme, and explain why the correct mark should be 1 rather than 0, and vice versa. Such points often led to a discussion between the markers and the trainer. The meetings concluded by the markers filling in the survey and being given the following handouts, with instructions: a) a sheet to record the times spent marking and the number of scripts marked during these times; b) a marking journal, consisting of a number of sheets to record their thoughts as they marked; c) The *Edexcel* 2006 Key Stage 3 Mathematics External Marking: Commentary for Training Scripts

The following set of instructions for marking the box of 90 scripts was also provided: a) the scripts will be marked in Batches of 10; b) for the first batch of ten

you are required to sort the 10 scripts into separate bundles of the four tiers with the lowest tier first. For each tier you mark each question separately (E.g., all questions 1 in the tier, all questions 2 in the tier, and so on); c) For each remaining batch of ten, mark all papers 1's in the batch of ten in the exact order of the given sequence, mark all papers 2's in the batch of ten in the exact order of the given sequence, mark all mental maths papers in the batch of ten in the exact order of the given sequence; d) bundle each batch of ten with the rubber bands keeping the same order within the bundles and keep the bundles in the correct overall order. These marking instructions were provided to ensure a common approach to marking and that the correct order of script marking was maintained. In order to follow the English marking procedures as closely as possible and also to control some of these variables, it was decided to insist that the first 10 scripts were to be marked by sorting into tiers and then marking question by question. In this way, the first 10 scripts would act as further training and consolidate marker familiarity with the marking scheme.

Home and centre marking

At the conclusion of the training day, home markers took away their scripts. They were instructed to return them within 11 days, after having completed all 90 scripts or marked for a maximum of 20 hours. They were also instructed to contact the English Marking Trainer if they needed any advice in interpreting the marking scheme. Centre markers did not take their scripts home. For each day they attended, the relevant boxes were given out to the markers who took them to the designated marking room. At the end of each session, each completed bundle of ten scripts was collected and stored in a separate area. For each of the scheduled sessions, all the markers marked in one room, where the English Marking Trainer was present to answer any enquiries.

Results

Marking accuracy

In terms of marking accuracy there was no significant difference between the Central and Home markers. Furthermore, no differences were found between the Australian and English markers. However, marking accuracy was not the main focus of this paper and is reported elsewhere in more detail (see Baker et al., 2006).

Marking rates

The mean number of student scripts marked was 85.3 for the Home markers and 79.9 for the Centre markers. Mean times spent marking were 22.3 hours for the Home group and 19.0 hours for Centre markers giving mean marking rates of 3.9 and 4.2 respectively. Under a t-test there was no significant difference in marking speeds between the two groups, $t(35) = 1.27$, $p = 0.21$.

Patterns of marking in the Home group

 Whereas the Centre markers were highly regulated in their marking patterns (Five 4-hour sessions), the Home markers were very much left to devise their own routines. Analysis of their Marking records revealed the following statistics. The group completed an average of 12.6 separate marking sessions: the highest being 24 and the least 7. On average a session lasted 1.9 hours. Whereas some markers were highly systematic in their approach (e.g. 11 sessions of 2 hours), others were far more irregular (e.g. 0.25, 4.25, 2, 1, 0.5, 0.75, 0.5, 1, 0.75, 1, 1.75, 2.75, 5.25 and 1 hour).

Generally Centre markers preferred to mark over small periods of time with substantial breaks.

Marking Journals Themes

Thirteen centre markers and 17 home markers returned their completed journals. These data were analysed using a simple content analysis technique. From each marker's report, a list of raw concepts (issues) was identified. If more than one marker documented the same issue then a frequency count was kept. Concepts with a common theme were then grouped together to form a category. For example, markers identified the 'tomatoes' and 'red kites' questions as difficult to mark. As a result both these questions were classified as members of the category named—Factors influencing speed and accuracy. Altogether, seven categories were identified: Training, Difficulties Interpreting the marking scheme, Positive aspects of the marking scheme, Factors influencing speed and accuracy, Comments on student performance, Increases in marking speed and Recommendations.

Training

A number of markers ($n = 10$) were concerned that they did not receive enough training, particularly home markers ($n = 8$), often referred to as 'Domestic' markers. One marker reported that too much time was spent arguing the pros and cons of the marking scheme rather than accepting it and moving forward. Four markers also expressed the view that having no quality control was disconcerting, affecting confidence and marking accuracy. Three observed that there was value in markers getting together to discuss the marking scheme, and one suggested that as a consequence the Centre markers would be better off. Several markers commented that reporting to a Senior Marker would have been beneficial, rather than working un-supervised. The following are some of the comments made:

> In Australia, 'domestic' markers still have the opportunity for clarification by discussing with their senior marker or with other team member via phone. Also, (there is) opportunity for this as you return boxes to the SM. In this study we were instructed not to communicate so some inconsistencies in applying the marking scheme are likely. (H11)

> There doesn't appear to be any quality control of what is happening with Home markers. Consequently I could mark very quickly but not accurately. (H13)

> Explanations that are "grey" – would be easy to get a "ruling" if senior markers there – can seem trivial to ring about one little thing. E.g. Is diagonal of enlarged square root (100) or root (200) student said " because" root (100) would show that the diagonal line and the side are the same length" seems reasonable but doesn't fit the manual. Will ring when I have another query as well. (H14)

> A longer briefing session or 2nd sample marking with a follow up meeting to discuss differences would have been beneficial. (H16)

> The process is very different as we are not being check-marked along the way – feedback as to how consistent I am marking would be helpful, although not really necessary as this marking does not effect the student's mark on the paper. (C8)

Difficulties Interpreting the Marking Scheme

A number of issues were reported in this category. Five markers reported that the marking scheme was far too complex and could have been simplified, with four reporting that summary information would have been useful. H16 commented in detail on this:

> Sometimes there is a lot of "jargon" to sift through in the "correct response" / additional guidance within. For example, Paper 1 p24 / 25, there is a lot in their 2 pages and if you want markers to keep speed up, maybe these guidelines need to be more concise / with a "tone" explained by what you are looking for. Checking up on some of the detail slows you down. We usually mark via 1) the solution and 2) rubric of intent from the people who set the exam not essays to be read through.

Eight markers believed that the scheme often penalised students who did not fit the expected answers and this made such scripts difficult to mark. Markers often reported that they felt students should have got a mark for correct working, but were unable to give it. For example H10 reports, "All the working is shown for Paper 1, Q1, but because no addition sign, therefore zero". Other similar comments on this theme included over-pedantic penalisation of marks for currency (£7.1, not given a mark), incomplete processing (7/2, no marks), substituting the wrong decimal symbol and probability (7 out of 10, no marks). Three markers thought that the scheme was inconsistent overall, sometimes ignoring an error that they had previously penalised. A possible explanation for this perception was the different emphases placed on conceptual and processing errors on specific questions. Three markers reported finding the idea that correct processing can be worth no marks given certain conceptual errors disturbing, as the following comments illustrate:

> I find the notion of "conceptual" errors a new one - one which was hard to stomach in some questions where a student lost 2 marks even though they knew the process. E.g. Q11 Paper 1 tier 3 – 5 (H11)

> Students drilled will more than likely leave out the "carry 7" How often do you write down your "carries" I never do. We penalise the better students here in the "design" of the question (i.e. the working is in the question). Do they know that they lose marks if they don't write the "7"? Do they know to show all the multiples etc? (H16)

There were other clashes in 'culture' reported. For example, three reported some difficulty with penalising a subsequent error. For example, if a student gave an answer as 21/4 and then incorrectly simplified to 5.1 this would be marked correct at the Higher School Certificate (HSC). It is worth noting that for HSC marking, there are acronyms that are widely used to express this situation, either CNE (correct numerical expression) and/or ISE (ignore subsequent error), although it should be pointed out that the HSC involves far more complex problems for older students. It was often voiced in personal anecdotes but not widely reported in the journals, that many believed that the English system looked to take away marks, rather than reward them. One marker reported that it was difficult following the marking scheme because of the underlying 'foreign' philosophy, another commented:

> The method of marking seems at odds with the way we mark in NSW e.g. marking complete paper as compared to marking a question, we look to give marks rather than deduct. (C10)

Three markers argued that the emphasis seemed to be on the final answer rather than the working, although a counter example reported by one marker was that the simultaneous equations question required working for a mark even though a correct answer might be given. Two markers reported that there were such subtle differences in some answers that they wanted to be free to make more judgements themselves, as the following quote illustrates:

> There was just some very subtle difference in some answers that were awarded 0 or 1 mark i.e. "multiples of 4 are even" and "4 is an even number" needed to use 'judgement flags' and would have liked to have the option where I felt the student

was more correct than wrong – but their option was not on the answer sheet.
(H17)

Finally, again illustrating the clash in cultures, two markers reported that a difficult aspect of the marking scheme was that it was predetermined and they had no part in its development and consequently no ownership.

Positive Aspects of the Marking Scheme

In contrast to the difficulties outlined above, there were some positive aspects of the marking scheme recorded, although not as many. However, six (5 home markers) reported that they found the comprehensive nature of the scheme very helpful. Two such comments were as follows:

> The mark scheme for all the papers was excellent. I liked the "minimally acceptable explanations" and the "incomplete or incorrect explanation." It certainly made the marking scheme easy to understand. (H6)

> Marked question by question (tiers 3 – 5) slow and laborious but starting to get a handle on the marking scheme (in particular the detailed guidelines given). I can see the need for the extra detail – particularly as a home marker. (H19)

Factors Influencing Speed and Accuracy

Responses on this theme were very similar. Twelve markers stated that there were far too many parts (questions per examination question, examination papers and tiers) to mark, interfering with their ability to remember the scheme and their marking confidence. Seven pointed out that the sheer number of pages to be turned was detrimental to speedy and accurate marking. Although not reported in the journals, anecdotal evidence suggested that an additional factor was the stapling of the scripts together, which made the flipping from page-to-page cumbersome. In contrast, the original student booklets were more user-friendly. Furthermore, four reported that if they discovered one of their own errors, it was again highly time-consuming to look back over previously marked scripts. Typical comments were as follows:

> Very slow having to turn pages so often – impossible to keep all the marking schemes in my head as we do in HSC marking. (H3)

> The marking of different papers and different tiers meant that I couldn't get into a flow of marking. Consequently I feel that I've probably made many more mistakes than I am used to. (H13)

> Remembering such a huge number of options makes marking incredibly slow. I also found it difficult to remember the tables and was slowed up by having to check continually. (C1)

Another major factor to influence speed and accuracy was the number of questions that required written explanations. Fifteen markers made this point directly or nominated particular questions that were most problematical. Chief amongst these were the questions labelled 'red kites' (n = 9), 'tomatoes' (n = 8) and 'odd or even?' (n = 4). Three markers added that they were very different types of questions to what they were used to. In addition, some non-explanation problems were also nominated as difficult to mark, such as 'five cubes' (n = 5), which required a number of 3-D perspectives.

Further, some markers (n = 4) reported that it was difficult to mark Tier 6-8 because they appeared so infrequently, particularly later questions, which many students did not try, and therefore they did not have sufficient exposure to get used to them. Finally, a major detriment to marking was the readability of the student scripts.

Eleven markers reported that there were many cases of poor photocopying/scanning which made it difficult or impossible to read. Two such comments were as follows:

> The indication that the question is worth 2 or 3 marks is good. Unfortunately, the extra line for marks was often missing due to the scanning process. (C1)

> MA803 Paper 1 Q8 Matching – too many crossed out – may have been different colours on the original, impossible to mark. (H10)

It was also reported that sometimes the markers guessed the student response, or simply left that question unmarked.

Comments on Student Performance

A number of observations were made about student performance on the examination. Generally, comments were not flattering. For example, individual markers reported that students were not good at the number plane, indices, simplifying fractions, Pythagorean theorem, graphs and circle geometry. More consensus was reached on algebra (n = 3), although one marker qualified this assessment by stating except Tier 6-8 students. Four markers highlighted English communication skills as a problem. Two made the assessment that some students were entered in too higher tiers, and two thought the standard overall was poor. In contrast, individual markers expressed the view that students were used to explanation questions, they were better at some content than others, they were very good at Q20, Paper 1 (Tier 3-5), and one reported enjoying students answers. Other observations included identifying common student errors on questions involving 'powers of 5' and 'three dice'.

Marker recommendations

A few markers made some recommendations mostly connected to splitting the marking into parts (n = 3) or marking only one tier (n = 3). One marker reported that it would be better to mark all of one tier first, before starting the next one. One marker also observed that the present format would be best marked corporately (at a centre), while one believed that a multiple-choice section would effectively cater for a number of the questions asked.

Conclusions

The information recorded in the journals indicated that the markers had experienced some difficulties on this task. Many felt, particularly home markers, that they did not initially receive enough training. Furthermore, a lack of further training or quality control had impacted on their confidence and ability to become adjusted to the marking scheme. Clearly, the task of marking whole exams, across 4 tiers, was very alien to their own marking experiences. The sheer number of pages to be turned in both student scripts and the marking schemes impacted on their speed and accuracy, as well as the sheer number of parts to become familiar with. Furthermore the Australian teachers reported that they would have liked more involvement in the development of the actual marking schemes, as 'ownership' was considered important. These findings indicate that the marking methods used by the KS3 markers may not be best practice.

Aspects of the marking scheme also took some adjustment. Different emphases on calculation accuracy, mathematical processes and conceptual understanding often created the perception that the marking scheme was inconsistent overall. Furthermore a high number of explain-type questions was also culturally different to their own experiences and led to difficulties in awarding marks. Other

positive aspects included the recognition by some that the marking scheme was very thorough and that some of the questions were creative and useful to the markers' own teaching practices. However, despite many reservations it was found that experienced Australian mathematics markers could adapt and successfully mark KS3 exams.

Finally this study may have some implications for the broader theme of assessment. Much of the research into assessment has focused on the importance of formative assessment and task selection, and not how such tasks are marked. It is clear that some items in the KS3 test were designed to test more than just facts. Yet the Australians had some difficulty marking particularly questions that emphasised conceptual processes despite being familiar with standards-based marking and formative assessment. Their methods of marking such questions were different, and they often felt the English method was more punitive. Here was a clear cultural difference where both countries could possibly learn from each other. It also suggests that providing good assessment tasks need to be backed up by equally good marking schemes in order to provide appropriate feedback. It is therefore suggested that further cross-cultural research into marking may be useful in supporting the bigger picture of assessment reform. As Clarke (2003) observed, assessment itself warrants more international comparative research.

References

Baker E. L., Ayres P, O'Neil H, Choi K, Tettey M and Sylvester R 2006. *KS3 Mathematics Marker Study in Australia: Report to the National Assessment Agency of England*. University of Southern California: Sherman Oaks, CA.

Beaton, A., Mullis, I., Martin, M., Gonzalez, E., Kelly, D., & Smith, T. 1996. *Mathematics Achievement in the Middle School Years: IEA's TIMSS.* : Chestnut Hill, MA, USA: Center for the Testing, Evaluation, and Educational Policy, Boston College.

Barksdale-Ladd, M.A. & Thomas, K.F. 2000. What's at stake in high-stakes testing. *Journal of Teacher Education*, 51: 384-397.

Clarke, D. 2003. International comparative research in mathematics education. In Second International Handbook of Mathematics Education, ed A. J. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick and F. K.S. Leung, 143-184. Dordrecht: Kluwer Academic Publishers.

Organization for Economic Co-operation and Development 2005. *PISA 2003 Technical Report*. Paris: OECD

Qualifications and Curriculum Authority (2006). *The KS3 (Levels 3-8) Mathematics Test: Teacher Pack.* London: QCA.

Qualifications and Curriculum Authority 2009. *Research into marking quality: Studies to inform future work on nation curriculum assessments*. London: QCA.

Acknowledgements

Developing a leading identity across the transition to mathematically demanding programmes at university

Laura Black

University of Manchester.

In a previous paper (Black et al. 2010) we drew on interview data with AS level mathematics (post-16) students to present the concept of 'leading identity' which, we argued, defines the student's motive for study and shapes their relationship with mathematics. We argued that whilst some students might focus on a leading identity of 'being a student' and thus, engage with the activity of 'studying' merely to gain qualifications, others focus upon 'studying' with a vocational future in mind and thus attend to the 'use value' of mathematical knowledge beyond the institution of schooling. In that paper, we presented the story of Mary, a student studying AS level Use of Maths, who had a leading identity of 'becoming an engineer'. We argued that this leading identity gave her enough motivation to persist in studying mathematics where others might have dropped the subject. In this paper, we wish to explore the sustainability of this leading identity for Mary as she experiences transition from college to university. We now have five interviews with Mary which cover the period from the start of her AS level studies to the end of her one year Foundation Degree in Mechanical Engineering at university. In telling Mary's transitional story, we ask: how does Mary experience transition in light of her 'leading identity'? And how does this affect her relationship with mathematics? Furthermore, we also seek to understand how Mary's transitional story is positioned within the landscape of cultural narratives about transition available to her.

Keywords: transition, leading identity, advanced mathematics

Theoretical background

In theorising Mary's transitional story we draw on Cultural Historical Activity Theory (CHAT) which focuses on transition and development in terms of movement between object orientated and socio-culturally mediated activities (Leont'ev 1981; Beach 1995, 2003). This line of theory implicates the construct of 'identity' since individual development is seen as a process of 'becoming' someone new – in line with their participation and transition between activities. As with many theorists, we recognise identity as multi-faceted and socially situated in practice or activity. Given the scope of activities we engage with on a regular basis, we have many different identities to draw on at a given moment in time. We refer to identity as our reflective understanding of who we are in relation to a given activity (Williams, Davis, and Black 2007); one 'becomes' what one 'does' and, importantly, one comes to 'think' what one 'says' through reflexivity. For example, we construct our identity as a mathematics learner upon reflection on our engagement in various forms of mathematical activity in the past. Thus, our notion of identity (or identities) is historical in origin and emerges from the subjectivities (how one views oneself) we experience in the process of doing activities.

In seeking to explore the notion of 'transition', we draw on the work of (Beach 2003) who emphasises the need to consider participants relation to transition – and the relationship between the different forms of activity they are transiting between. He provides us with a typology of different types of transition – of relevance here is 'lateral transition' – involving the individual's movement between two historically related forms of activity in a linear direction. Thus we can see students move from schooling to university and eventually to the workplace as a lateral transition since a) individuals move through these activities in a fairly linear fashion (albeit with some exceptions e.g. returning to education after a period of work) and b) these forms of activity are historically connected through the positioning of education (both schooling and university) in terms of its 'value' as preparation for future labour (Williams 2008). Participation in one form of activity is superseded and replaced by activity in another and some notion of progress is embedded in the individual's transition. (Beach 2003) argues that by acknowledging the unidirectional nature of an individual's lateral transition between activities (and the sense of progress and development which goes along with that) – we can begin to understand the ways in which students might engage with mathematical knowledge differently as they experience transition from schooling to 'working'.

Furthermore, (Beach 2003) argues that the notion of 'leading activity' is an important methodological tool in terms of understanding transition both in terms of individual progression and in terms of macro level change (these two are entwined in such a way that you cannot have one without the other). He draws on (Leont'ev 1981) to define leading activity and its role in development suggesting that human life
"is not built up mechanically ... from separate types of activity. Some types of activity are leading ones at a given stage and are of greater significance for the individual's subsequent development, and other types are less important. Some play the main role in development and others a subsidiary one. (1981, 95) Thus, we might argue that schooling is a leading activity since for many it plays a significant role in our development. Nevertheless, what counts as 'significant' is not merely defined in terms of the individual but "is as much to do with whether it is leading or not as the societal sequence of activities. It is in this way that activities mediate between large-scale societal change and the local coupling of individuals with activities." (Beach 2003, 125) It is from this framework that we developed the notion of leading identity in our previous paper (referred to earlier). We argued that the identity made available by one's leading activity at a given moment in time is a *leading identity*. This reflects a hierarchical organisation of motives in understanding the self and directs the organisation of all other identities available. For example, we might suggest that Mary's leading identity as 'a student wanting to become an engineer' hierarchically subordinates the other identities she constructs and which are available to her (e.g. being female).

However, whilst this approach places emphasis on the students' motive for engaging with mathematics – it should be distinguished from constructs such as 'motivation' and 'goal orientation' which have also been referred to in the literature (e.g. (Dweck 1986). The latter reflect types of behavioural approaches in relation to task (e.g. mastery goal orientation, performance avoidance etc.) which are internal to the individual and appear generalisable across tasks. A CHAT approach on the other hand, places motive in the dialectic between individual and the activity (along with its socio-cultural and historical baggage). For instance, students' motives for studying mathematics are often defined in terms of the 'grades' they can achieve and the 'status' such grades bring in what is apparently an elite subject. However, this motive

is only made relevant by the activities which take place in mathematics classrooms (e.g. cramming for tests) and the wider institutional discourses in which they are situated.

Methodology and Data Analysis

This data comes from two connected projects. The first investigated students' participation in post compulsory mathematics education (aged 16-17 years) in England with the aim of comparing two Advanced Subsidiary (AS) level programmes: Mathematics and Use of Mathematics with a view to understanding how pedagogy impacts on students' dispositions towards studying mathematics, and electing to study mathematically demanding courses, (e.g. Science, Technology, Engineering etc.) at university. As part of this project, we conducted interviews with 40 students focusing on their background history, their experiences with mathematics, career aspirations and disposition towards future study.

(http://www.education.manchester.ac.uk/research/centres/lta/LTAResearch/tlrp/)

The second project is currently investigating how students experience mathematics education across the transition from A-level to mathematically demanding degree programmes at university. It looks to explore how such experiences interact with students' background social factors (e.g. class, gender etc.) to shape learning outcomes, identities, dispositions, and the key choices made surrounding transition into university. Mary was a participant in both of the above projects so we have 5 interviews spanning a period of 3 years.

(http://www.lta.education.manchester.ac.uk/TLRP/summary.htm)

Mary's interviews have been analysed using narrative analysis. This draws on the work of Bruner (1996) who emphasises the importance of narrative not only in construing how we understand ourselves in the world but also the 'reality' in which we operate. Thus, we present our students' interviews as biographical narratives, made up of inter-connecting sub-stories which can then be connected (or disconnected) through a reformulation process. The latter involves the identification of a central 'plot' within or across a number of interviews and sub-stories told by the student are considered in terms of their proximity to this (Goodson and Sikes 2001). On this occasion, we have used the constructs of leading activity and leading identity as the central plot. A leading activity is identified where we see a significant shift in the student's motive to engage with a particular activity or others like it and where this shift in motive is implicated as significant in shaping the student's trajectory. This can be seen within a particular sub-story the student recounts where s/he reflects on the shift in some way or it may be more apparent when comparing sub-stories regarding the same activity(ies) at different points in time (i.e. between two different interviews). The construct of leading identity can be identified where the student makes 'I statements' pertaining to the afore mentioned leading activity. Identity statements may refer to either their state of being (in the past, present or future) (e.g. I am, I will be, I was etc.) or themselves in action (e.g. I do, I got, I will do). The use of leading activity and leading identity to formulise a central plot means that our analysis connects the sub-stories students tell to present the 'whole story' as it emerges across the interview(s). Inevitably, some sub-stories told within the interviews are not included in our analysis; we have selected data which is of significance to the overall plot (i.e. the student's leading identity).

Mary's Story: Prior to University

In our previous paper, we presented Mary's earlier experiences and attitudes towards education and mathematics as a 'canonical' story which was shared amongst a small

number of students in our sample (5 out of 40 students). This story was identified as 'when troubles come, aspirations remain the same' and in Mary's case, we told of her leading identity of becoming an engineer which enabled her to persist with studying mathematics despite various troubles (i.e. being placed in a low GCSE set, dropping statistics at AS level (Use of Maths) due to likely failure). We argued that this leading identity emerged out of Mary's engagement with her GCSE Engineering programme (Double Award) during which she undertook various design projects (the leading activity). Mary, herself, identified such projects as crucial in shaping her aspiration to become an engineer and spoke of how this experience developed an awareness of her 'needs' as a student ('I like hands on stuff') and her future potential self.

> "That whole process [making something from scratch] and that accomplishment and that feeling I got, I loved it and I just thought I really want to keep that and be part of it. One of my dreams is to do something massive, and be like, "Yes, I did that"

In our previous paper, we argued that Mary's relationship with mathematics was shaped by the motive provided by her leading identity of becoming an engineer. Thus, she spoke of its use value in an engineering context in addition to its exchange value in terms of the grades she might obtain and it was this perception of the use value of mathematics which enabled her to persist with the subject even in the face of potential failure. This contrasted with other students in the sample who were more focused on the exchange value their grades could provide and who spoke of dropping mathematics and adjusting their aspirations in the face of similar troubles.

Mary at University

We catch up with Mary near the end of her Foundation Degree in Mechanical Engineering. She is attending X University, although this was not her original choice (more on this later), after obtaining 140 UCAS points through her A-level results. As in previous interviews, Mary remains highly committed to engineering as a career, although her aspirations have become a little more refined. Nevertheless, she remains keen to keep her options open and see what options a mechanical engineering degree might provide:

> "You know maybe do aeronautical. I wasn't, I'm still not 100% sure because there's a lot of choice in the stuff— […] But I think what I was going to do was stay with mechanical engineering and do maybe a Master's in aeronautical aerospace you know, something like that.

However, whereas before university Mary struggled to gain the grades she needed/wanted, she now describes herself as a successful student. She tells us she is maintaining 'a B or A sort of level' which 'is good and a bit surprising'. On her mid year exam she tells us she gained an A/A- and describes the transformation 'From like D's and E's to like an A so I was kind of like, 'ok.' 'I thought they did a mistake'. Thus, we can see a distinct transformation in Mary's identity from struggling to successful student.

This change in Mary's narrative seems quite counter intuitive – given her troubles at A and AS level and the perceived importance of A-level grades for entry and performance at university, we might expect her to present a more troubled account of her move to university. However, as mentioned above, transition from one organisation to another can be an affordance for an individual in that they experience becoming someone or something new (Beach 2003). Indeed, as Beach argues a sense of progression is embedded in uni-directional, lateral transition since it this principle which historically connects the two organisations concerned (in this case, school and university). The sense of a new beginning for Mary was certainly evident in her interview data where she tells us of the changes in her identity that she has made:

> "….getting onto the course, getting onto it I think wasn't hard but doing it was a bit of a different manner. I had to change my whole attitude where I thought, ah, maybe I'm not gonna be that good, or you know, looking at my past grades this is crap you know, but I had to kind of change my attitude and think, 'wait, this is a new start'.

Here, Mary comments 'I **had** to kind of change my attitude' which suggests she felt her 'new start' was a necessary part of her transition in order that she progress in line with the activities she participates in.

Leading identity and transition to university

When asked why this change has come about, Mary directly implicates the motive derived from her leading identity in the narrative. She tells us:

"I think it was just because I finally knew what I wanted to do. […] I've always wanted to do a sort of Maths and I knew I liked making things[…] I can't wait to get my job and you know, call myself an engineer. It'll be such a great moment in my life I think."

However, as noted above, this motive is not solely the product of Mary's past experience but is also partly located within the structure of the activities and institutions with which she engages. As such, Mary talks of her progression in terms of finally finding a place where she belongs – a place where her perceived potentiality aligns with the motive generated by the activities which constitute her engineering course. This sense of belonging comes through most strongly in her description of her relationship with her fellow students on the Foundation degree:

> "Yeah, I've got very nice guys who I hang out with […] it's really nice to kind of have that support from you know, really good friends to have that support from them, and you can see that they're the kind of people that really wanna get somewhere on the course, and it's really nice to have that."

When asked why she thinks they are such a good group of people to work with she tells us: "I think cos they all wanna get somewhere in their lives. […]And we're all kind of there for the same reason, so that kind of brings us together."

Thus, Mary tells of belonging to a community of participants who engage in the same activities with the same motive in mind – for Mary, her leading motive is now shared with her fellow students (i.e. knowing where they want to go in life) - a community generated by the engineering activities which make up the Foundation Degree programme. Thus, we see what appears to be a dialectic relationship between Mary's leading identity and her sense of 'togetherness' with the community and activities which constitute her course. On the one hand, she is able to find this sense of togetherness precisely because she has a leading motive to become an engineer but simultaneously, the alignment of her leading motive with the activities and community which constitute her course further resources her leading identity and thus her motive for studying ('I can't wait to get a job..and call myself an engineer').

Leading identity, transition and Mary's relationship with mathematics

So does this transition bring about a new relationship with mathematics for Mary? In our earlier analysis, we highlighted how Mary's motive to 'become an engineer' shaped her relationship in a way which enabled her to perceive the use value of mathematics beyond the walls of the classroom. Yet there was an apparent contradiction in Mary's account between this notion of use value which she saw as powerful in terms of her future as an engineer and the limited exchange value which her low grades provided in the here and now (at AS level). We argued that this represents wider conflicts within the education system with the current emphasis on 'performance' in mathematics being mostly disconnected from its eventual 'use' in the labour market (Williams 2008). However, Mary's transition to university has resolved this conflict in some sense - Mary is now a successful student and therefore,

her positive relationship with mathematics is not just framed in terms of its use to her future but also draws on a new identity – that of, successful student. – " [I'm] suddenly getting good grades without even trying sometimes". As such, we might suggest that Mary feels a new sense of alignment between the exchange value her 'good grades' provide and the use value of the mathematics she is learning. This is referred to in the quotation above where she suggests that doing the kind of mathematics she likes ('the maths behind the designing and making') is one reason for her new success in the subject.

A sense of alienation in developing and maintaining a leading identity
However, although Mary's transition to university may be seen as a fairly positive story – her interviews contained repeated references to a family narrative regarding her 'designated future' and the sequence of leading activities which this implicates. Mary comes from an immigrant Pakistani family which she describes as: "we have a very backwards sort of family. It's a very Asian sort of, you know, girls should stay at home to do the cooking— […]Where the guys go out and work." Thus she suggests that a more acceptable sequence of leading activities under this narrative may be seen as 'schooling followed by motherhood/domestic work'.

On the one hand, Mary speaks of her resistance to this expected sequence of activities. She speaks of having to convince her parents and her sister that 'Engineering was something I've always wanted to do' and informs us that 'my step-granddad he doesn't agree with it still.' But she tells us '… I've just started to ignore him now because it is just, I'm so happy doing this course…' So Mary tells us that her trajectory of 'becoming an engineer' has implicated resistance to a designated identity laid out in her family narrative. However, a sub-story regarding why she chose to go to *X university* and therefore, undertake a Foundation Degree (involving an extra year of study) rather than go to her first choice of institution, suggests that this resistance has created tensions for her. Mary tells us that she felt X university would have better resources ('flight simulators' and an 'aeroplane cut in half') which she was not sure her first choice of institution would have (having never visited it). She then tells us

> "and then I thought about my grades and how you know, obviously I didn't do the Maths and stuff and I've heard from my older sister, who had friends who did Engineering—[…] Quit the course because they found the Maths too hard […] and I thought to myself well if I spend an extra year kind of getting used to it then hopefully I have two years to kind of be good. […] It kind of took me to, I kind of, wasn't really pushed but I was more kind of towards *X university* anyway. […]So I went with *X university* instead."

So it seems that in order to sustain her leading identity of 'becoming an engineer' Mary is required to negotiate and even subordinate the various other identities available to her (e.g. being female and Asian). Mary's account of why she ended up at *X university* provides evidence of this – she has reached a compromise with her family by 'staying near home' (thus fulfilling her family's wishes & being the dutiful Asian daughter) whilst still pursuing her engineering dream. Thus, to some extent she seems aware that her leading identity of 'becoming an engineer' necessarily implicates alienation from the culturally normative expectations her family holds for her as an Asian female. Nevertheless, Mary resists the idea that the 'Foundation Year' is a 'cost':

> "Yeah I thought to myself taking an extra year would that make me look dumb, would that you know, just waste my time or something but, doing the actual course I found that, you know I actually needed it cos I thought if I go into first year not knowing what integration is and differentiation was—[…]   Then I'll be really stuck."

Thus, in some sense, the compromise she has made with her family has provided access to mathematical knowledge (integration and differentiation) which she sees as necessary for her future studies and it has provided a slower pace to her progression and perhaps even eased her transition to university.

Conclusion

In this paper, we have highlighted the transitional story of one student, Mary, as she progressed from school/college and into university. We have shown that, by adhering to a particular leading identity (becoming an engineer), Mary has found a deeper motive to the activity of studying. For Mary, studying is not simply about gaining a qualification but rather it is seen as preparation for the next stage of her life – working as an engineer. We have argued that the motive provided by her leading identity may be resourcing a positive transition providing her with enough confidence to make a new start. Therefore, we feel Mary's story makes a valuable contribution to previous research on student learner identities. This has tended to explore the identities of highly successful students on mathematically demanding courses in elite institutions in Higher Education (Solomon 2007; Brown and Macrae 2003) or has provided generic accounts of transition which do not specifically focus on mathematically demanding courses and the role mathematics plays in transition (Ball, Maguire, and Macrae 2000).

Running alongside our representation of Mary's transitional story, we have highlighted Mary's ongoing relationship with mathematics which is framed in terms of its use value to the activity of 'doing engineering' both inside the classroom (education system) and in the workplace. This appears no longer disconnected from the 'exchange value' her mathematical knowledge provides. The question we might then ask: is how can we encourage students to find ways of understanding their leading identity and its relationship with 'studying' as an activity? How might we encourage students to reflect on their motives for study? And consequently, how might we help them to see the 'use value' of mathematics?

Additionally, we have also argued that Mary's transitional story is embedded in wider cultural narratives about future aspirations and the sequence of and relations between leading activities which such narratives adhere to. Mary expresses a desire to follow a normative pathway by moving from schooling into the workplace (both can be classed as leading activities). Furthermore, in describing this pathway she draws on a liberal humanist stance suggesting she has 'choices' (and thus agency) which she makes on the basis of what she 'enjoys'. But this jars with her family narrative (and possibly the sequence of leading activities which may be expected of her in that narrative i.e. schooling – domestic work - motherhood). Ultimately, it seems that she has to negotiate this narrative and we have shown in the data how she has sought a compromise in 'doing engineering' in a way that appeases her family. For Mary, undertaking a Foundation Degree provides a slower rate of change and progression in her transition and provides a pathway to university which allows her to ease a sense of alienation from her family.

References

Ball, Stephen J., Meg Maguire, and Sheila Macrae. 2000. *Choice, pathways and transitions post-16*. London: RoutledgeFalmer.

Beach, King. 2003. Consequential Transitions: A Developmental View of Knowledge Propagation Through Social Organisations. In *Between school and*

work: New perspectives on transfer and boundary-crossing, edited by T. Tuomi-Gröhn and Y. Engeström. Amsterdam/Oxford: Pergamon.

Black, L., J. Williams, P. Hernandez-Martines, P. Davis , and . G. Wake. 2010. Developing a 'Leading Identity': The relationship between students' mathematical identities and their career and Higher education aspirations. . *Educational Studies in Mathematics* 73 (1):55-72.

Brown, Margaret, and Sheila Macrae. 2003. *Full Report of Research Activities and Results: Students Experiences of Undergraduate Mathematics: Reference Number: R000238564.* Swindon: Economic and Social Research Council.

Bruner, Jerome. 1996. *The Culture of Education*. Cambridge, Massachusets: Harvard University Press.

Dweck, C. S. 1986. Motivational Processes Affecting Learning. *American Psychologist* 41 (10):1040-1048.

Goodson, I.F. , and P. Sikes. 2001. *Life history research in educational settings. Learning from lives*. Buckingham: Open University Press.

Leont'ev, Aleksei Nikolaevich. 1981. *Problems of the development of the mind*. Moscow: Progress.

Solomon, Y. 2007. Not belonging? What makes a functional learner identity in the undergraduate mathematics community of practice?' *Studies in Higher Education* 32 (1):79-96.

Williams, J.S. 2008. Towards a political economic theory of value in education. *TLRP-WP-Maths working paper.*

Williams, J; , P; Davis, and L. Black. 2007. Sociocultural and Cultural–Historical Activity Theory perspectives on subjectivities and learning in schools and other educational contexts. *International Journal of Educational Research* 46 (1-2):1-7.

Research and Developments in Probability Education Internationally

Manfred Borovcnik and Ramesh Kapadia

Institut für Statistik, Alpen-Adria-Universität Klagenfurt, Austria, Institute of Education, University of London, England

In the topic study group on probability at ICME 11 a variety of ideas on probability education were presented. This paper provides a brief summary of the main threads of research in probability education across the world; it is intended that this will help consolidate developments. Further areas for work and research are also presented.

Background

Whether probability education needs to be seen as discrete and separate from statistics has been an ongoing debate for many decades. Nowadays, statistics seems to be dominant in school education and data handling has been a key theme as part of the movement of mathematics for all. Conversely probability is thought to be harder and less relevant. Nevertheless, probability is an important discipline in its own right, and does contain the key underpinning concepts to understand and use data sensibly.

This paper focuses on the international research on probability issues in education, mainly derived from ICME 11 (2008) in Mexico. This study group, also linked to IASE included all the major themes being studied internationally in probability education. A thorough system of peer review, using an international panel of experts, ensured that the papers at ICME 11 were both well written and also covered the key areas of research being undertaken across the world. The full papers from the conference are available on the ICME 7 site in the links below. Most of the papers have been developed further, including many interactive features (such as links to related and background research ideas) and can be found in Borovcnik & Kapadia (2009), which can also be accessed electronically. These themes of international research provide a valuable bridge between international research and themes in Britain on the teaching of probability, which is often subsumed under data handling or statistics.

Probability education has not been a central focus of the research community in the last three decades since the theoretical framework espoused by Kapadia & Borovcnik (1991) appeared. Jones (2005) on "Exploring probability in Schools" has largely followed the new paradigm in educational research, which is empirically oriented. Designs of teaching sequels are administered to students and analysed. Sometimes, beliefs and attitudes of teachers are empirically investigated. Only one of the contributions in Jones is philosophically oriented: Batanero, Henry, & Parzysz (2005) give a summary of the philosophical debate on the interpretations of probability and discuss its implications on teaching.

Analysis of the subject matter is still dominated by Heitele's fundamental ideas (1975), which seem to be more a description of the main chapters of a probability textbook than an analysis of the concepts from a more general perspective and their purpose. The educational debate is being revived by the more recent endeavours to explain the concepts of risk (see Gigerenzer 2002), which comes from societal needs taken up by cognitive psychologists and thereby attracts more attention in the community of educationalists. However, the fundamental ideas of probability,

in line with those discussed for statistics by Wild & Pfannkuch (1999) are still awaiting elaboration. Some starting points may be found in Pratt (2005) though the CERME working group, too, is mainly devoted to the empirical paradigm.

The authors support the ideas and results of Fischbein (1987) who elaborates on intuitions and their impact on understanding (and accepting) probabilistic concepts. Raw primary intuitions of individuals are revised by teaching interventions and changed to secondary intuitions, which should and could help to handle with the formal sides of the concepts as a companion. Such ideas have the potential to describe any learning process; research begins with the learning process of an individual or a group on the research topic of interest and later becomes a learning process within the wider community when research papers are published, Fischbein's ideas might also orientate the way a research community exchanges its results and enriches the discussion and makes progress.

Despite the fact that a multitude of new technologies is available now in the era of information technology, and multi-media is spreading to all corners of life, publication in research has hardly changed. Textbooks for study are changing gradually, a few hypertexts make use of the possibilities of new media but for research publications it seems that times have not changed yet. Borovcnik (2007) has analysed the consequences of new technologies on applications and on educational endeavours; more endeavour is needed from the research community to improve its communication. Kapadia & Borovcnik (2009) have taken up the challenge of innovative publication (including multi-media and more) in the age of web 2.0. In what follows, we will focus on the content, the main streams of research in educational probability by classifying the international endeavours to obtain new insights on the teaching of probability.

Central themes in international probability education research

Probability and statistics have been part of school mathematics for less than 40 years and complement the traditional topics of arithmetic, algebra and geometry. Statistics is part of the curriculum in virtually all countries but ideas of probability may only be introduced for older pupils. Application-oriented statistics is undisputed in its relevance, but the place of probability is more ambivalent. Reduction of probability to the classical conception, mainly based on combinatorics, and its perception as a solely mathematical discipline with its connection to higher mathematics, are sometimes used as arguments to abandon it in favour of the statistics part. However, there are key reasons for a strong role for probability within mathematics curricula:

1.   Misconceptions on probability affect people's decisions in important situations, such as medical tests, jury verdicts, investment, assessment, etc.
2.   Probability is essential to understand any inferential procedure of statistics.
3.   Probability offers a tool for modelling and "creating" reality, such as in physics.
4.   The concepts of risk (not only in financial markets) and reliability are closely related to and dependent upon probability.
5.   Probability is an interesting subject in its own right and worthy of study.

The challenge is to teach probability in order to enable students to understand and apply it, by creating approaches that are both accessible and motivating. Both, the

frequentist and subjectivist views of probability, and connections of probability to practical applications should be taken into account. Simulation is one such strategy, as is visualisation of abstract concepts. The use of technology helps to reduce the technical calculations and focus the learner on the concepts instead. The world of personal attitudes and intuitions is another source for success or failure of teaching probability. The main themes in the research, which nevertheless do overlap are: Conditional Probability and Bayes' Theorem; The School Perspective: Pre- and Misconceptions; The Teachers' Perspective: Pre- and in-service Courses; Impact of Technology; Fundamental Ideas.

Conditional Probability and Bayes' Theorem

Conditional probability and Bayesian inference are important ingredients of university teaching, including courses for non-mathematical students. Many different types of errors have been investigated in isolation. According to C. Batanero and C. Diaz (Spain), there is neither a study investigating connections between various types of misconceptions, nor an analysis whether misconceptions are related to mathematical knowledge, i.e. whether they decrease with better achievement in mathematics. Consequently, they have developed a test with (mainly familiar) items, and administered it to university students. Results were analysed with factor analysis. Though some phenomena remained even with higher mathematics education, there was a significant decrease in misconceptions with a higher level of mathematics. For interrelations between several misconceptions, the results were less optimistic as these misconceptions seemed to be quite isolated. As a consequence, endeavour in probability education has to be fostered and misconceptions still have to be continually and repeatedly stressed in teaching in order to facilitate students' understanding.

P. Huerta (Spain) identifies a serious flaw of some existing research which does not take the structure of the posed problems into account. He has classified the mathematical structure of "ternary problems" into 20 different types of problems with conditional probabilities of which only one subclass (and from it mainly one type of task) has been used in existing research. A graph with all problems has been developed to visualise the grade of difficulty of a special problem at hand. Applying this deep structural analysis, Huerta has developed a plan for future empirical research to cover all types of conditional probability problems to enhance the insights which might be gained.

The School Perspective: Pre- and Misconceptions

There has been a trend away from misconceptions, which may be changed by suitable teaching, towards pre-conceptions. Such a change of focus in research may be traced in current empirical research.

D. Abrahamson (USA) designed an experiment with a single child (Li, 11 years), using in an in-depth interview after a teaching phase in a classroom environment where an urn experiment was replaced step by step by the computer environment. Abrahamson analysed the learning trajectory of the child and how the interaction of the representation of the notions by different media influenced learning (see below), starting with a 4-block scooper to pick out blue and green marbles, as illustrated below. As a valuable side effect of the approach, the histogram can be visually linked (with a "greenish" impression) to the proportion of green marbles in the urn.
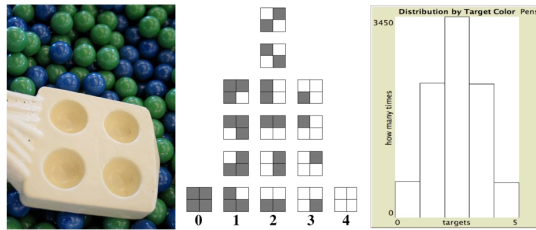
Figure 1. Abrahamson's "four block scooper" as a unifying element.

The empirical studies by F. Chiesi & C. Primi (Italy) and L. Zapata (Colombia/USA) dealt with heuristics in the tradition of Kahneman & Tversky, especially the changes in "negative" and "positive recency" with age. They compared 9, 11, and 25 years olds in order to imitate a longitudinal survey. Interestingly, their study showed an increase of the normative (correct) solution first (from age 9 to 11), which dropped again (age 25). They found that the bias towards "negative recency" decreased (from age 9 to 11) first and then increased again (age 25). However, the "positive recency" decreased (9 to 11) and was unchanged amongst adults (age 25). More in-depth investigations would clarify what is happening, and whether such a "development" can really be confirmed (of course it is not truly longitudinal).

Conditional probabilities are considered to be difficult; Bayesian inference is no less difficult. L. Martignon and S. Krauss (Germany) have worked with 10 year olds on these ideas and their precursors. They started with Wason cards to initiate learning steps in the children.

Which cards do you necessarily have to turn around in order to check, whether the following rule holds for the set of 4 cards? "If one side of a card exhibits a vowel, its other side must exhibit an odd number"
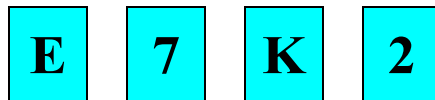
Figure 2. The original Wason cards.

Children had difficulties but improved with contexts closer to everyday life. The researchers moved on to use tinker cubes (multi-cubes) to build towers, so that both sides could be seen simultaneously and then promoted learning steps in proportional thinking, reporting encouraging results from their pilot projects.

K. Rolka and S. Prediger (Germany) studied 12 year olds playing a game of luck with tokens moved forward on a playing board by the result of various sorts of dice. The discussion amongst the children revealed how they interacted and argued fiercely in favour of preferred strategies, learning from their *common* struggle for an optimum strategy. The social situation of the class with the children interacting in their discussion featured strongly.

The Teachers' Perspective: Pre- and in-service Courses

There are pitfalls in the interpretation of results from statistical tests or from confidence intervals. These originate from the reduction of the interpretation of probability to situations, which may be repeated independently in the same manner. On this issue there has been a vigorous debate not only in the foundations of statistics but also in the didactical community. Ö. Vancsó (Hungary) has developed a parallel course in classical *and* Bayesian statistics. He believes that it is a false dichotomy, to teach *either* classical statistics *or* Bayesian statistics, as both offer a consistent theory of probability. He has tried and refined his ideas in several cycles in teacher pre-service education and reported his positive experiences: "Now I have really understood what is meant by confidence intervals" one of his students exclaims.

An interesting extra-curricular activity was explored by H. Trevethan (Mexico) who described a project in the context of a science fair. A pair of students worked on a project to present a game of chance. There were several advantages. These included the autonomous activity of the students, their own responsibility, presenting in public etc. The game was "Shut the box", which is certainly open to varied stochastic strategies as different dice can be chosen to play the game. This authentic (and not artificial) transfer of responsibility could well be taken up more often in teaching in class. Mathematically, conditional probabilities and Bayes' theorem are the key concepts to develop winning strategies.

K. Lyso (Denmark) used a battery of standard tasks covering the main primitive concepts including two-stage experiments. The distinctive feature was on the discussion about which solutions are feasible, or which reconstruction of the task made sense and therefore led to a sensible solution even if it did not coincide with the "normative" solution. One result was the documentation of an inclination to reformulate tasks. The students sometimes reformulated two-stage experiments into one-stage tasks getting a wrong answer but found it hard to see why their reconstruction was misleading:

L. Zapata (Columbia/USA) investigated well-known tasks from Kahneman & Tversky in order to clarify what may be learned from new as well as more experienced teachers. She has tried to derive meta-knowledge from her in-depth interviews with teachers. Surprisingly, or possibly unsurprisingly, novice teachers repeated the same misleading intuitive conceptions as their students and thus were not really able to help them. It may be that probability is much more prone to such difficulties than other topics in mathematics. This is confirmed by V. Kataoka (Brazil) who ran a series of workshops of in-service education. One special experiment used in the workshop illustrates the importance of suitable models and data sampled by randomness (when do you really have data from *random* samples?)

> We break a stick randomly into three pieces. Afterwards, the subjects are asked to form a triangle of the three pieces. Finally the success rate is determined with which triangles actually could be formed. Try it with spaghetti – without explaining in advance what you plan.

Success rates of 75% are not rare. In contrast to it, there are (at least) two models for randomly breaking the stick (with 25% and 19% success rates). The obvious discrepancy between the theory and the model lets us gradually start to doubt whether we can break the stick truly randomly into 3 parts. As a conclusion, relative frequencies might sometimes be of no value to estimate an unknown probability. This enriches the usual discussion about the convergence of relative frequencies by focussing on the underlying assumption of randomness of the data. Analogous examples are abundant but are less emotionally laden than spaghetti.

S. Anastasiadou (Greece) has developed a battery of simple items to research relations between algebraic and graphical skills in student teachers. She used similarity diagrams and came to the conclusion that there is a widespread lack of skill to change between different representations of a task or a notion. This lack of conceptual flexibility seemed to hinder a deeper comprehension of the notions. With different representations, students seem to learn *different* concepts – they do not necessarily notice that the representations deal with the same notion only in a different form.

Impact of Technology

Technology can be viewed in at least two very distinct ways. In one aspect is the media used such as Powerpoint or interactive use of computers by students. The other aspect relates to the software tools used. Some software is generic (eg Excel) and some software is designed specifically for probability such as Fathom. In practice there is more software relating to statistics, though probability software is growing. Fathom and Tinkerplot can be used for efficient calculations and for illustrating key ideas such as the concept of distribution and the law of large numbers, as noted by S. Inzunsa (Mexico) and R. Peard (Australia).

New media indirectly form the backbone of the research of D. Pratt, writing with R. Kapadia (England) on shaping the experience of naïve probabilists. Sequences of the programme ChanceMaker supplied new and challenging experiences to learners in order to shape their intuitions and strategies. There are new challenges for designers of software and teachers using this software. In a fusion of control over the initial parameters (via randomness) and representations of results (histograms for the distribution of data or statistics like the mean), new insights into randomness have been generated. A new world of up-to-date unknown intuitions might emerge, which would affect concepts and their understanding.

J. Watson and S. Ireland (Australia) reported the results of in-depth interviews covering issues on the relations between empirical and theoretical aspects of probability. The class of 12 year olds undertook coin tossing and tabulation of results, followed similar experiments with Tinkerplot, which is becoming a popular piece of software. This widened the children's experience. Some questions remained open for further scrutiny. Can the computer really generate randomness? How can one read diagrams from the software correctly? How can we ensure that the children have sufficient experience in proportional thinking?

Fundamental Ideas

The fundamental ideas of probability include random variables, distribution, expectation, and relative frequencies, as well as the central limit theorem. R. Peard (Australia) has gone to the roots of the subject with questions from games of chance. Games of chance have been partially discredited by their closeness to combinatorics (which is not always easy to understand) and by their artificiality (we want to teach real applications to our students). However, games of chance have spread widely, such as lotteries, and developed to become an important business sector, which is still growing.

M. Borovcnik (Austria) has studied some peculiarities of stochastic thinking, which make it so different from other approaches:
- There is no direct control of success with probabilities – the rarest event may occur and "destroy" the best strategy.

- Interference with causal re-interpretations may lead a person completely astray.
- Our criteria in uncertain situations may stem from "elsewhere" and may be laden with emotions – probability and divination have a common source in ancient Greece.

With these features of stochastic thinking in mind, paradoxes like the stabilising of relative frequencies, even though new events have full-fledged variability, may not seem special. One difficulty lies in a primitive attribution of an ontological character of probability to situations. Probability does not exist – it is only *one of many views* to reflect on phenomena of the real world.

Perspectives for the future

We end with some assertions requiring further research endeavour in probability education:

- People use their experience in order to judge probabilities incompletely and – even worse – in a haphazard manner.
- People have difficulties in judging very small and very high probabilities especially if these are connected to adverse consequences.
- People are inclined to attribute equal chances to the – given or seen – possibilities, especially if there are just two.
- People attribute probabilities and process these into new ones neglecting even the most basic rules (e. g. all probabilities sum to 1).

We believe that sharing and testing ideas across different countries will help promote deeper understanding. In particular, further empirical testing using shared instruments will yield deeper insights.

References and Brief Bibliography

Batanero, C., M. Henry, and B. Parzysz. 2005. The nature of chance and probability. In *Exploring probability in schools: Challenges for teaching and learning*, ed. G. A. Jones , 15–37. New York: Springer.

Borovcnik, M. 2007. New Technologies revolutionize the applications of statistics and its teaching. Invited Paper Session 38: How modern technologies have changed the curriculum in introductory courses, 56. *Session of the ISI Lisboa*. Retrieved from www.stat.auckland.ac.nz/~iase/publications/isi56/IPM38_Borovcnik.pdf July 10, 2009.

Borovcnik, M., and R. Kapadia. eds. 2009. Research and Developments in Probability Education, *International Electronic Journal of Mathematics Education* 4 (3). www.iejme.com

Borovcnik, M., and R. Peard. 1996. Probability. In *International Handbook of Mathematics Education*, ed. A. Bishop, et.al., 239–288. Dordrecht: Kluwer.

Fischbein, E. 1987. *Intuition in Science and Mathematics. An Educational Approach*. Dordrecht: Reidel.

Gigerenzer, G. 2002. *Reckoning with risk: learning to live with uncertainty*. London: Penguin Books.

Heitele, D. 1975. An epistemological view on fundamental stochastic ideas. In *Educational Studies in Mathematics* 6: 187–205.

Jones, G., ed. 2005. *Exploring probability in school: Challenges for teaching and learning*. New York: Springer.

Kapadia, R., and M. Borovcnik eds. 1991. *Chance Encounters: Probability in Education*. Dordrecht: Kluwer Academic Publishers. Online at Google books: books.google.com/books?id=doqeKAICpQ4C&pg=PP1&dq=kapadia+borovcnik&hl=de&cd=1#v=onepage&q=&f=false.

Pratt, D., ed. 2005. Working Group 5 on "Stochastic Thinking". In *European Research in Mathematics Education IV*. *Proceedings CERME 4*, ed. M. Bosch , 481–630. Barcelona: Fundemi IQS – Universitat Ramon Llull. CD ROM. Retrieved from ermeweb.free.fr/CERME4/CERME4_WG5.pdf, July 10, 2009.
Wild, C., and M. Pfannkuch . 1999. Statistical thinking in empirical enquiry (with discussion), *International Statistical Review* 67: 223–265.

Links

Topic Study Group 13 on "Probability" at ICME 11: tsg.icme11.org/tsg/show/14; see also the website of IASE at: www.stat.auckland.ac.nz/~iase/publications.php
Topic Study Group 14 on "Statistics" at ICME 11: tsg.icme11.org/tsg/show/15
Joint ICME/IASE study: www.ugr.es/~icmi/iase_study/

Software

ChanceMaker (n. d.) people.ioe.ac.uk/dave_pratt/Dave_Pratt/Software.html.
Fathom (n. d.) www.fathom.com.
Tinkerplots (n. d.) www.keypress.com/x5715.xml.

The struggle to achieve multiplicative reasoning 11-14

Margaret Brown, Dietmar Küchemann and Jeremy Hodgen

King's College London

Multiplicative reasoning is a key competence for many areas of employment and everyday life, and for further mathematical study. It is however a complex conceptual field. The ICCAMS project, with multiplicative reasoning as one of its two focus themes, has in Phase 1 conducted a broadly representative survey of attainment which suggests that standards in this area have not risen since the 1970s and that relatively few students are achieving competence in the relevant areas of the national strategy Key Stage 3 framework. Student difficulties are illustrated by evidence from group interviews in Phase 2 of the project.

Keywords: Multiplicative reasoning, ratio, assessment, key stage 3

Background: multiplicative reasoning

The ESRC project 'Improving Competence and Confidence in Algebra and Multiplicative Structures' (ICCAMS) selected these two areas of the curriculum because of their pivotal role in the Key Stage 3 (age 11-14) mathematics curriculum, and especially for the further study of mathematics and for its functional application.

While algebra centrally underpins the whole of the further study of mathematics, it is the basis for only the more sophisticated applications. In contrast multiplicative reasoning is the foundation of most mathematical applications and is relevant to all pupils. This paper will examine aspects of performance in multiplicative reasoning that students find difficult and which form a major barrier to developing competence and confidence in functional applications.

The delineation of the conceptual field of multiplicative reasoning is complex (Harel and Confrey 1994, Confrey et al. 2009). While there are other aspects of multiplicative reasoning e.g. those concerned with combinatorics or calculation of areas and volumes, applications of the ratio/rate model are by far the most common and only these will be discussed in this paper.

Broadly speaking, the main contexts of ratio/rate application are those where two or more values are being compared, and/or where one value is being scaled up or down to give another. Sometimes these comparisons or operations are appropriately expressed additively and involve the ideas of difference ('a is d more/less than b'); more often they are multiplicative ('a is r times bigger/smaller than b').

The values being compared may be of essentially the same quantity and the comparison may be related to two or more different 'things', or to one 'thing' at two or more different times (numbers of boys and girls; changing numbers of boys). Especially where the quantity is a discrete variable, or where the multiplying factor is a whole number or familiar fraction, this relationship is often expressed as a ratio a:b. Where the comparison is between two variables which refer to different quantities measured in different units, the relation is usually expressed as a rate a/b which takes the form of a single number to which is attached a composite unit like miles per hour or £ per capita derived from the units in which the two variables are measured. These

comparisons are equivalent in reverse to scaling up a value to get another value, respectively by using a ratio or scale factor, or by using a rate.

Whereas a specific value of a rate is a single number with a composite unit, a specific ratio a:b can be regarded as a set of pairs of numbers which is associated with two 'dimensionless' rates a/b and b/a. Such a dimensionless rate is often described as a proportion, especially when it is expressed as a fraction or percentage and refers to a comparison between one contributing part and a whole collection ('what proportion of the class are boys?'), or to similar geometrical figures ('have the same proportions') . 'Direct proportion' is also used more generally to describe a multiplicative relation between two variables.

Rates often but not always involve the variable of time and are very commonly used in finance and economics (e.g. GNP per capita, rates of interest, and exchange), in the physical sciences (e.g. speed, density, power, pressure) and in health (e.g. rates of growth, medicine doses). Other applications use dimensionless rates (proportions), for example probability and risk, and enlargement through scaling in spatially focused professions such as architecture, design, and engineering.

Thus all the ratio/rate applications have in common the two processes which constitute multiplicative thinking: the derivation of a rate (or ratio or proportion) from two corresponding values (a/b) of two or one variables, and the use of a rate (or ratio or proportion) to calculate an unknown value of one variable given a corresponding value. These require respectively the operations of division and multiplication, with division if anything taking the predominant role.

In our primary curriculum the aspect of multiplication which is still most emphasised is that of repeated addition ('add three five times') rather than that leading to multiplicative reasoning and ratio ('five for every one of three', 'five times larger than 3'). This is even when the ratio meanings have been demonstrated to be easier (Nunes and Bryant 2009).

Multiplicative reasoning starts in the primary school with whole number quantities and whole number scale factors. Later in primary schools and especially in Key Stage 3, it becomes tied in strongly with rational number reasoning. Dickson et al (1984, 287) note there are two meanings of rational numbers, those related to measurement (1.7 metres) and those to operators – essentially rates or scale (multiplying) factors (1.7 times as large); Confrey et al. (2009) separate off from the latter a further meaning, that of ratio, but there seems little justification for doing so since ratios are so closely related to the operators of rates and scale factors.

Again, the non-ratio 'measurement' meanings of rational numbers as a measure of a fractional part of a spatial whole, or as a decimal number on a number line, are the ones that tend to be emphasised in the primary curriculum, although there is some mention of meanings concerned with ratio ('3 for every 5', '3/5 of').

Perhaps not surprisingly, given the complexity of the conceptual field on which it depends, and current emphases in primary schools, in the early years of secondary school students experience problems with multiplicative reasoning. This paper explores some of these using early results from the ICCAMS project.

Methods: the ICCAMS project

Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) is a 4-year research project funded by the Economic and Social Research Council as part of a wider initiative aimed at identifying ways to participation in Science, Technology, Engineering and Mathematics (STEM) disciplines.

Phase 1 of the project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England. This is followed in Phase 2 by a collaborative research study with teacher-researchers who are using the results of Phase 1 on their own classes as part of extending the investigation to classroom/group settings. The aim is to examine how formative assessment can be used to improve attainment and attitudes, and finally how the work can be disseminated on a larger scale. Currently we have had the first year of the survey and the exploratory year of the collaborative study.

In Phase 1, comparison with the Concepts in Secondary Mathematics and Science (CSMS) study (Hart et al. 1981) will also enable us to examine how students' understandings have changed since 1976. The survey consists of three of the CSMS tests, Algebra, Ratio and Decimals, and an attitudes questionnaire. Some selected items from the original CSMS Fractions test were appended to the Ratio test. Results discussed here relate mainly to the Ratio test and part of the Decimals test.

In late June or early July 2008, tests were administered to a sample of approximately 3000 students from approximately 90 classes in 11 schools as shown in Table 1. Each student took two of the three tests so as to provide comparative information between tests but not to overload students. The numbers of students in each year-group taking each test is therefore roughly two-thirds of the total number of students involved in that age group in 2008.

Test	Year 7	Year 8	Year 9	Total
Ratio(+fractions)	680	754	588	2022
Decimals	717	767	598	2082

Table 1: Numbers taking the two tests in the 2008 sample

The sample of 20 schools involved in the survey across 2008 and 2009 is a stratified random sample drawn from MidYIS, the Middle Years Information System. MidYIS is a value added reporting system provided by Durham University, which is widely used across England (Tymms and Coe, 2003). Because some schools were not able to complete the testing in 2008, the 2008 sample for which results will be reported on an interim basis is not fully representative and is slightly skewed towards higher attaining students. The original CSMS test on Ratio was administered to a sample only from Years 8-10, although Decimals were used with Years 7-10 and Fractions with Years 7-9. In the current study, because the focus was on Key Stage 3, all the tests were administered to students in Years 7-9.

Ratio items, some Decimals items, and almost all of the small set of Fractions items used in this study, were designed to assess whether students could apply multiplicative reasoning in increasingly complex situations.

In the original CSMS data analysis in the 1970s, items were selected from each test to form a series of hierarchical levels of difficulty. These items included 20 out of the 27 items in the Ratio test and 39 out of 72 items in the Decimals test. The criteria for selected items and levels were consistent performance across different age-groups in the sample, high levels of correlation within the items for each level and strong hierarchical relationships between items in different levels (Hart et al. 1981). The remaining items that were retained were for diagnostic purposes. There were different numbers of levels associated with each test, since the levels were derived empirically; Ratio had 4 levels and Decimals had 6. The additional 15 items on fractions which were added to the Ratio test to better assess multiplicative reasoning, were drawn from items which appeared in the hierarchy of the CSMS Fractions test.

However in the case of Fractions only single item comparisons with the 1976 data will be possible since the full set of Fractions items was not used.

Students were judged to have been successful at a specific level if they had successfully answered two-thirds or more of the items at that level. Students who had not achieved two-thirds of Level 1 items were said to be 'at Level 0'. It was possible to broadly describe the type of mathematical understanding required for the items in each level in each topic, although these were not always neat descriptions since the items and levels were assigned on empirical not theoretical grounds.

As part of the ICCAMS study, the CSMS tests were checked against the current version of the National Curriculum and appeared to form just as appropriate assessments as they had done in the 1970s. Piloting indicated that only minor updating of language or context for a very small number of items was required which would be unlikely to significantly affect their difficulty.

Phase 2 represents the intervention part of the project in which the aim is to design, trial and disseminate an intervention based on formative assessment practices in Year 8 in the areas of algebra and multiplicative reasoning. Two Year 8 teachers from each of four schools are part of the research team, with the schools having a range of intakes and attainment. The classes involved in 2008/9 covered almost the full range of Year 8 attainment. The work in the first year has been partly exploratory, with all members of the research team undertaking several school-based activities and reporting back in team discussions. These activities have included observing student responses in lessons relating to algebra and multiplicative reasoning, interviewing small groups of Year 8 students about aspects of their mathematical understanding, and trying out starters and other class activities as diagnostic tools.

Results and discussion

Because the full analysis of CSMS test results on the complete representative sample including those tested in Summer 2009 is not yet available, this account of Phase 1 results must be interpreted with some caution as it may differ somewhat from the final outcomes, especially because the mean MidYis score of the 2008 sample was a little higher than the national average.

The proportional bar-charts for the ratio results for Years 8 and 9 are shown in Figures 2 and 3 respectively. These suggest that in both year groups, as in other tests, there is a higher proportion below Level 1 in 2008 than in 1976. The reasons for this general finding are not yet clear. At the top levels the picture is more encouraging, although the improvements over 1976 are small enough to possibly disappear when the sample is larger and better balanced.

The verbal description of the levels is:

Level 1: Simple integer rates like ×2 or ×3 but including halving

Level 2: Rates like ×1.5 set in contexts that allow a 'building up' approach such as taking an amount then half as much again

Level 3: Problems involving fractional quantities or rates such as ×1^2/3 set in contexts that allow an indirect 'building up' approach or which prompt the unitary method

Level 4: Non integer enlargement (e.g. 5:3) or problems set in a context involving scaling continuous quantities where a 'building up' approach (or the unitary method) is not meaningful
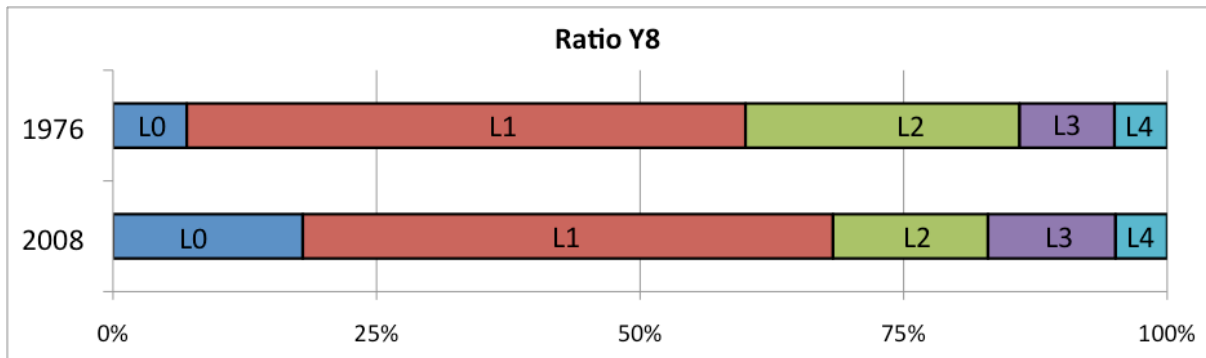
Figure 2: Proportions of Year 8 students assessed at the different CSMS levels for Ratio in 1976 and 2008   [N= 754 (2008), 800 (1976)]
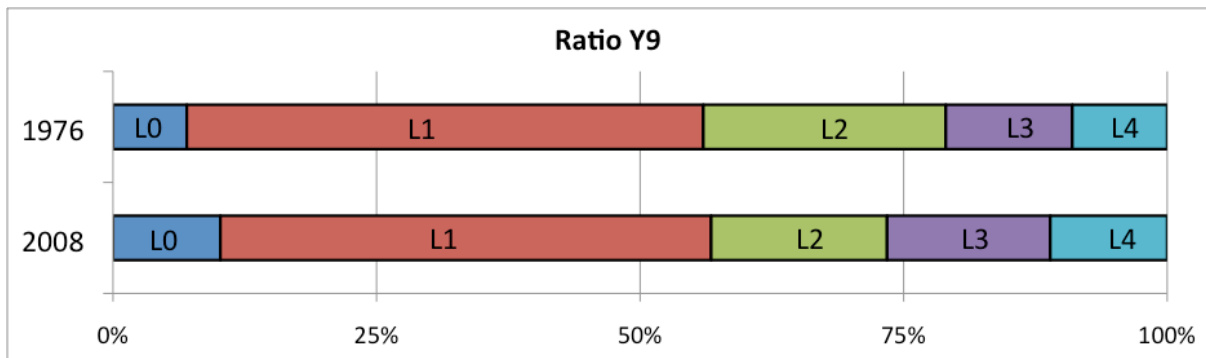
Figure 3: Proportions of Year 9 students assessed at the different CSMS levels for Ratio in 1976 and 2008   [N= 588 (2008), 767 (1976)]

So the results in Figures 2 and 3 suggest that two-thirds of Year 8 students and well over half Year 9 students cannot consistently manage anything in terms of rate and ratio other than reasoning involving whole number multiplication, or division by 2, with about 20% at Year 8 and about 10% at Year 9 not even managing this reliably.

Currently it looks as though over 15% at Year 8 and over 25% at Year 9 can manage non-unitary ratio where some intermediate step is reasonably easy to spot. While only about 5% of Year 8 and 10% of Year 9 are fully operational with ratio, this proportion is equivalent at Year 8 and slightly better at Year 9 than results for 1976.

Comparing these levels with the national strategies framework for Key Stages 3 and 4, there is not a direct match but it appears that in Years 7 and 8 students are expected to be taught the content at Level 3, and Level 4 is intended to be covered by the end of Year 9. (Some aspects of enlargement only appear in Year 10 but the examples given for Year 9 seem to suggest that calculation and use of scale factors like 0.7 is to be covered there.) The results therefore suggest that under 20% of students are succeeding on this material.

The proportions at different levels for the Decimals test were generally better than in 1977, except at the highest and lowest levels in Year 9. Most of the levels were defined in relation to understanding of the decimal place value system which arise in the measurement aspect of decimals, but part of the highest level is closely related to the idea of multiplicative reasoning:

Level 6: Decimals which result from division (and the existence of an infinite number of decimals)

The proportion who reached Level 6 was about 10% at Year 8 and 15% at Year 9. The Year 8 result was better than in 1977 and the Year 9 result was worse, but again the margins were small and are thus unreliable. This result ties in well with the results for ratio as the understanding that you can obtain a decimal answer from a whole number division (and the ability to find it with an easy divisor, such as 20) is needed for the highest level, Level 4, of the Ratio test. Again this Level 6 content is covered in the national strategy framework by the end of Year 8, but by the end of Year 9 still only 15% seem to have grasped it.

Looking at the proportions of items in each test on which there is significant improvement, significant reduction, or no significant change (Figure 4), it becomes clear that while there are significant improvements on over 50% of Decimal items, there are hardly any items with improvements on the Ratio test and none from the Fractions test.
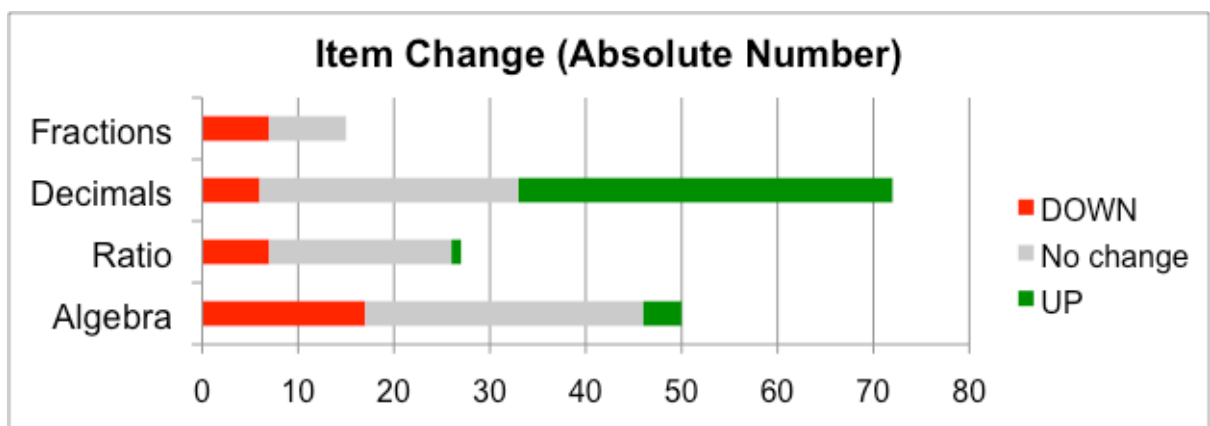
Figure 4: Numbers of items on which success rates have significantly increased or decreased between 1976/7 and 2008 in each mathematical topic using Year 9 data

Again many of the Decimals items with significantly improved scores were related to Decimals used for measurement rather than as operators. This leaves us therefore with the result that multiplicative reasoning seems to be generally weak and if anything to have deteriorated since the 1970s.

This is illustrated in Figure 5 by detailed results for one of the Level 4 items in the Ratio test. Although the ratio is an easy one of 2:3, the fact that the problem cannot be readily solved by the unitary method or by an additive strategy makes it difficult. Although enlargement first appears in the Year 8 strategy framework, and with non-whole number factors in Year 9, their answers indicate that many students at the end of Years 8 and 9 do not seem to appreciate that enlargement requires multiplicative rather than additive reasoning.

The results for both parts of the question are similar, as are the results for 1976 and 2008; the fact that the small initial lag in the 2008 results at the start of Year 9 closes by the end may be due to the inclusion of non-whole number enlargement in the Year 9 framework. However, even at the end of Year 9, only 20% of students can solve the problems.

To illustrate student thinking a summary is given of discussions on this 'Kurly Ks' item with two groups of Year 8 students, both from London comprehensive schools with GCSE results well above the national average.

These two letters are the same shape. One is larger than the other.
AC is 8 units. RT is 12 units.

The curve AB is 9 units. How long is the curve RS ? . . . .

The curve UV is 18 units. How long is the curve DE ? . . . .

——7a: K: RS 12/8 x 9
——7b: K: DE 8/12 x 18
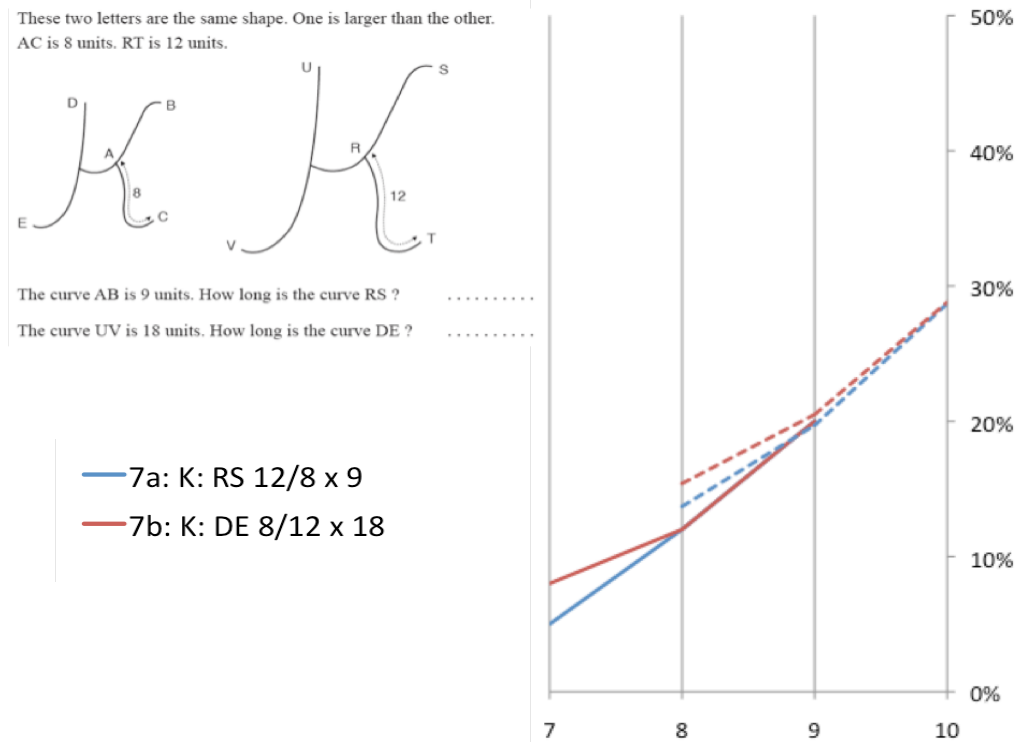
Figure 5: Success rates for Years 7 to 10 on an enlargement item in 1976 (dotted lines) and 2008 (unbroken lines)

Dietmar and Margaret with Nasreen, Ahmed, Maniyan and Susan (middle set) 28/11/08.   Ahmed quickly launches in with '13 units… 4 more there... cos that's 9, that will be 4 more, the same...'. Others agree… They all went for an addition strategy, either 9+4 or 12+1, to get 13. Dietmar then suggests: 'Say instead of this being 12 we made it 16... and this bit's still 9...'. Nasreen sticks to the addition strategy: 'You add 8 now'. Ahmed agrees. Margaret suggests that instead of adding 8 you could double both numbers. Three of them say that this isn't right, you have to add 8. (Not sure about Maniyan, whose English is weak.) Margaret asks what the answer would be if you did double the 9. Ahmed: '18, and it's supposed to be 17'. (Susan agrees, and Nasreen I think.) They seem to regard doubling as a special case. Ahmed: 'You don't just do times 2 because it works on one answer'. Later Susan adds 'You're going to have to double if it's twice as big, but if it is just under twice as big you can't double everything.' When Dietmar then sketched similar rectangles (8 by 9 and 16 by an unknown number) they tended to prefer 'doubling' to find the unknown side i.e. preferred the answer 18 to 17.

Dietmar and Jeremy with Bethan, Zack and Danny (top set) 10/11/08. Starts with a long silence...little whispers...Bethan: '8 is kinda equal to 12, in the same way that 9 is equal to RS'... pause…Dietmar asks what the next step would be. Bethan says she is still thinking... Zack: 'Don't really get... I was thinking... if the 8 here (points) and that's 9, you plus it to find out that? But...' [I now think he is working just on the little K and wants to add the 8 and 9 to find DE, but then thinks it might not work as the lines are curved – but at the time we didn't appreciate this] He returns to RS on large K and now suggests 13. Bethan says she's still not sure.... Jeremy to Zack: 'Why 13?' Zack: 'Difference is 4... larger by 4... so this (RS) should be larger (than the 9) by 4.. if it's an accurate enlargement'. Bethan: 'I think it's 13 and a half... cos 8 up to 12 is two-thirds... so 9 is two-thirds… so I halved 9 which is 4.5 then add it onto 9 so you get…

Jeremy: 'You think 13, you think 13.5...' He asks Bethan to explain again. She now uses multiplication rather than adding in the final step: 'You halve it, then times it by 3 to get 3 thirds'. Zack later sticks to his additive strategy: 'Like 8 and 12... the difference 4, hey... if you enlarge the smaller shape to get the bigger shape, wouldn't everything have to have the same... difference?' Bethan: 'I'm not sure if you have to add something or times it by something...'.

These interview quotations confirm that students from the middle of the attainment range, and also some from nearer the top, tend to use an addition strategy for enlargement. They reluctantly allow some exceptions, for example doubling, but do not see that they could use a multiplicative factor other than a whole number. (Susan's final comment is revealing). It is only a few higher attaining students, like Bethan, who are able to even if tentatively see their way through to multiplication strategies, even when easy scale factors as here allow 'building up' strategies.

Conclusions

Multiplicative reasoning is clearly an area which is key to both a large number of mathematical applications and to further study in mathematics. Yet the evidence presented here suggests that students have a very weak grasp of important aspects of the conceptual field and that understanding in this area has not improved since the 1970s. This is not because these things are not taught as most of the students in the sample will have experienced teaching of the relevant ideas. We have to conclude that the teaching has not been very successful, and it is possible to speculate on many reasons why this should be so. For example it may not have been appropriately related to students' prior understandings, or may not have been sustained enough and broad enough to have had a permanent effect. In Phase 2 of the project we will be developing an intervention based on formative assessment and more sustained periods of teaching to see if this improves the learning of these important ideas.

References

Confrey, J., A. P. Maloney, K. H. Nguyen, G. Mojica, and M. Myers. 2009. Equipartitioning / splitting as a foundation of rational number reasoning using learning trajectories. In *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, edited by M. Tzekaki, M. Kaldrimidou and H. Sakonidis. Thessaloniki, Greece: PME.

Dickson, L., M. Brown, and O. Gibson. 1984. *Children learning mathematics: A teacher's guide to recent research*. Eastbourne: Holt, Rinehart & Winston Ltd.

Harel, G., and J. Confrey, eds. 1994. *The Development of Multiplicative Reasoning in the Learning of Mathematics*. Albany, NY: SUNY.

Hart, K., M. L. Brown, D. E. Küchemann, D. Kerslake, G. Ruddock, and M. McCartney, eds. 1981. *Children's understanding of mathematics: 11-16*. London: John Murray.

Nunes, T., and P. Bryant. 2009. Paper 3: Understanding rational numbers and intensive quantities. In *Key understandings in mathematics learning*, edited by T. Nunes, P. Bryant and A. Watson. London: Nuffield Foundation. Available from www.nuffieldfoundation.org, accessed 9 October 2009.

Tymms, P., and R. Coe. 2003. Celebration of the Success of Distributed Research with Schools: the CEM Centre, Durham. *British Educational Research Journal* 29 (5): 639-653.

Connecting mathematics in a connected classroom: Teachers emergent practices within a collaborative learning environment

Alison Clark-Wilson

The Mathematics Centre, University of Chichester

During 2008-9 seven secondary mathematics teachers from England, Scotland, Netherlands and Sweden began to use a wireless classroom network to link their students' handheld ICT devices. This paper focuses on the teachers' reported uses of the Screen Capture feature, which were coded to reveal patterns in the emerging classroom practices. Analysis of the data revealed: increased opportunities for purposeful classroom discourse; improved formative assessment practices; and highlighted the need for teachers to choose rich examples on which to build the mathematical tasks.

Mathematics, ICT, Teacher development,

The Research Context

Following on from the English TI-Nspire[TM] handheld pilot evaluation in 2007-8 (reported in Clark-Wilson 2008) Texas Instruments funded a second project which introduced the TI-Nspire[TM] Navigator[TM] wireless classroom network system to the existing handheld classroom environment. Two of the teachers involved in the English project joined teachers from Sweden, Netherlands and Scotland in beginning to develop viable classroom approaches and evaluate their associated outcomes. Of particular interest were the ways in which the teachers perceived that the introduction of the network changed their classroom environment and the nature of the tasks that they could set. The focus of the pilot was limited to the following TI-Nspire[TM] Navigator[TM] functionalities: File management, Screen Capture; Quick Poll; Live Presenter; Class Analysis. This paper will report specifically on how the teachers used the Screen Capture feature. Whilst the cultural aspects of mathematics teaching within the different country settings is of great interest, this is not addressed within this paper due to the limitations of space.

Theoretical framework

At the heart of this study was an exploration of teachers' classroom practices, and in particular those which enhanced and promoted opportunities for student (and teacher) learning, resonating strongly with research into formative assessment practices. The interpretation of formative assessment adopted by the researcher throughout this study (and shared with the teachers) is best articulated by Black and William who write:

> [Assessment] practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited. (Black and Wiliam 2009, p. 9)

Consequently, the research focused on providing opportunities for teachers to describe and evaluate their practices, and in particular on the way in which the

information gained through the use of the technology impacted upon their decisions and actions in the classroom. A number of previous research studies into the use of TI-Navigator[TM] with TI-84 graphing calculators in the United States, Canada and France had concluded that the opportunities for formative assessment had been enhanced (Dougherty 2005; Hivon, Pean, and Trouche 2008; Sinclair 2008). For example, a pilot study on one classroom in Hawaii concluded,

> The use of TI-Navigator technology supports the development of a collaborative classroom environment by enhancing student interactions, focusing students' attention on multiple responses, and providing opportunities for students to peer- and self- assess student work. The ability to display a full class set of data or task responses supports a problem-solving approach to developing skills and concepts. (Dougherty 2005, p.28)

In addition, the notion of a 'collaborative classroom' was a reported outcome of a French study (Hivon, Pean, and Trouche 2008) which also suggested that the TI-Navigator[TM] had changed the nature of the mathematics classroom environment by:

> offering an opportunity to change the nature of classroom mathematics tasks;
> offering new opportunities for both cooperative and collaborative group learning;
> and increasing the complexity of the teacher's role with respect to 'orchestrating'
> the lesson. (Hivon, Pean, and Trouche 2008)

The notion of 'orchestration' expands upon the complex nature of the teacher's role in 'conducting' a lesson using networked handheld technology, with a number of researchers offering emergent theories on this theme (Trouche 2004; Drijvers and Trouche 2008; Roschelle and Pea 2002).

Finally the nature of the task design with technology was an important consideration with the study, and in particular the choice of mathematics examples (which constituted the task) and the resulting classroom discourse (Ainley, Pratt, and Hansen 2006; Goldenberg and Mason 2008). The classroom network offered a new opportunity for the design of tasks, although this paper is not reporting specifically on this outcome (see Clark-Wilson 2009 for a fuller account).

The Research Approach

The research sought to collate evidence from the teachers about the aspects of the classroom use of TI-Nspire[TM] Navigator[TM] that promoted desirable classroom pedagogies and the nature of the rich mathematical starting points that might lead to enhanced student engagement and achievement in mathematics. A set of subsidiary questions were developed for each of the system's functionalities and, for the Screen Capture feature, these were:

- What was the nature of the 'rich' mathematical starting points for which Screen Capture stimulated pupils (and teachers) to learn mathematics?
- How could Screen Capture be used to maximise the opportunities for students' peer communication with (and independent from) the teacher;
- What classroom strategies did the teachers devise to use data from Screen Capture to support them in all aspects of formative assessment?

The teachers met for an initial one day meeting at which they shared some of their early classroom approaches and the project community begun to be established. At this meeting they were introduced to some of the technical aspects of the system

and the research protocols through which they would report their classroom experiences. The group also spent time, with expert support, developing some starting points for mathematical activities which used the collaborative sharing of students' handheld screens as a fundamental aspect of the task design. I stimulated this discussion by offering a number of simple activity ideas which exemplified this approach. For example, by asking students to all enter a function which they thought would pass through a particular coordinate point on a common set of axes.

Each teacher was asked to complete a detailed lesson evaluation (Less-eval) for at least 3 selected lesson taught during the Spring and Summer terms 2009 and submit them with any relevant accompanying evidence which typically included students' work, Screen Capture images, students TI-Nspire[TM] handheld files, lesson plans and the teacher's own reflective comments. The nature of this lesson evaluation process was reported in an earlier BSRLM paper (Clark-Wilson 2008). In brief it provided a structure for teachers to record information about the class, the mathematics to be taught and to tell the story of the lesson and its activity, whilst encouraging teachers to reflect upon the multiple and various outcomes. This data was analysed using a constant comparison approach, influenced by grounded theory (Glaser and Strauss 1967) through which a number of emergent practices became evident. This data analysis was supported by the use of Nvivo7 software (QSR International 2008) which facilitated the data analysis process and the systematic coding and grouping across the range of data types.

In addition, each teacher was observed at least once using TI-Nspire[TM] Navigator[TM] with a class and the teachers also participated in a semi-structured interview (Int) directly after each lesson which probed specific choices and actions they had taken during the lesson.

So what is the Screen Capture feature within TI-Nspire[TM] Navigator[TM]?

Normally, when working with a class who are using a handheld device such as a scientific calculator, graphical calculator or TI-Nspire[TM] handheld, the individual screens are private to the user and teachers would observe the students' mathematical activity by moving around the classroom or, with some devices, inviting students to connect their device to a whole class display.
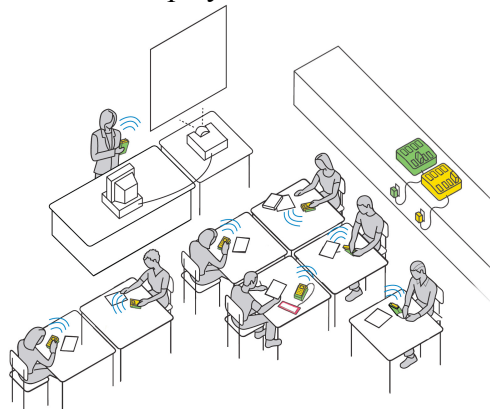
Figure 1 The TI-Nspire[TM] Navigator[TM] classroom network

The TI-Nspire[TM] Navigator[TM] classroom network requires students to log into the 'class' which enables the teacher to use a range of features, one of which is to choose to display all of the students' handled screen simultaneously. Using one of the reported lesson examples, Figure 2 shows an example of this view.
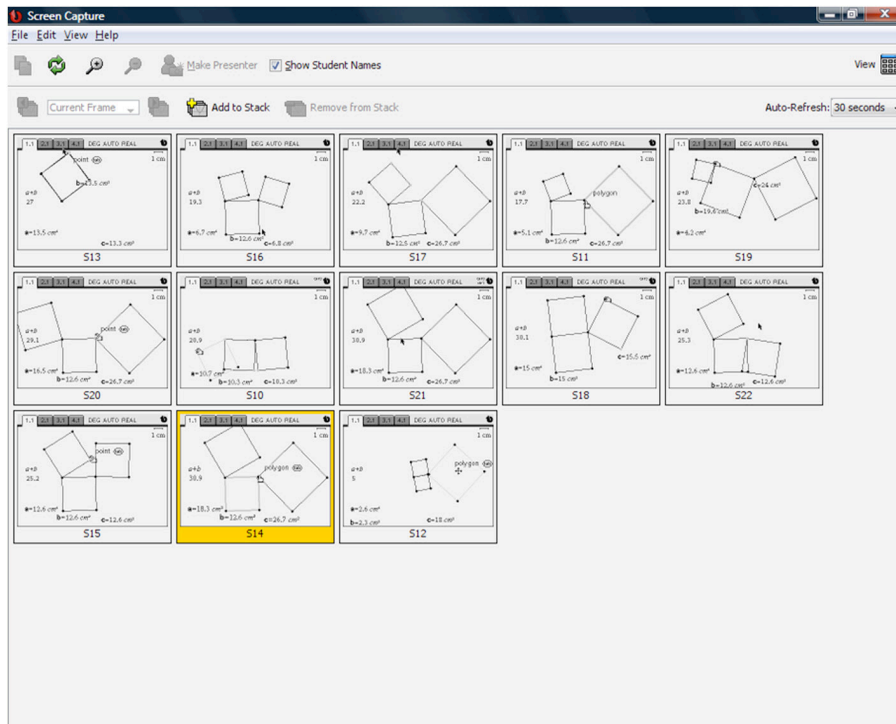
Figure 2 A Screen capture view of the students responses to a mathematical task

When using Screen capture, each teacher chose whether to display this privately on their own computer or display it to the class using a data projector. This paper will focus on examples where the teachers chose to make it a central feature of their task design and associated mathematics pedagogy.

Research Results

The Screen capture was used in 23 of the 25 lessons that the teachers reported to the study and teachers were unanimously positive about their experiences. The data analysis led to the following categories of use: monitoring students activity during the lesson; supporting teachers to know when to intervene; promoting and initiating whole class discourse; promoting and supporting peer- and self- assessment; privileging mathematical generalisation; increasing sample sizes (within statistical work); and enabling mathematical sorting.

What follows is a description of each of these usage categories with supported by the teachers' evaluations based on their classroom experiences.

Monitoring student activity during lessons

All of the teachers reported that the use of Screen Capture had enabled them to monitor students' activities whilst they were working with TI-Nspire[TM] during lessons. The rationale for this monitoring varied, with some teachers using this functionality in a passive way to check 'that every student is working – doing what they were supposed to do' [Int] and 'to ensure that everyone got to the first page okay' [Int]. There were a small number of lessons where the teachers used Screen Capture privately on their own computer (e.g. whilst the students were completing a test). However, in the vast majority of lessons the Screen Capture display was on public view in the classroom. This prompted a much more active use about which the teachers were highly enthusiastic, namely the insight that Screen Capture gave into how their students were engaging with the tasks they had been set.

Some teachers said that they were monitoring the students' work for pace, both to observe the students' natural work rates and to impose an expected work rate. One teacher said it was to 'spur on the learning' [Less-eval] and another 'to selectively sample the class work and drive the activity forwards by challenging them with new situations they had constructed' [Less-eval]. One of the notable comments made by several teachers related to an appreciation that the pace at which their students worked was much more widely distributed than they had previously appreciated.

Several teachers commented specifically on the value of Screen Capture as a window into their students' progress through the lesson. In one lesson evaluation, the teacher commented,

> Screen Capture gave me instant feedback throughout regarding student progress. As a teacher you develop a sense of how a lesson is progressing based on the level of noise, the snippets of conversation that you hear, etc. However, in this lesson I was able to 'see' what students were doing. This adds to the other sensitivities that I have so that I am able to make better judgments. [Less-eval]

Other teachers reported that they were monitoring specifically for the purpose of identifying exemplar solutions and common (or particular) mistakes, which in some cases led to an individual or whole class intervention by the teacher.

Supporting teacher interventions

In all cases, the reported use of Screen Capture led to some action or intervention on the part of the teacher. These ranged between interventions involving individual students (or groups of students) to the initiation of focusing activities involving the whole class.

By far the majority of teacher interventions related to those of a pedagogical and mathematical nature concerning the task itself. In many cases the teachers used the initial Screen Capture view to identify particular students (or their screens) as the focus for the whole class discourse. In some cases this led to the identification of one student to become Live Presenter[1], and in other cases to highlight particularly interesting solutions or responses.

Some teacher interventions following the use of Screen Capture arose from their observation of issues the students were experiencing with their use of the TI-Nspire[TM] files. For example 'I adjusted my advice for how to keep the triangles on the screen by reducing all of the lengths to smaller numbers' [Less-eval] and 'I had to use Screen Capture to see why they did not find any connection... I could see they had dots everywhere so I had to resend the file.' [Less-eval]. This also supported the teachers to appreciate issues within the design of the TI-Nspire[TM] file by 'learning from how the tns file was interacted with and what things could too easily be 'broken' with it' [Less-Eval].

There was evidence in their lesson evaluations to suggest that, as the project progressed, the teachers were becoming more aware of knowing when to intervene and who to intervene with and also used the information from Screen Capture to inform them as to when the students were ready for them to progress with the lesson content. This took the form of suggestions as to how they might manage different pedagogic situations differently or how they would redesign tasks to build in opportunities to solicit students' responses.

---

[1] The Live Presenter feature enabled the selected handheld display (and key presses) to be wirelessly displayed to the whole class through the data projector.

Promoting whole class discourse

A common reason cited by the teachers for the use of Screen Capture was with the intention that it would promote discussion and communication in the classroom. The outcomes of this were positively reported by one teacher, who said 'Seeing each others' work gave a wider discussion through the class than what normally happens between students seated next to each other' [Less-Eval]. The nature of this discourse varied. In the observed lessons it tended to be a more teacher dominated discourse of instruction and explanation, but there was evidence in two classrooms of the teacher using the Screen Capture view as a prompt for the students to discuss amongst themselves prior to feeding back their thoughts to the whole class.

Promoting and supporting peer assessment

The diversity of student outcomes that Screen Capture made public resulted in several examples from the classrooms in relation to the opportunities for both peer- and self-assessment. The following quotes provide rich evidence:

> The students could 'learn from each other and see their mistakes'. [Less-eval]

> Some students also watched to get ideas and support. [Less-eval]

> The sharing of thoughts at the half-way stage led in several cases to students checking some of the declarations that had been made by their peers. [Less-eval]

> It is really fantastic to see how much the students learn by just looking at each other on Screen Capture. [Less-eval]

Other teachers were more specific about how the use of Screen Capture had supported the students to communicate their mathematical ideas and associated thinking, again leading to improved responses as a result of viewing the responses of their peers.

Privileging mathematical generalisation

An initial aim for this pilot study was to actively encourage the teachers to consider the mathematical topics for which sharing a variety of students' responses would have a clear mathematical purpose. The teachers responded to this challenge very positively and some rich mathematical examples were generated. The underlying principle for all of these tasks was to create a shared learning space through which all students' contributions contributed to the 'big picture' and, with careful teacher questioning and prompting, the students were able to make sound mathematical generalisations. The sort of mathematical aims that the teachers proposed were for students to explore the function family y=ax 'to see how the value of a will have a different impact on the graphs' [Less-eval] and 'that there are infinite number of straight lines through a certain point' [Less-eval].

The major feature of these lessons was the emphasis on similarity and difference (two fundamental mathematical concepts) and the teachers made particular comments in relation to both of these with respect to how Screen Capture facilitated a new approach to the mathematics:

> After some exploration there were 16 screens with all different situations. [Less-eval]

> I wanted the students to 'see the different lines that they had all drawn through the same point' [Less-eval]

> I wanted Screen Capture to 'show a variety of results... the idea being that we could all see that right angled triangles gave the right solution'. [Less-eval]

One teacher justified this approach by saying 'I needed every student to see many different screens to come to the generalisation. If they only see their own screen they lack a global view.' [Less-eval]. In these examples Screen Capture enabled the use of the shared learning space to focus the students on the common part of the task. This sense of a collaborative learning environment was captured by another teacher who wrote, 'We are in this together – the knowledge is shared – not 25 individuals' [Less-eval]. Several of the teachers commented that they would be keen to further develop the mathematical TI-Nspire[TM] activities which would exploit this use of Screen Capture.

Increasing sample sizes (within statistical work)

An interesting mathematical use of Screen Capture emerged from three lessons in which the Screen Capture view was used with purpose within lessons involving work in statistics. In these three cases the teachers explicitly commented on how the mathematical relevance of sample size could be promoted to the class by seeing everybody's data. In one case this was 'to show that when students showed the same RandSeed number, all screens in the next Screen Capture showed the same random number' and in another to 'take a snapshot of how many students were getting a better estimate – to tally up-front'. This suggests a potential use of Screen Capture which might be worth developing further within the design of TI-Nspire[TM] activities for use with TI-Nspire[TM] Navigator[TM].

Enabling mathematical sorting

One lesson attempted to use the students' Screen Capture displays as objects for sorting into those which did and did not have a particular mathematical feature. However, the current functionality of TI-Nspire[TM] meant that, as the screens automatically 'snap to grid' it was not so easy to make obvious to the students where the boundaries of the groups lay as the screens were continuously reordered in fixed grid layout. This also highlighted the facility to colour code the individual screens as an alternative way to draw attention to particular mathematical features as potential future functionality within Screen Capture view.

Conclusions

The analysis of the emergent practices that the teachers developed revealed that TI-Nspire[TM] Navigator[TM] was used to:

- develop new and support existing formative assessment practices;
- enable the development of innovative mathematics tasks;
- support teachers' lesson planning to include desired pedagogical approaches, lesson organisation and classroom management strategies;
- support the use of the TI-Nspire[TM] handhelds for individual and whole-class work.

In conclusion the research reported in this paper has provided an insight into the emergent practices of teachers' uses of one of the features of the TI-Nspire[TM] Navigator[TM] system. The findings suggest that the ICT offers opportunities for

teachers to both rethink the design of ICT-based mathematical tasks for collaborative learning approaches and their underlying pedagogical approaches.

References

Ainley, Janet, David Pratt, and A Hansen. 2006. Connecting engagement and focus in pedagogic task design. *British Educational Research Journal* 32 (1):pp. 23-38.

Black, Paul, and Dylan Wiliam. 2009. Developing the theory of formative assessment. *Educational Assessment Evaluation and Accountability* 21 (1):pp. 5-31.

Clark-Wilson, Alison. 2008. Evaluating TI-Nspire[TM] in secondary mathematics classrooms: Research Report. Chichester, UK: University of Chichester.

———. 2008. Teachers researching their own practice: Evidencing student learning using TI-Nspire. In *BSRLM Proceedings 28 (2)* University of Southampton.

———. 2009. Connecting mathematics in the connected classroom: TI-Nspire[TM] Navigator[TM]. Chichester, UK: The Mathematics Centre, University of Chichester.

Dougherty, Barbara. 2005. TI-Navigator™ Technology and Algebra I. Hawaii: Curriculum Research & Development Group, University of Hawaii.

Drijvers, Paul, and Luc Trouche. 2008. From Artifacts to Instruments: A Theoretical Framework behind the Orchestra Metaphor. In *Research on technology in the learning and teaching of mathematics: Syntheses, cases and perspectives. Vol 2 Cases and perspectives*, edited by G. Blume and K. Heid. Charlotte: National Council of Teachers of Mathematics/Information Age Publishing.

Glaser, B., and A. Strauss. 1967. *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine NY.

Goldenberg, Paul, and John Mason. 2008. Shedding light on and with example spaces. *Educational Studies in Mathematics* 69 (2):pp. 183-194.

Hivon, L, M Pean, and L Trouche. 2008. From a network of calculators to collaborative knowledge construction in the class. *Repères-IREM* 72.

QSR International. 2008. *Nvivo8*. Melbourne: QSR International.

Roschelle, Jeremy, and Roy Pea. 2002. A walk on the WILD side: How wireless handhelds may change computer-supported collaborative learning. *International Journal of Cognition and Technology* 1 (1):145-168.

Sinclair, Margaret 2008. TI-Navigator Study Interim Report. Ontario, Canada: York University.

Trouche, L. 2004. Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning* 9:281-307.

Roles of Research in the Professional Development of Mathematics Teachers

Els De Geest

The Open University

This paper reports on an empirical study which investigated ways in which research is presented in CPD initiatives and the impact research utilisation in professional development has on teachers. Data consisted of qualitative responses to on-line and electronic questionnaires, field notes of discussions and observations which were validated by the participants. Data was analysed using a process of constant comparison (grounded theory). The analysis offers descriptive categories for ways in which research is used in CPD and for effects research utilisation has on teachers. We conclude that using research is an effective means, although perhaps not the only one, for teachers to becoming aware of different perspectives about teaching and learning, to engage in deep thinking and to gain confidence in their thinking.

Keywords: research utilisation, professional development

Introduction

Hargreaves (1996) opened the annual lecture to the Teacher Training Agency in 1996 with the statement "Teaching is not at present a research-based profession" and he argued that teachers were not using research in their professional decision making as they considered research to offer little help with or give useful insights into the practical issues of the classroom.

There is a wealth of recent research on professional development for mathematics teachers. Joubert and Sutherland (2008) conducted a review of such literature including some 200 publications and categorised the literature in five sections: what is meant by CPD, complexity of professional knowledge; professional knowledge for teachers of mathematics; change and professional development; and important characteristics of 'effective' CPD. None, however, seemed to address research utilisation. In a broader context of education in general, there seems to be some although limited research about the use of research in the professional development of teachers. This is confirmed in the systemic literature review on the use of research to improve professional practice in education by Hemsley-Brown & Sharp (2003) who argue "while the subject of practitioners' use of research has attracted considerable debate, there is limited empirical evidence on this subject" (Hemsley-Brown & Sharp, 2003, p454). They considered Zeuli's work (1994) on how 13 US teachers use research findings as offering the most pertinent evidence based study. Zeuli suggests teachers only consider using research findings or find research credible when it matches their own personal experience in classroom practice. He therefore advocates more sustained opportunities to link findings from research with classroom practice. Rhine (1998) argues similarly when he explores the value of research for the development of teachers' knowledge base in professional development, referring to research-based professional development programs such as the American Cognitively Guided Instruction (CGI) and Integrating Mathematics Assessment (IMA). He also suggests that as research offers so much knowledge and

teachers can not retain, process and utilise this at one moment in time, research should be an available resource for teachers when they feel they need to and teachers should be offered professional development to help them accumulate that knowledge.

The study we report on in this paper was part of the Researching Effective CPD in Mathematics Education (RECME) project, a large research project funded by the National Centre for Excellence in the Teaching of Mathematics (NCETM) and explored factors of effective CPD for mathematics teachers. One of its five aims was to identify the different roles, if any, research plays in professional development undertakings. In this paper we aim to give further insights into these roles through descriptive categorisation of empirical evidence we obtained from researching 30 ongoing CPD initiatives.

The Study

The RECME project was a short term (15 months) project with a data collection period from January to July 2008. The non-interventionist project investigated 30 ongoing CPD initiatives representing different models of CPD for teachers of mathematics in England. Four of these initiatives did have some other involvement with the NCETM: one had received a grant, in two the NCETM was involved as one of several organisations supporting a network, and one concerned a network that was being led by the regional NCETM coordinator. However, no one of the RECME research team was involved in these NCETM activities. Overall, about 250 teachers in pre-primary, primary, secondary, further and adult education settings were involved in these initiatives. The project adopted the theoretical framework that all human activity, including the learning of teachers, is historically, socially, culturally and temporally situated (Vygotsky 1978). This suggests that the experiences and contexts of teachers will have a major influence on their learning and implies a need to pay attention not only to the situation, the opportunities and the context of sites of learning (in this case initiatives of professional development), but also to the individuals taking part in professional development. Importantly, the philosophical underpinning of the project was one of co-constructing meaning with teachers, researchers and other stakeholders. The data we obtained for the study we report on in this paper reflects this and contains self-reported data from teachers and CPD organisers. We are very aware that such data should be treated with caution, however we feel its use is appropriate here because we wanted to find out how participants to CPD were experiencing research utilisation. Details and descriptions of the full data set, the research design and case studies can be found in the final RECME report and other publications (NCETM 2009; De Geest *et al* 2008).

To find out the different roles of research in professional development for teachers of mathematics we addressed the following research questions:
1. What is the effect of 'research involved CPD' on teachers taking part in the PD?
2. In which ways is research presented and used by participants in the CPD initiatives?

Different data sets were used to address the two research questions. The study was conducted using a constant comparison approach from grounded theory with clearly defined analysis questions (mentioned later) looked at from a social constructivist theoretical perspective (Vygotsky 1978), i.e. what are the opportunities and the context of the initiatives of professional development and what are the experiences of the individuals taking part in professional development.

Data collected and analysed for the first research question consisted of qualitative answers to questions in the online questionnaire, which was completed by

92 of the teachers participating in the RECME project. It had one specific section related to research: "Are you aware of any research evidence underpinning any CPD which you have been involved in?" If yes, the follow-up question was asked: "You indicated that you are aware of research underpinning some of the CPD you have undertaken. We are interested in whether this influenced the way you felt about the CPD. Please explain here". Further data was found by searching for the word 'research' in all the qualitative responses to this questionnaire. The analysis question used in the constant comparison of this self-reported data was: "what change are teachers reporting as a result of working in a research based or research informed CPD initiative?"

Data collected and analysed for the second research question consisted of information supplied by the thirty organisers in electronic questionnaires, notes of discussions, and observation notes of the CPD meetings made by the research team which were sent back to the organisers and participants for validation and amendments. The analysis question used in the constant comparison was: "in what format(s) does 'research' present itself in the participating CPD initiatives?"

Findings

Overall, about three-quarters of the CPD initiatives (23) in the sample entailed some form of research in their set-up and running. We call this 'research–involved CPD'. About half of these 23 initiatives that involved some form of research also had HEI involvement. Of the 92 respondents to the on-line questionnaire precisely half stated that they were aware of research underpinning the CPD and offered insights in how this had influenced the way they felt about the CPD. We will first report on our findings regarding the effect of research-involved CPD on teachers. Then we will consider in which ways research was presented and used by participants in these CPD initiatives.

The effect of 'research involved CPD' on teachers

As a result of taking part in a research-involved CPD initiative, teachers reported changes in various ways relating to their classroom practice, their further professional life and their knowledge development. In particular, using research seemed to offer teachers effective means of becoming aware of different perspectives about teaching and learning, to engage in deep thinking, to gain confidence about their own thinking, to trust the validity of the ideas offered and to have thus the courage to try out these ideas on the classroom, and to give status and credibility to the CPD initiative itself and to the teacher. We expand on these descriptive categories in more detail:

Raised awareness
Teachers reported raised awareness about research, current thinking and information on existing practice is 'out there' relating to teaching and learning which can also be of use in the classroom. For example:

> I am taking part in a Masters in Maths Education and it has made me realise what a wealth of research there is to read and use in teaching.

> I liked knowing that I am aware of current thinking, research and best practice.

> I have taught for 21 years and have become a far better teacher in the last few and am only now becoming more aware of what is going on out there [as a result of using research]!

Stimulate thinking about teaching and learning

Teachers reported feeling stimulated in their thinking about teaching and learning, feeling stimulated to research their own practice and to engage with research literature as a result of the research-involved aspect of their PD. For example:

> "It's like a zoom lens that is based around everyday practice and involves observations and recording that I would do in any case. It has stimulated my own thoughts re practice and introduced me to academic research I might not otherwise have accessed. It has stimulated thinking and debate amongst us three participants in the research"

> "I have read about how Chinese teachers are taught how to enable children to learn mathematics, particularly with regard to ensuring that their own subject knowledge is sufficiently developed to enable them to explain concepts correctly. This has caused me to question my own subject knowledge and develop it further in the context of enabling me to teach maths more effectively"

> "The National Strategies made very explicit some of their source materials. I like this. There's an integrity about it and it also invites challenge and different viewpoints. To simply be told a way of doing something without acknowledging the source feels patronising. I like it when people expect me - and trust me - to think"

Affirm and/or develop the professional self

The teachers reported how the research aspect of their CPD affirmed their perceptions of their professional self: how they think about and evaluate themselves as teachers, leading to confidence in their professional self. They also reported how working on their existing interests and understanding led to a deepening development of their professional self which felt satisfying. For example:

> "The research validated what I already knew and therefore gave me increased confidence to use new techniques, for example Swan"

> "I found [some previous CPD that used research] so fulfilling, leading to me doing some class based research into self esteem. This was definitely a high in my professional career and I was asked to write part of a national strategy as a result of this work"

> "It has made me research an area of the curriculum about which I am strangely passionate, reflect on my own understanding and practice, collect and collate evidence and share this with fellow maths enthusiast within my school and the group"

Confidence to act

Teachers reported that knowing that the CPD they were undertaking was underpinned by evidence from research made them feel confident to act: to take further part in CPD and to apply research ideas to their practice as it has been shown to 'work'. For example:

> "I feel much more motivated by CPD that is underpinned by research as I know that people have really tried things out with children rather than made something up and hoped for the best!"

> "XXX [name of project] project provided the evidence that Collaborative practice was the way forward in improving mathematics teaching - gave me and the other teachers confidence"

> "Having research that underpins the CPD that we are doing shows us that there are already results which will prove that what were doing is likely to benefit both teachers and pupils"

Gain status and credibility

Teachers reported that research-involved CPD gives credibility and status to the CPD initiative itself and to the teacher's professional thinking and activities. For example:

> "I think it gives the programme more status it isn't just a maths club it is something important"

> "It adds more credibility to what you are learning"

> "It makes me feel that I can justify my interest in the approach to those above me in the line management structure"

What surprised us in this study was the lack of negative responses to research utilisation by the teachers– only one response contained a negative element and mentioned not finding it easy to read papers. Perhaps a question such as 'what do you consider barriers to engaging with research evidence in your professional development' would have given more insights into negative aspects of research utilisation.

'Research–involved CPD'

This section reports descriptive categories of how research presented itself to the participants of the CPD in the 30 professional development initiatives that were part of the RECME project. Our analysis suggests five descriptive categories: reading research literature; using resources that have been developed based on research; research-inspired CPD; being part of a research project; doing research as CPD and research-informed CPD which we will discuss next in more detail. These categories are not mutually exclusive and indeed in several initiatives a combination of these were present.

Reading research literature

This refers to participants reading specific research literature. This literature can be in the form of published research papers for example Hallam & Ireson (2006); summaries of learning theories compiled by organisers of the CPD who have a background of working in HEIs, for example on Piaget (J. Piaget 1950, 1953; J. Piaget & Szeminska 1952); summaries from other documents, for example from Swan (Swan 2006, 2005); chapters from books or short publications, for example Black *et al*. (2002). The literature tended to be selected by the organisers of the CPD who had more (expert) knowledge of existing research than the other participants. The data indicated that in most cases the selected literature was of immediate and clear relevance to issues worked on in the CPD initiative and was often chosen as a result of issues raised, or interests declared in certain topics by the participants. In a few cases the readings included also non mathematics-teaching research, such as research methods and other more academic research. Although this was not therefore 'directly relevant' to issues raised by the participants, organisers of the CPD considered this relevant to the aims of the CPD initiative, for example the reading of research methodology was part of a masters course and thus required by university standards and regulations. 'Reading' varied from reading literature as 'gap'[1] task between

---

[1] 'gap' tasks are activities that the teachers were asked to engage with between their CPD sessions. These gap tasks might involve trying out an activity with students in their classrooms, looking at students' work or reading something related to the CPD, such as in this instance, a research article.

sessions to reading in meetings, discussing and interpreting the text, and discussing how it applied to practice.

Using resources that have been developed based on research

These are resources that have been developed as part of research into pedagogies and didactics related to the learning and teaching of mathematics and/or have been refined and evaluated as being effective learning resources through a research process. Examples of these resources and the research they are based on are an 'active' learning approach (Swan 2006), Realistic Mathematics Education originating from the Freudenthal Institute in The Netherlands (Van Den Heuvel-Panhuizen 2003).

Research inspired CPD

These are instances where research findings and literature are used implicitly rather than explicitly. Often these concern theories about 'ways of working', such as learning collaboratively, without making explicit reference to which research or publications it was based on. An example is when the organiser of the CPD, who might have in-depth knowledge of the research that inspired the CPD, plans the CPD activities by building on that research but without exploring the research itself with the other participants of the CPD. At other times ideas from research papers were used as a starting point, as a vehicle for triggering discussion, exploration, reflection and experimentation. Although the findings of the research could be mentioned, it does not involve the reading of papers by all participants. In these cases the original meaning of the research and its findings might get re-constructed.

Being part of a research project

Three of the initiatives involved teachers taking part in research projects. One was an action-research project funded by an outside organisation and led by an HEI which offered optional accreditation towards a Masters degree. The research project focused on teachers evaluating the introducing of an ICT resource on their teaching practices and students' learning. The second research project consisted of being part of a doctoral research study. In this case, the doctoral researcher who was also the organiser of the CPD worked collaboratively with the teachers who are acting as research assistants. The third case concerned a two-year Master's in mathematics education course that at the same time was a research project. The project researched teachers' professional and personal development while undertaking postgraduate research study with teachers acting as co-researchers.

Doing research as CPD

We could identify clearly eight initiatives where teachers were given the opportunity to develop professionally by doing research. In these instances doing research was part of the organisation and planning of the initiative in that they led or could lead to accreditation towards a Master's degree which required the candidates to conduct their own research projects. Three were part of a Master's degree, five were potentially so in that the teachers could opt for masters level accreditation. There also was one initiative that was without accreditation but part of a research project and involved the teachers as active co-researchers (this initiative is also described in the category 'being part of a research project). Although other initiatives included, to different extents, trying out new ideas and reflecting on professional activities we did not include these in this category of 'doing research as CPD' because we could not ascertain these activities were systematic.

Conclusion and limitations of this study

We found that three quarters of the CPD initiatives which were researched as part of the RECME project entailed some form of research in their set-up and running, offering a different picture to the one Hargreaves reported on in 1996 of little research utilisation in the teaching profession and professional development. This could indicate that professional development provision has changed since the mid 1990s and research now has entered its discourse. We found that the use of research in CPD does not present itself in one format, but that there are wide-ranging ways. Our analysis suggests five descriptive categories. Interestingly, only half of the initiatives that involved some form of research also had HEI involvement, suggesting research is also considered useable and relevant by those who are not part of the research community which tends to be based at HEIs.

Contrary to observations from the mid 1990s (Hargreaves 1996) teachers in this study considered the use of research to be helpful in ways that relate to their classroom practice, their further professional life and their knowledge development which could be because in all cases in this study research was used from a practical point of view, confirming research findings from Zeuli (1994) and Rhine (1998). The analysis suggests that in the CPD initiatives that involved a research aspect, the teachers were asked to think and reflect about the research that was presented in the CPD and were as such cognising and reflective agents (Brown and Burko, 1992 reported in Cooney, 1994). Teachers evaluated such opportunity to think and reflect positively. Of course, these findings do not mean that similar effects could not be achieved in other ways than through research utilisation. However, we suggest that using research is an effective means, although perhaps not the only one, for teachers to becoming aware of different perspectives about teaching and learning, to engage in deep thinking, and to gain confidence in their thinking.

Our analysis flagged up four major issues for discussion:

- We found that at times the original meaning of the research and its findings may have been reconstructed, interpreted or 'watered down' without the original text being studied by all the participating teachers. Although such interpreted research was still used as a starting point or vehicle for exploring of and reflecting on issues in the classrooms, it raises the question to which extent this can still be considered as the use of research, and how the reconstructing of research findings affects the teachers' knowledge base.
- We identified effects of the use of research in CPD as reported by teachers. The question remains to which extent these differ from CPD that does not involve research aspects.
- The study did not identify barriers to engage with research utilisation in professional development. What are these and can they be overcome?
- What is meant by 'doing research'? For example, when does trying out new ideas in the classroom, reflecting on the effects of the changes become research? What are the boundaries between reflective practice and doing 'research'?

Acknowledgements

References

Black, P., C. Harrison, C. Lee, B. Marshall and W. Dylan. 2002. *Working inside the black box: assessment for learning in the classroom*. London, nfer Nelson.

Brown, C. A., and H. Borko. 1992. Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, 209-239. New York: Macmillan.

Cooney, T.J. 1994. Research and Teacher Education: In Search of Common Ground. *Journal for Research in Mathematics Education, 25*(6), 608-636

De Geest, E., M. Joubert, R. Sutherland, J. Back and C. Hirst. 2008. Researching Effective Continuing Professional Development in Mathematics Education (RECME). *International Conference of Mathematics Education*, Monterrey, Mexico.

Hallam, S., and J. Ireson. 2006. Secondary school pupils' preferences for different kinds of structured grouping practices. *British Educational Research Journal, 32*(4), 583-599

Hargreaves, D. H.1996. Teaching as a Research-Based Profession: possibilities and prospects. *Teacher Training Agency Annual Lecture*, London, Teacher Training Agency

Hemsley-Brown, J. and C. Sharp. 2003. The Use of Research to Improve Professional Practice: a systematic review of literature. *Oxford Review of Education, 29*(4), 449-470

Joubert, M. and R. Sutherland. 2008. *A perspective on the literature: CPD for teachers of mathematics*. Sheffield, NCETM.

NCETM. 2009. *Final Report: Researching Effective CPD in Mathematics Education (RECME)*. Sheffield, National Centre for Excellence in the Teaching of Mathematics

Piaget, J. 1950. *The psychology of intelligence*. London, Routledge.

———1953. *The Origin of Intelligence in the Child*. London, Routledge.

Piaget, J. and A. Szeminska. 1952. *The Child's Conception of Number*. London, Routledge

Rhine, S. 1998. The Role of Research and Teachers' Knowledge Base in Professional Development. *Educational Researcher 27*(5), 27-31

Van den Heuvel-Panhuizen, M. 2003. The didactical use of models in Realistic Mathematics Education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics 54*(1): 9-35.

Swan, M. 2005. *Improving learning in mathematics: challenges and strategies*. Department for Education and Skills, Department for Education and Skills Standards Unit.

———2006. *Collaborative Learning in Mathematics: A challenge to our beliefs and practices*. London, NRDC.

Vygotsky, L. 1978. *Mind in Society: The Development of Higher Psychological Processes*. London: Harvard University Press.

Zeuli, J. 1994. How do teachers understand research when they read it? *Teaching and Teaching Education, 10*(1), 39-55

Using Realistic Mathematics Education with low to middle attaining pupils in secondary schools

Paul Dickinson, Frank Eade, Steve Gough, Sue Hough

Manchester Metropolitan University Institute of Education

This paper provides an account of two projects involving the trialling of a new approach to teaching in secondary schools in England. The method being trialled is based on Realistic Mathematics Education (RME), originally developed in the Netherlands. The paper focuses on the underpinning of RME, provides an overview of the associated projects, the research methods and initial findings, and explores emerging issues from the projects.

Introduction

The Smith Report (2004) stimulated a lot of interest in applications of mathematics and supporting learners in becoming mathematically literate. Consequences of the Smith Report included the introduction of GCSE assessment in mathematics which makes greater use of contextual questions and the piloting of a 'twinned pair' of GCSEs, one of which focuses entirely on the application of mathematics. An overriding issue for many mathematics educators is how we support learners in developing application skills as well as learning the normal content associated with mathematics without separating these out and increasing the time allocated to mathematics teaching.

This paper explores the conjecture that it is possible to develop both content knowledge and problem solving skills using an approach based on Realistic Mathematics Education (RME).

Background

Mathematics in England

The mathematics curriculum in England has undergone radical changes in the last fifteen years with the introduction of a variety of forms of formal and informal assessments, emphasis on functional skills and the advice offered to teachers in how they might support learners. A recent study would suggest that despite the investment in mathematics there is little evidence that the standards in mathematics have improved (Hodgen et al. 2009)

Clearly the strategy has been effective in changing some of the patterns of behaviour of teachers and in shifting the emphases on different parts of the mathematics curriculum. However, the work of Anghileri et al. (2002), Brown et al. (2003) and Hodgen et al. (2009) would suggest that there may be grounds to doubt that these changes have been as effective as the government would wish us to believe. Despite apparent short-term improvements as measured by the end of key stage assessments, Smith (2004), Brown (2003), Anghileri (2002) and Hodgen et al. (2009) all highlight worrying concerns about longer-term conceptual understanding, procedural fluency, and the ability to apply mathematics. Indeed Askew et al. (2010)

argue that, in England, procedural fluency and conceptual understanding are largely seen as mutually exclusive aims.

The Smith Report (2004) suggests the need for "… greater challenges… harder problem solving in non-standard situations, (and) a greater understanding of mathematical interconnectedness …". The report also indicated that the mathematical skills developed by pupils age 16 are not concerned with " the growing mathematical needs of the workplace… mathematical modelling or ... problems set in the real world contexts." Smith also suggested that in comparative terms "England seriously lags behind its European competitors" in terms of the number of pupils achieving an appropriate level 2 qualification. Hodgen et al. (2009), suggest that, while exam passes have risen dramatically in the last 30 years, pupils' underlying understanding of mathematics has changed little.

In summary, we see the above as evidence of a need to explore and develop a practical pedagogy of mathematics education that supports pupils' conceptual understanding, problem-solving skills and the use of these in real world situations.

Mathematics in the Netherlands

The Freudenthal Institute, University of Utrecht was set up in 1971 in response to a perceived need to improve the quality of mathematics teaching in Dutch schools. This led to the development of a research strategy, an approach to teaching and to a theory of mathematics pedagogy called Realistic Mathematics Education (RME). RME uses realistic contexts and a notion of progressive formalisation to help pupils develop mathematically. A strong feature of RME is the simultaneous and integrated development of conceptual and procedural knowledge. Pupils engage with problems and scenarios using common sense/intuitions, collaboration with other pupils, well judged activities and appropriate teacher and textbook interventions. (See Treffers (1991) and Treffers et al. (1999) for further discussion of RME.)

At a surface level, RME resonates strongly with progressive approaches used in England where investigative and problem-solving strategies are utilised and where pupils are encouraged, as a whole class, to discuss their work to resolve important issues. One difficulty with this approach to teaching in England is that pupils may stay with naïve mathematical strategies and are often unwilling to move to more sophisticated strategies and procedures. The need for a teaching and learning trajectory is clear. Through intensive research, trialling and re-evaluating materials and approaches, Dutch mathematics educators have developed a variety of ways of encouraging and supporting pupils' mathematical progress. So, for example, pupils remain in context throughout and stay with a topic for a much longer period of time than would be usual in England.

Associated Projects

MiC in the US

In 1991, The University of Wisconsin (UW), funded by the National Science Foundation (USA), in collaboration with the Freudenthal Institute, started to develop a curriculum and pedagogy based on RME (See Romberg and Pedro (1996) for a detailed account of the developmental process and van Reeuwijk (2001) for an account of the care taken in developing one aspect of the scheme.) The first version of Mathematics in Context (MiC), together with comprehensive teacher materials, was published in 1996/7 and has undergone several revisions since then.

MiC in the UK

The Gatsby Foundation agreed to fund Manchester Metropolitan University (MMU) to run a project based around trialling RME (utilising MiC) over a three year period. The Economic and Social Research Council (ESRC) also agreed to fund an examination of how teachers' beliefs and behaviours change as a result of engagement in the project (see Hanley et al. (2007) for an account of the research into the changes in teachers involved in the project).

The project focused on three main issues: developing an understanding of RME in an English context, understanding how learners develop, and supporting teachers to develop practical skills and a deep knowledge of RME.

In terms of pupil development over three years the project team saw evidence that pupils' approach to solving problems changed and that this influenced how they understood the mathematics. More details of this are given below; for other findings of the project see Dickinson and Eade (2005).

MSM in the UK

In 2007, as an extension to the work of the Key Stage 3 project, The Making Sense of Mathematics (MSM) project began. This was aimed at Foundation Level GCSE students (Years 10 & 11) with new resources being produced as a result of collaboration between the Freudenthal Institute and MMU. These resources consist of 11 booklets which together cover the Key Stage 4 Foundation level curriculum. These booklets build upon the experiences gained from the Gatsby project and take account of difficulties highlighted by the Key Stage 3 teachers, such as the need for RME based materials which feature British contexts and are more closely linked to UK national tests.

The MSM project has involved Foundation level classes from 6 schools in the first cohort and 10 schools in the second cohort. MMU has supplied resources to these schools and has provided ongoing support in the form of twilight training sessions and school based observations. Feedback given by the teachers has been used to revise the materials which are currently in their second version.

Key findings from the MSM project to date show influences on both teachers and pupils. For the purposes of this paper, we focus on pupils, and the similarities between KS3 pupils working with MiC and KS4 pupils working with MSM.

Data Analysis

As mentioned above, in both UK based projects (MiC and MSM), data was collected from project pupils and 'control' pupils. In terms of the control, pupils were drawn from parallel classes in the project schools and also from non-project schools whose intake and performance were similar to those in the project. Project and control were matched either in terms of their KS2 SATs scores (for the MiC project) or in terms of GCSE target grades (for MSM).

While data was collected from project and control groups in a number of ways (including national test data and attitudinal questionnaires) the focus for this article is on what we termed 'problem solving'.

In the problem solving test, pupils were given ten minutes to complete each question. Consequently, much more detailed solutions were produced, and it was possible to analyse methods and approaches in addition to whether or not pupils had arrived at a correct answer. Since one of the aims of the project was to construct an account of how pupils develop mathematically, we needed an assessment which would provide data on this. The programme team was also mindful to take account of

Hawthorne related issues (Landsberger 1958) associated with simple tests of content gains and was therefore looking to find evidence of significant differences in pupils' approaches to solving mathematical problems. After substantial trialling, five questions were produced across the attainment targets and all project and control pupils worked on these individually over two lessons. The questions come under the general heading of 'problem-solving' in that they required pupils to 'mathematise' situations which they had not met previously. For further details of the original MiC problem solving tests, see Dickinson and Eade (2005)

The data discussed here are from Year 7 pupils for MiC (sample sizes 100) and Year 10 and 11 pupils for MSM (sample sizes 50). With so many schools and pupils involved, we are confident that the data gathered is reliable.

We wish here to focus on two topic areas, area and fractions. We have chosen these firstly because the traditional treatment of these topics at secondary level tends to be procedurally dominated and secondly, because they exemplify nicely the differences between project and control pupils that were noticed across all curriculum areas. We have also focussed on results from Year 7 pupils who were in the lower 40% of the attainment range for KS2 SATs scores) so that results from the two projects can be compared more easily.

Area

Here, the identical question was given to pupils in both projects, with remarkably similar results. The question was

Find the area of the shape shown below.

Show carefully how you worked it out.
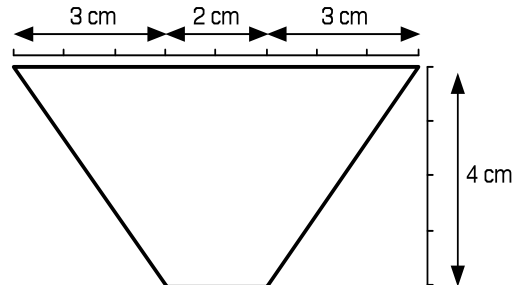
If we class 'drawing cm squares and counting', 'splitting into rectangle and triangles', and 'moving a triangle to create a rectangle' as 'sense-making strategies' (as against simply performing a numerical calculation), 74% of MiC and over 80% of MSM pupils attempted to make some sense of the problem (with over 50% achieving the correct answer). The corresponding figures for control pupils were 32% and 37%. Interestingly, control pupils who 'made sense' of the problem tended to get the correct answer; it would seem that for these pupils they can either do the question or they can't, there is no middle ground. In project groups, we see around 25% of pupils who cannot get the correct answer yet are able to make a sensible attempt (for example through counting squares).

Indeed the vast majority of control pupils adopted a purely numerical method, often simply adding, multiplying, or in some way averaging the numbers on display. This is despite the fact that all will have counted squares when initially introduced to area. There is a strong suggestion here that introducing formal mathematical knowledge at too early a stage in a pupils' development will replace any more informal ideas and leave the learner with no knowledge at all once a rule is forgotten

(see Anghileri (2002), Hart et al. (1989), and Boaler (1998) for similar conclusions in other mathematical topics).

Fractions

In the Key Stage 3 (MiC) project, the following questions were set.

a) Tape is sold in pieces *1/3* of a metre long.

Show how many pieces you can cut from a piece 4 metres long.

b) *3/4* of this glass is full of orange.

1/3 of this glass is full of orange.

Explain whether you can pour the second *glass* into the first without it overflowing.
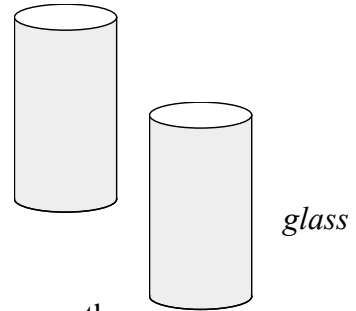
In terms of correct responses for lower ability groups, the following results were obtained (n=100)

Question Number	Project Pupils	Control Pupils
2(a)	42%	7%
2(b)	54%	12%

While the differences here are quite stark a further significant feature, in addition to the proportion of correct answers, was the number of pupils who drew something in order to make progress with the problem. 50% of project pupils attempted to draw something in part (a), compared to only 23% of control pupils. Similarly in part (b), the figures were 74% and 49% respectively.

In the Key Stage 4 project (MSM) a different fractions question was used, as evidence from another trial in a local school had suggested some interesting results.

The question was simply

Find $\frac{1}{4} + \frac{1}{2}$

Do you think you have got this right? Explain why.

Given that these pupils would have studied fractions for at least 7 years, the data gathered was somewhat shocking. In terms of pupils getting the correct answer, results were (n=50 in both cases)

	% correct Target grade C	% correct Target grade D/E
Project	83%	57%
Control	72%	30%

A striking feature when analysing pupils' work was that in the control groups one only had to look at the work of a small number of pupils to know exactly the method that had been taught to them and the procedure that their teacher wished them to follow. Script after script was identical in how pupils attempted to solve the

problem (two of the most common ways are shown in Appendix 2).It then appeared that more able Foundation level pupils remembered the method and got the answer correct, but if they forgot the method (which was often the case for D/E level pupils) they produced something that amounted to mathematical nonsense. In contrast to this, project pupils seemed to have a variety of approaches at their disposal and were able to make sense of the problem.

It was also interesting that when asked for reasons, project pupils often drew something, talked about 4 quarters making a whole and hence a half being two quarters, or referred to pizzas, cakes, etc. Control pupils invariably referred to a numerical method, often simply describing what they had done. At target grade D/E, over 50% of pupils got the answer 2/6 with the vast majority believing that they were correct and citing the fact that 1+1 =2 and 2+4=6 as their justification for this. No project pupils at this level got the answer 2/6. We see this as further evidence that pupils taught an RME based curriculum are more able to make sense of their mathematics, both in achieving answers, and in reasoning why they feel they are correct. On the other hand, pupils taught formal algorithms have no other mathematical resources to fall back on if the algorithm is not remembered correctly. One control pupil (Year 10 target grade D/E), when faced with ¼ +½ commented tellingly that "I am stuck with this question because I forgot the method".

Why do RME based approaches lead to different results and (in particular) different methods of solution?

Use of contexts

In RME, contexts are used not only to illustrate the applicability and relevance of mathematics in the real world, but also as a source for the learning of mathematics itself. Contexts can be taken from the real world, from fiction or from an area of mathematics that students are already familiar with. It is important that they should be sufficiently real for students to be able to engage with them so that they are solving problems which make sense, but also critical that they reflect the mathematical structures that we want the students to work with. The contexts used are extensively researched and differ significantly from those found in standard UK textbooks.

Students are encouraged to make sense of the context using their experiences, intuitions and common sense. They then stay in context, and remain at a sense-making level, while they develop mathematically. The word 'realistic' is used to emphasise that students are able to imagine the situation.

Experience shows that, through staying connected with the context, students are able to continue to make sense of what they are doing, and do not need to resort to memorising rules and procedures which have no meaning for them. 'Mathematics' and 'context' are not separated – to experience success in one implies success in the other. So, for example, when working with area, the notions of reallocating parts of a shape and visualising arrays are predominant. If a formula or rule emerges, this is referred back to the context for validation. We believe that the benefits of this are clearly seen in the trapezium question analysed earlier-far fewer project pupils appear to view area as simply another numerical procedure.

Use of 'models'

RME provides a different view on how contexts should be chosen, and also on how these can then be used to support mathematical development. The use of 'models' is

crucial here (see van den Heuvel-Panhuizen (2003) for a thorough analysis of the use of models under RME).

A model emerges from a context. Initially it may be little more than a representation, for example a picture, suggested by the context. Later, however, these models become more sophisticated mathematical tools such as the number line, ratio tables, etc.

Models bridge the gap between the informal and the formal and so teachers feel less pressure to replace students' informal knowledge with formal procedures. As pupils begin to formalise their mathematics, models and contexts support the process of vertical mathematisation while retaining the 'sense-making' element. In this way, the formal and informal are more likely to 'stay connected' in the minds of the pupils. This is evidenced, for example, by project pupils working with a common denominator in a way which makes sense to them, something which non-project pupils seem unable to do. Models also allow students to work at differing levels of abstraction, so that those who have difficulty with more formal notions can still make progress and will still have strategies for solving problems.

An important part of a student's mathematical development is the recognition that the same model can be used in a variety of situations and to structure solutions to many kinds of problems. As Boaler (1998) suggests, a traditional approach is unlikely to lead to pupils being able to recognise situations which are 'mathematically similar'. The mathematics levels in the 1999 National Curriculum for England characterise students' development by means of the procedures which they are able to perform and the difficulty of the numbers they are using. However, in RME, teachers use a richer range of descriptors to gauge students' progress. This includes observing students' use of models, insights and reflection as well as mathematical landmarks and procedures (see Fosnot and Dolk (2002) for discussion of this in the context of fractions).

Multiple Strategies

One aspect of being 'functional' in mathematics is being able to choose the most appropriate strategy to solve a problem, rather than always relying on one strategy or algorithm. The contexts in RME are chosen to elicit many different strategies and students are constantly encouraged to reflect on these and refine them. Lessons will involve comparison and evaluation of different student strategies; a fundamental tenet of RME is to build sophistication into student-generated procedures rather than for teachers to impose a 'standard method' or algorithm. Evidence of this is clearly seen in the Key Stage 4 data discussed here.

RME encourages the development of more formal methods from students' informal methods. However, it also allows students to continue to be able to use informal methods, where appropriate, rather than relying on being taught a method of solution that fits a certain type of problem.

We feel that there is significant evidence here that average and lower attaining UK pupils can benefit significantly from following an RME based curriculum and that through doing so they will be able to make more sense of their mathematics and hence become more functional in the subject.

References

Anghileri, J., M. Beishuizen and K. van Putten. 2002. From informal strategies to structured procedures: mind the gap! *Educational Studies in Mathematics* 49: 149-170.

Askew, M., J. Hodgen, and S. Hossein. 2010. *World's Apart. International review seminar* 11th January. Awaiting submission.

BERA website. 2009. Pupil maths performance similar to 1976 despite sharp rise in grades, study shows. http://www.bera.ac.uk (accessed 17th December 2009)

Boaler, J. 1998. Open and Closed Mathematics Approaches: Student Experiences and Understandings. *Journal for Research in Mathematics Education* 29 (1): 41-62.

Brown, M., M. Askew, A. Millett and V. Rhodes. 2003. The key role of educational research in the development and evaluation of the National Numeracy Strategy. *British Educational Research Journal* 2: 655-672.

Dickinson, P. and F. Eade. 2005. Trialling Realistic Mathematics Education (RME) in English secondary schools. *Proceedings of the British Society for Research into Learning Mathematics* 25 (3).

Fosnot, C.T. and M. Dolk. 2002. *Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents*. Portsmouth, NH: Heinemann.

Hanley, U. and Darby S. 2007. Working with curriculum innovation: teacher identity and the development of viable practice, *Research in Mathematics Education* 8: 53-66.

Hart, K. et al. 1989. *Children's Mathematical Frameworks*, Windsor: NFER-Nelson.

Hodgen, J., D. Küchemann and M. Brown. 2009. *Secondary students' understanding of mathematics 30 years on*. Conference presentation, British Educational Research Association Conference, 2-5 September.

Landsberger, H. A. 1958. *Hawthorne Revisited*. Ithaca, NY: Cornell University.

Romberg, T. A. and J. D. Pedro. 1996. *Developing Mathematics in Context: a research process*. Madison, WI: National Center for Research in Mathematical Sciences Education.

Smith, A. 2004. *Making mathematics count*. London: HMSO.

Treffers, A. 1991. RME in the Netherlands 1980-1990. In *RME in primary school*, ed. L. Streefland. Utrecht: Freudenthal Institute.

Treffers, A. and M. Beishuizen. 1999. RME in the Netherlands. In *Issues in teaching numeracy in primary schools*, ed. I. Thompson. Buckingham: Open University Press.

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: an example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics* 54: 9-35.

Van Reenwijk, M. (2001). From informal to formal, progressive fomalization: an example on "solving systems of equations". Utrecht: Freudenthal Institute <http://www.fi.uu.nl/publicaties/literatuur/4465.pdf> .

Students' Experience of Mathematics Enrichment

Wai Yi Feng

Faculty of Education, University of Cambridge

This paper presents findings on students' experience of mathematics enrichment, arising from a recent and more extensive study designed to inform understanding of different forms and practices of mathematics enrichment in the UK. Four case studies were conducted, each focusing on a distinctive enrichment programme, namely: one set of residential Mathematics Summer Schools, offered by the National Academy for Gifted and Talented Youth; one series of Mathematics Master-classes, run by a Royal Institution Master-class group; an after-school outreach and enrichment programme, targeted at students from a disadvantaged, inner-city area, run in collaboration with NRICH; and the United Kingdom Mathematics Trust's (Junior and Intermediate) Maths Challenge competitions, undertaken in one school. Data were collected through interviews with student participants, informed by observations of enrichment practice. Whilst students reported a range of enrichment benefits, broadly related to their mathematical, and personal and social, development, support for mathematics learning in school, and exposure to higher education, their experience was more subtly related to the characteristics of the programme in which they had participated, interpreted according to more familiar experiences of learning mathematics in school.

Keywords: mathematics enrichment, students' experience, enrichment programmes, gifted education

Background

Students' apathy towards mathematics has received much public attention in recent years: many are reported to be disaffected with school mathematics and to grow more so through their years of schooling; numbers recruited into post-16 mathematics education had been falling between 1996 and 2007, following longer-term decline across the wider science fields (Smith 2004; The Royal Society 2008). In the most recent EU survey on attitude to 'science' among young people aged 15 to 25 (European Commission 2008), mathematics was the least popular discipline, even among the minority who would 'consider' studying 'science' subjects; interest in studying mathematics in the UK was among the lowest in EU Member States.

Against this background, students' participation in, and attitude to, mathematics have become key concerns for policy-makers and the mathematics and mathematics-education communities (Clarke 2004; Smith 2004). This has led to an increase in the number and variety of 'enrichment' activities, aimed at engaging young people in mathematics in general and at supporting and extending the mathematics learning of specific groups of students (e.g. those identified as being 'mathematically-gifted' or belonging to 'disadvantaged' social groups) in particular, above and beyond provisions prescribed by the school Mathematics curriculum. Such activities include:

- efforts by individual teachers or Mathematics Departments to make mathematics learning in lessons more interesting for students;
- school-based mathematics clubs run by teachers to supplement and enhance students' learning;
- competitions to stimulate interest in mathematics in large numbers of students and to identify and cultivate talent;
- summer study programmes, popular lectures, workshops and master-classes to engage students in mathematics and to expand their mathematical horizon; and
- university outreach and 'widening participation' activities to attract students to study mathematics-related courses, and to recruit, and raise the aspirations of, students from social groups currently under-represented in higher education or in mathematics-related studies and careers.

Despite the increasing amount of activity taking place under the 'enrichment' label, the 'enrichment' area is under-theorised and under-researched. The term 'enrichment' is often used common-sensically, masking the range of activities which operate under its label (Feng 2005); outcomes of enrichment are also not well-understood. The large numbers of enrichment activities — not just in mathematics, but across STEM (science, technology, engineering, mathematics) fields — and their role and impact in promoting the STEM disciplines to young people are currently receiving increased attention from policy-makers (DfES and DTI 2006; DIUS 2008).

Design and method for the study

This paper presents findings on students' experience of mathematics enrichment, arising from a recent and more extensive study which developed and refined a conceptual framework for understanding 'mathematics enrichment' in the UK. The main study itself synthesised available literature and provided new empirical evidence, gathered through four case studies (Bassey 1999), to inform understanding of different forms and practices of 'mathematics enrichment' emerging from the literature (Feng 2006): 'enrichment' which focuses, respectively, on the development of mathematical talent (e.g. Roberts 2005), the popular contextualisation of mathematics (e.g. Zeeman 1990), the enhancement of mathematical proficiency and learning processes (e.g. Piggott 2004), and outreach to the mathematically underprivileged (e.g. Bouie 2007). Each case study focused on a distinctive enrichment programme as follows:

- one set of residential Mathematics Summer Schools for 'gifted' students (in the top 5% of the attainment range) lasting three weeks, offered by the National Academy for Gifted and Talented Youth;
- one series of Mathematics Master-classes, composed of five Saturday-morning workshops led by different speakers each week, run by a Royal Institution master-class group at its local university;
- a year-long outreach and enrichment programme, composed of weekly, after-school classes, targeted at students from a disadvantaged, inner-city area and held in their local university, run in collaboration with NRICH; and
- the United Kingdom Mathematics Trust's (Junior and Intermediate) Maths Challenge competitions, undertaken in one school.

Within each case study, the ideas and experiences of enrichment held by the staff organising, and the students participating in, the respective programme were explored through analysis of official documents and interviews with organisers and students, informed by observations of enrichment practice.

The data reported in this paper were collected through end-of-programme, semi-structured interviews with 40 student participants between 2005 and 2006, supported by observations of enrichment practice made over a more extended period (2004–2006). Participants were chosen in consultation with enrichment providers or their teachers so as to cover, potentially, a range of views and experiences of participating in enrichment. Interviews lasted, on average, 50 minutes. Themes were developed inductively from the data as well as deductively from the literature, and refined iteratively through constant comparison.

'Enriching' experiences of mathematics

Participants from each of the four programmes identified a range of experiences which they valued and perceived as 'enriching'. Typically, students reported investing sustained effort into solving mathematical problems: salient features of the way they worked include thinking mathematically to find ways forward and identifying strategies from their own repertoire. This contrasts with their reported experience of solving school-mathematics problems, which they usually completed very quickly using known techniques:

> [You can't] come up with an answer [straight-away]. You have to be systematic [and] build upon it. [In school], it won't take [long to answer a question. If] they give [you] 6 marks, then [you do] 6 bits. […] You know what method to use. [Here], you don't know. You have to work out [what's best to do].
>
> (Outreach participant)

As such, students typically found enrichment problems more 'challenging' and 'difficult' than problems commonly encountered in school. Although a small number preferred the 'security' of solving 'known problems', students generally found the material introduced to them interesting and relished the substantive challenge.

In addition, a key part of the Summer Schools, Master-classes and Outreach involved participants in collaborative work (e.g. discussing mathematical ideas and working with, and learning from, adult 'specialists' and fellow participants):

> It was all about finding out new things and watching what other people did and seeing how you can improve yourself. […] You could just be together and work as a group, discussing [the problems], seeing what each can find out about the answers, sharing [ideas] in a group.
>
> (Master-class participant)

In these three programmes, emphasis was placed on exploring mathematically and on elucidating mathematical thinking:

> [You] have to think about [the problem and] explore it a lot. It wasn't just [getting to the] answer. […] The whole idea was that you explored your answers a bit more: [….] to get at its nature rather than just leaving an answer — to question it a bit more and think why.
>
> (Master-class participant)

Such experiences were valued by participants of these three programmes as an integral part of their 'enrichment'. Where students were able to work on Maths-Challenge problems in class, the experience was similarly considered 'enriching'.

Enrichment 'benefits'

Participants from the four programmes also identified a number of ways in which they felt they had 'benefited' from their enrichment experience. These relate broadly to:

their mathematical, and personal and social, development; support for mathematics learning in school; and exposure to higher education. Looking beyond these broad categories of 'benefits', however, specific experiences were more subtly related to the characteristics of the programme in which the student had participated, interpreted according to more familiar experiences of learning mathematics in school.

For example, all Master-class and Summer-School participants interviewed felt that they had benefited mathematically, and personally and socially, from their experience. However, in the Master-classes — a programme where enrichment exposure was limited to a small number of weekly workshops — identified benefits were primarily in terms of gains in mathematical knowledge. In the Summer Schools — a programme where participants were in contact with one another over a relatively-long uninterrupted period and in wider social contexts — personal and social benefits and mathematics-related gains were identified in almost equal measure; references to both were interwoven in participants' accounts:

> [I've gained] a lot of knowledge, [a] lot of understanding and a lot of friends. I suppose I've gained some independence as well. I definitely understand maths more. And I suppose [I've] gained knowledge [about] how to challenge myself.

The Maths Challenge competitions, meanwhile, were thought to be engaging and exciting. The quality of the Maths-Challenge problems was consistently praised:

> I didn't know there [could be] that much fun working through these problems. I think that's why I found [the Maths Challenge] enriching. [...] It's just using your mind so much more. [When] you open the textbook, you know what topic [the questions are] on. But you don't with the Maths Challenge. [That's] what makes it more exciting.

This notwithstanding, the students interviewed gauged the 'benefits' they had derived primarily in relation to mathematics learning in school, and in some cases, particularly as 'revision' or 'practice' for high-stakes public examinations and as a means for judging their performance — even 'ability' — relative to other students nationally and in their school. The latter reflects the culture of assessment in schools through which the students understood the Challenges. Where 'enrichment' was not followed up and students simply 'sat the test', learning was thought to be limited. Indeed, some students had mixed feelings about the competition and its associated rewards:

> If you do well, then it [helps]. But if you don't do so well, then you're really congratulated for taking part, but you still might feel inside that you've not done very well compared to [others].

Similar reactions are likely to occur wherever competition is a salient feature of an enrichment programme.

The enrichment 'benefits' identified by students, the relationship between 'benefits' and programme characteristics, and students' interpretations of their 'enrichment' experience in the light of more familiar experiences with school mathematics are further illustrated in the following subsections.

Personal and social 'benefits'

The personal and social benefits most commonly identified across the four programmes were increased confidence and sense of capability, and opportunities to meet and work with other students.

For Outreach participants, increased confidence and sense of capability was attributed to positive problem-solving experiences and the sharing of mathematical

ideas; seeing other participants' endeavours and achievements was motivating and encouraged participants to recognise possibilities for themselves:

> [I saw others] working on [the problems. It] gave me a thirst to [prove myself], because at the end of the day, [we're] the same age [and] we've got the same abilities. [You see people working] and you just think, 'I can do it as well!' […] When you actually accomplish the puzzle, [you get] self-satisfaction. [That] boosts your confidence.

Meeting and working with (like-minded) students from other schools was also thought to be enriching in its own right.

For Summer-School participants, confidence was derived from the whole (social and academic) programme, which made them feel more independent and motivated (generally and towards studying mathematics). Participants also valued opportunities to work with like-minded students of similar levels of attainment:

> Everyone wants [to] work, [which] makes it a much better [working] environment. [The other students are of] equal ability to you, so they're pushing you [to] achieve the highest standards possible.

Through these opportunities, participants reportedly developed better interpersonal skills and began to feel less isolated.

For Maths-Challenge participants, confidence and sense of capability stemmed from 'successful' participation (defined differently by different students according to their prior experience) and favourable comparisons with other competitors:

> [What I particularly like is getting] a certificate at the end — that's good. And you can compare your scores with your friends. […] For a certificate, you feel like you've done well at something, [and] most of the time, I do better than my friends, so it feels good.

To some extent, this is consistent with enrichment goals to recognise and reward achievement, and to improve participants' confidence and self-esteem through the competitive process. The potentially-negative connotations which may be associated with such comparisons, however, were not intended. No social benefit was mentioned by Maths-Challenge participants; neither were social experiences a part of the intended provision.

Finally, Master-class participants made very few references to stimulating social exchanges or to having benefited socially from their experience; none reported any increase in mathematical confidence. This may be due to the limited time available for participants to build comfortable relationships with speakers and fellow participants, wherein stimulating social exchanges could take place. The Master-classes were successful primarily in providing an introduction to a variety of mathematical topics (see below).

Mathematical benefits

Mathematically, participants from all four programmes felt that enrichment had broadened their mathematical experience and horizon. In the Master-classes and Summer Schools, this was brought about primarily through introducing participants to stimulating aspects, and unfamiliar applications, of mathematics not normally taught in school; in Outreach, participants were encouraged to think mathematically by way of the problems posed and to apply their knowledge flexibly to solve challenging and unfamiliar problems. In accordance with these methods, the specific way in which Master-class and Summer-School participants felt that their mathematical experience

and horizon had been broadened was through becoming more aware of the breadth and pervasiveness of mathematics and its applications:

> You were introduced to maths in lots of different contexts. […] It makes you see that maths isn't just [about] numbers; it's in many different things.
>
> (Master-class participant)

Outreach participants, meanwhile, reported increased perseverance and improvements in the way they went about tackling unfamiliar problems:

> I've learnt a lot [in] the way I approach a question. [At] the beginning, I'd pick a random number [to try]. But now, [I'm] more systematic. […] And previously, if I couldn't do [a problem, I'd just] sit there until the answer comes. But as the sessions have gone by, [I've tried harder] to get the answer.

Those Outreach participants who reported greater appreciation for problem-posing and problem-solving also reported greater satisfaction and progress.

In addition, the majority of Master-class, Summer-School and Outreach participants interviewed felt that their interest in mathematics had increased. The majority of Outreach and Summer-School participants also reported improvements in their perception of mathematics. Maths-Challenge participants, however, reported limited mathematical gains, perhaps because enrichment exposure was only brief; the majority interviewed felt that the Challenges had made no difference to their interest or perception of the subject.

Support for mathematics learning in school

With the exception of the Master-classes, which participants perceived primarily as an opportunity to experience stimulating mathematics that is not necessarily associated with the mathematics they learn in school, the remaining programmes were each thought to 'benefit' mathematics learning in school in some way.

For Maths-Challenge participants, a key 'benefit' of participating in the competition was the support they gained for school-work through revision and practice afforded by the Challenges and the experience of working under test conditions in preparation for high-stakes public examinations:

> It gives you practice in [doing maths under] exam conditions. So it's helpful [for seeing] how you would react if some exam [is] sprung on you: [to] see what you already know and what you don't. [So regardless of how well you do], it's just an experience that will help you for other exams.

This confounds the fact that the Challenges were never conceived as such a means of enhancing school performance; interpretations of 'enrichment' as examination practice (in light of the culture of assessment in schools) were wholly unintended.

Almost all Outreach participants felt that their enrichment experience had consolidated mathematical ideas taught in school, and that this was helpful to them in their school-work. Indeed, some students participated in the programme in order to remedy perceived shortfalls in their school mathematics education, treating their enrichment experience as extra tuition. Whilst remedying shortfalls in school mathematics provisions was not a part of the intended 'enrichment', such an interpretation can be understood given that participants were drawn from specifically-targeted 'disadvantaged' schools.

A minority of Summer-School participants reported becoming more aware of connections within mathematics — a subject which had previously been presented to them in a more fragmented way in school. That said, there was no evidence that students felt any more appreciative of school mathematics after participating in the

Summer Schools. Indeed, from the many contrasts participants from all four initiatives had made between school mathematics and enrichment, enrichment was unlikely to have enhanced students' appreciation or liking for school mathematics in any of the four cases.

Exposure to higher education

Raising participants' educational aspirations and enabling participants to gain insights into higher education were central to the enrichment intentions of the Outreach programme and the Summer Schools. Both programmes were hosted in universities expressly for the purpose of bringing participants into higher-education settings. The benefits of being in a university, however, were felt only by Outreach participants and those (older) Summer-School participants at or towards the end of their secondary education. In addition to being able to see inside a university, these older Summer-School participants also valued the subject expertise from the institution:

> [This] is quite a good university for maths. So maths would be a good course to do here [because] the lecturers will know more about the subject [and] be better at teaching. [It's amazing] just how interesting it is and how [deeply] you go into [the maths. Being] a maths lecturer, [our tutor] has told us quite a bit about what we're doing and how it relates to degree and graduate work.

Outreach participants, on the other hand, valued the prestige associated with 'studying' at university and the change in environment that this afforded. The latter, in particular, encouraged them to adopt a more mature attitude towards learning and enabled them to shed the constraints associated with being interested in mathematics:

> [Being at university] makes you [feel] older [and more grown up]. People don't just sit there and do nothing. […] Everyone's working and everyone's interested in [thinking] about [maths].

In the case of the Master-classes, although the act of exposing students to a university environment was thought to be enriching, exposing participants to higher-education settings was not a central part of the conceived enrichment. Predominantly, participants acknowledged that the Master-classes had given them an opportunity to visit a university (which they thought of as a comfortable space for meeting and learning), but the experience of being inside a university was not associated with their experience of mathematics or learning.

Conclusion

The general consensus among participants was that enrichment presented valuable opportunities for them to engage in mathematics which they would not normally experience in school. Participants from all four programmes were able to identify a range of enrichment 'benefits' and experiences which they perceived as 'enriching'. These relate broadly to their mathematical, and personal and social, development; support for mathematics learning in school; and exposure to higher education.

Beyond this level of generality, participants' experiences were more subtly related to the characteristics of the programme in which they had participated, interpreted according to more familiar experiences of learning mathematics in school. So whilst 'enrichment' can offer valuable opportunities to extend students' mathematics learning, it can only really be 'effective' if offered alongside rich 'everyday' experiences of learning mathematics in school, because that 'daily diet' of mathematics is the basis on which students interpret and understand any additional 'enrichment' experience.

Acknowledgement

References

Bassey, M. 1999. *Case study research in educational settings*. Buckingham: Open University Press,

Bouie, A. 2007. *After school success: academic enrichment strategies with urban youth*. New York: Teachers College Press.

Clarke, C. 2004. *Making mathematics count: the Department for Education and Skill's response to Professor Adrian Smith's inquiry into post-14 mathematics education*. Nottingham: DfES Publications.

DfES and DTI. 2006. *The Science, Technology, Engineering and Mathematics (STEM) programme report*. London: HMSO.

DIUS. 2008. *A vision for science and society: a consultation on developing a new strategy for the UK*. London: DIUS.

European Commission. 2008. *Young people and science: analytic report*. Flash Eurobarometer Survey No. 239.

Feng, W. Y. 2005. *Conceptions of Enrichment*. Paper presented at the Cambridge Education Research Annual Conference (CamERA) at the University of Cambridge, April 21, in Cambridge, UK.

———— 2006. Conceptions of Mathematics Enrichment. Paper presented at the British Educational Research Association (BERA) Annual Conference at the University of Warwick, September 6–9, in Coventry, UK.

Piggott, J. 2004. *Mathematics Enrichment: What Is It and Who Is It For?* Paper presented at the Annual Conference of the British Educational Research Assoication, University of Manchester, UK.

Roberts, J. L. 2005. *Enrichment opportunities for gifted learners*. Texas: Prufrock Press Inc.

Smith, A. 2004. *Making mathematics count: the report of Professor Adrian Smith's inquiry into post-14 mathematics education*. London: The Stationary Office Ltd.

The Royal Society. 2008. *Science and mathematics education, 14-19: a 'State of the Nation' report on the participation and attainment of 14-19 year olds in science and mathematics in the UK, 1996-2007*. London: The Royal Society.

Zeeman, C. 1990. Christmas Lectures and Mathematics Master-classes. In *The popularization of mathematics*, ed. A. G. Howson and J.-P. Kahane. Cambridge: Cambridge University Press.

Reasoning on transition from manipulative strategies to general procedures in solving counting problems

Lourdes Figueiras and Maria C. Cañadas

Universitat Autònoma de Barcelona and Universidad de Granada

We describe the procedures used by 11- to 12-year-old students for solving basic counting problems in order to analyse the transition from manipulative strategies involving direct counting to the use of the multiplication principle as a general procedure in combinatorial problems. In this transition, the students sometimes spontaneously use tree diagrams and sometimes use numerical thinking strategies. We relate the findings of our research to recent research on the representational formats on the learning of combinatorics, and reflect on the didactic implications of these investigations.

Introduction

The work on combinatorics at school is restricted in many cases to the use of formulas that limit the development of reasoning. English (1991) and Fischbein and Gazit (1988) emphasised the interest of students' reasoning processes when solving combinatorial problems and their educational implications. At the time of such investigations, Piagetian theory affirmed its relevance to cognitive psychology, considering combinatorics as an essential component of reasoning. Currently, the interest in discrete mathematics and, particularly in combinatorics is increasing the research on this content (Jones, 2005).

Much previous research related to our interests has focused on early education, and the detected strategies emerge from a context where students solve counting problems whose solution is usually a number small enough to be obtained by enumerating all possibilities, and counting one by one afterwards. (Empson and Turner, 2006; English, 1991; English, 1993; Steel and Funnell, 2001). Within this context, English (2007) concludes that 7- to 12-year old students "with no prior instruction and receiving feedback only through their interaction with the physical materials, the children were able to apply their informal knowledge of the problem domain to their initial solution attempts." (p. 152) She suggests that activities with tree diagrams and systematic lists lead 11- and 12- years old children to derive the basic formula for combinations (p. 154).

In this paper, we analyse the case of five selected students of this age to explore how they use their previous knowledge to develop strategies leading to the use of the general, and highlight how numerical reasoning of the students arises using sophisticated representations different than tree diagrams. Consequently, we call into question attempts to start directed instruction of tree diagrams at a too early stage.

Reasoning strategies and scheme in counting problems

English's investigations (English, 1991, 1993, 2007) allowed characterising the strategies used by children between 4 and 12 for counting the possible arrangements of two and three elements. The identified strategies grant an important role to

manipulation and may be arranged in increasing order of complexity, from a resolution based on a trial and error approach to the odometer strategy.

For more sophisticated tasks, Fischbein and Grossman (1997) refer to a scheme as a program that allows the problem solver to interpret a certain amount of information and prepare the corresponding reaction. They consider that the procedure performed by students leads them to compute the solution to a counting problem as a scheme that allows expressing the total number of possibilities. The formulae for calculating the number of n-permutations or (*n,k*)-combinations are examples of schemes. The work carried out on combinatorial reasoning by Fischbein and Grossman (op. cit.) and Kollofell et al. (in press) recognise four basic patterns (schemes) of counting for solving these kinds of problems:

•Permutations of *n* elements.

•Arrangements without replacement: Number of selections of *k* elements which one may obtain from *n* given elements, considering that no element may be used more than once in a selection and that the order of elements is relevant.

•Arrangements with replacement: Number of selections of *k* elements which one may obtain from *n* given elements, considering that every element may be used more than once in a selection and that the order of elements is relevant.

•*k*-Combinations of *n* objects

These schemes have immediate didactic consequences, as combinatorial problems used to be classified also under four types, emphasising the presence of one of those schemes, which, once discovered, allow the student to find the solution. However, many combinatorial problems do not admit an approximation by these schemes because the criteria of repetition and order are not obvious. Moreover, many students fail to solve problems where those schemes must be modified or when the statement of the problem requires to reproduce the steps involved in the scheme construction. This fact led us to go back to the much more basic scheme involved in combinatorial formulae, namely the multiplication principle, which can be stated generally as follows:

> The total number of arrangements of k elements having n_1 possibilities for the first one, n_2, for the second and in general n_k for the *k*-th, each position being independent of the others, is $n_1 \cdot n_2 \ldots n_k$.

The multiplication principle of two sets is the simplest scheme that underlies those of permutations and arrangements with and without replacement, and together with basic arithmetic operations leads to that of combinations. Thus, we focus our attention on the way how students acquire this basic scheme by themselves, and pay special attention to the use of different representations. We use the theoretical concept of representational format as used by Kollofel (2008).

Representational formats in the combinatorial problem solving

From a cognitive perspective, Holyoak and Morrison (2005) emphasise the relationship between problem solving and representations performance of subjects and conclude that the representation used to solve a particular case is a key factor in solving the general problem. Rico (2009) characterised the notion of representation as all those tools -signs or graphics- which are present mathematical concepts and procedures and with which the subjects dealt with and interact with the mathematical knowledge, i.e., record and communicate their knowledge about mathematics.

There is agreement in mathematics education to distinguish between internal and external representations. Although both types of representation should be seen as separate domains, from the genetic viewpoint, external representations are

characterised by acting as a stimulus for the senses in the process of building new mental structures and allow the expression of concepts and ideas to individuals who use them. Ideas must be represented externally in order to communicate them (Hiebert and Carpenter 1992). We focus our attention on the external representations as those that have a trace or tangible support even when this support has a high level of abstraction (Castro and Castro 1997).

In the specific case of combinatorics, Kolloffel et al (2008) focus their research on three representational formats: (a) arithmetic, (b) text and (c) diagrams. Diagrams are considered to help students to understand new situations. Its functionality and, particularly, tree diagrams, has been analysed in several studies. For example, Fischbein and Gazit (1988) give the maximum benefit to the tree diagrams in its instructional programs. They consider that tree diagrams are representative of a state of maturity in the counting strategies. The effectiveness of this type of graphical representation has been questioned by Kolloffel (op.cit), arguing that the benefit is restricted only to conceptual learning, and taking into account the possibility of combining two or more representations. Moreover, in our investigation, we will deal with a new category of representational format which also integrates two or more representations, but under the additional condition that none of them by themselves make sense of the problem. We'll call this new type of representation synthetic representation.

Research objectives and methodology

Our research aim is to describe how students become able to use the multiplication principle in the context of a combinatorial problem. This general objective is broken down into two specific objectives:

- To characterise the strategies used for solving combinatorial problems which solution is a number larger enough so that the students can not calculate it by enumerating all possibilities and then counting them one by one.
- To characterise the process of solving a problem leading to generalize the multiplication principle in terms of the representation used.

We selected five cases from a previously selected sample of 25 students. Previous research on children's strategies for solving combinatorial problems (English, 2007) informed the first decision on the selection of this sample from a cognitive point of view, that was consider a group of students about 12 years. On the other hand and in order to minimize other contextual variables, it was decided to gather data under optimal conditions for the students' involvement and interest for the activity. This informed a second decision on the selection of the sample. In respect of it and to make it clearer to the reader, we need to expose briefly about a national project seeking to stimulate mathematically talented students. This project's main objective is to identify, guide and stimulate interest of students aged 11 and 12, who are particularly attracted by the beauty, depth and usefulness of mathematics (Hernandez and Sanchez 2008). Mathematic teachers are informed through teacher's associations about the project and asked to propose possible candidates to joint it. There are also public calls in newspapers and the internet. From about 300 candidates, only 25 are selected by a test of mathematical problem solving and by interviews to ascertain their interest in participating. Selected students show a good aptitude and attitude toward mathematics, but they are not necessarily gifted.

The group of 25 students participating in the project last year became our research sample and data were collected during the programmed sessions within the context of the project. They had not worked previously in combinatorics and were at

the beginning of the first three sessions devoted to this subject. The objectives for these sessions included reasoning and deducing counting methods, and the first step was to inquire about the multiplication principle. In this paper, we focus on analysing the first three questions proposed (Figure 1), as they were especially oriented towards obtaining information about student's depth of knowledge on the multiplication principle. We asked them to elicit all their actions by writing and then analysed students' reports on the solution of the problem. For the selection of cases to be analysed, the first step was to consider if question 1 was solved correctly or not, and if the correct solutions made use of the multiplication principle immediately. Cases would be interesting for us if the students solved the problem correctly but did not immediately use that rule.

You can construct different animals with the cards that you have on the table. (There are a total of 48 cards: 16 heads, 16 bodies and 16 tails)

Q1. How many different animals can you make?

Q2. How many of them have the head of a crocodile and the tail of an elephant?

Q3. How many of them have the body of a chameleon?
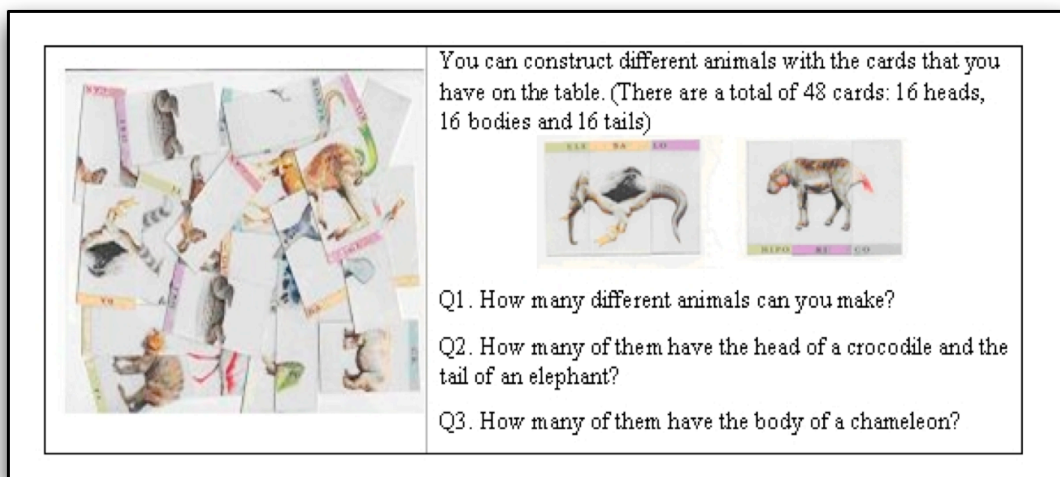
Figure 1: Proposed problem with three questions. The cards were provided to the students.

Twenty students solved the problem by directly multiplying 16·16·16 and elicited no more actions, but five did something different. These five cases were considered for further research. The fact that in a more or less homogeneous group in terms of age, interest and capacities most of the students use immediately the multiplication principle, allows us to assume that those who do not do it are still in the process of consolidate it. This justifies why the selected cases were considered especially relevant to our investigation.

A bird's eye on the selected cases

Ana

Ana uses a tree diagram to represent the statement, which permits her to organise all possibilities (see Figure 2). In order to create this diagram, Ana sets a first card for the animal's head and covers all possibilities for this. For the particular case of two cards in the second position, the student represents all possible options. Following the categories proposed by English (1993), Ana is using the most effective exhaustive strategy for counting.

The tree diagram allows the student to represent the easier two dimensional problem (1-16-). Then Ana introduces a textual representation of this particular case, which permits her to identify a multiplicative pattern expressed arithmetically (1·16),

thus solves the two-dimensional problem, and leads to extend the tree diagram to analyse the dimensional situation -16-16. Again, a textual representation leads to represent the solution mathematically: 16 (2nd position) x 16 (3rd position) x 16 (1st position), following the order reflected in the tree diagram.

Ana also comes to use a symbolic representation of three-dimensional patterns. After solving the problem, the solution of questions 2 and 3 is given immediately by multiplying the number of possibilities for each position and we could infer that the multiplication principle has been interiorised.
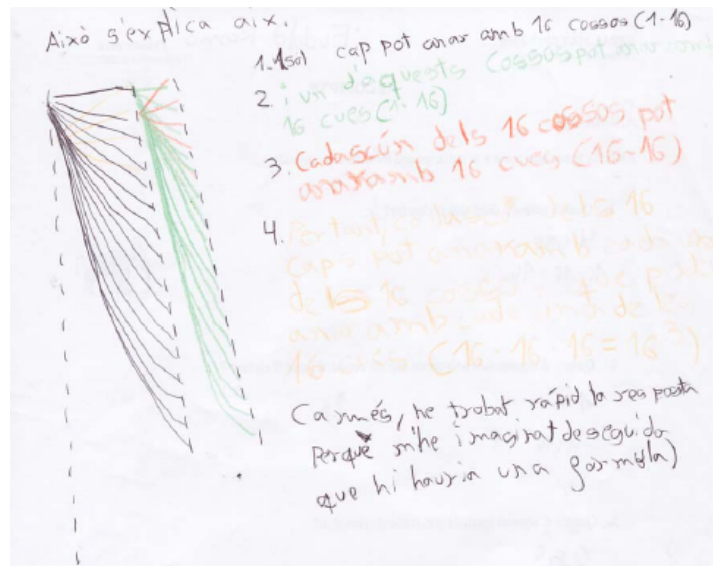
Figure 2. Tree Diagram used by Ana. (Translation: 1. Each head can combine with 16 bodies // 2. And one of these bodies can be combined with 16 tails // Each one of the 16 bodies can be combined with 16 tails // Then each of the16 heads can be combined with each one of the 16 bodies, which can be combined with the 16 tails.

Biel

Biel starts by solving the particular problem in which the number of cards for each item is 3. In contrast to the previous student, Biel has taken this step without a prior external representation. Then, he uses a comprehensive representation of all possibilities through the tree diagram that allows counting all options one by one, and leads him to detect some pattern, which is expressed as 3^3.

After solving an easier problem using cards, Biel generalises his arithmetic representation of 3 cards to the problem with 16 cards, and writes 16^3 as the solution of the proposed problem. Then he also expresses immediately the solutions to questions 2 and 3 in terms of multiplication of possibilities for each position.

Carles

Carles identifies a multiplicative pattern in the statement and write down two conjectures: 16·3 and 16^3. Afterwards, the student proceeds to justify them. To this end, Carles uses a representation of the conjecture 16^3 in which we find elements of a tree diagram and a textual representation. Each form of representation does not give meaning to the problem independently, but taken together, they do, what reinforce our theoretical position about consider a new category of representational format integrating two representations.

After obtaining an answer to the problem, Carles tries to refute the guess 16·3 with a textual representation of the operation in the context of the problem:

> The 3 are the parts they have, but have no relationship,

thus suggesting that the multiplicative pattern that responds to a sum repeated 3 times does not fit the problem. As Ana and Biel did, after the solution of the first question, Carles immediately expresses the solution of questions 2 and 3 using the multiplication principle.

David

David addresses the problem by combining six cards belonging to two real animals. He solves it by counting one by one 8 animals. He then seeks to solve the problem for the case of three real animals by extending the previous result, and writes down the following false conjecture:

> With the three selected animals, 3 different pairs of cards can be made. For each pair 8 possibilities were obtained [before], and therefore the total number of possibilities will be 8 x 3 = 24.

Furthermore, he counts one by one the 27 possibilities for this case, what leads him to refute the initial guess. Thereafter, David uses a synthetic representation (textual-arithmetic) that allows him to argue what was wrong.

Finally, he tries again to look for a new pattern from the particular cases of two and three animals, and finds the correct answers 2^3 and 3^3. From that concludes the correct answer 16^3. However, in contrast to Ana, Biel and Carles, he fails to identify the multiplication principle to solve the second and third question immediately.

Eva

Eva's reasoning is similar to that of David, beginning with one particular problem, guessing three different conjectures, and looking for their justification or refutation through arithmetical representations (Figure 3). She also faces the solution of questions 2 and 3 as new independent problems.

Discussion of results

A first approach to the work of these five students permits us to identify that three of them (Biel, David and Eva) start by a certain inductive process of reasoning, while the other two (Ana and Carles) use a specific representation of the problem. Based on this initial difference, we have identified the different steps that each of the students performed, and the representational formats they used in solving problems (see Table 1). David and Eva show a very rich inductive process from a heuristic point of view. This is very effective for reasoning in mathematics, and allows them to effectively solve the first question. However, they do not respond directly to questions 2 and 3 in terms of the multiplication principle as the others do. Their reasoning enables them to generalise about the number of possibilities for each of the three placements (16) but not about the number of placements. As a consequence, they do not generalise the multiplication principle. David and Eva do not use tree diagrams.

In contrast, the reasoning process followed by Ana, Biel and Carles does lead to generalise the number of placements. Focusing on the students who use the tree diagram, its use is more effective for Biel, who uses this diagram to represent a particular case. In this sense, we suspect that the failure to address the problem by working with individual cases increases the effectiveness of this representation for the generalisation of the multiplication principle.

Table 1. Each cell shows the representational format used at different steps of students' reasoning.

	Cases				
	Ana	Biel	Carles	David	Eva
Complete representation of the statement	Graphic (diagram)		Arithmetic		
Study of particular cases	Synthetic (Arithmetic - textual)	Graphic (diagram) Arithmetic		Synthetic (Textual-Arithmetic)	Arithmetic
Organisation of particular cases				Synthetic (Textual-arithmetic)	Arithmetic
Pattern detection	Arithmetic	Arithmetic	Arithmetic	Synthetic (Textual-arithmetic)	Arithmetic
Conjecture			Arithmetic	Synthetic (Textual-arithmetic)	Arithmetic
Justification of the conjecture			Synthetic (Graphic-textual	Synthetic (Textual-arithmetic)	Arithmetic
Generalisation	Algebraic	Arithmetic	Arithmetic		

Conclusions

This research responds to theoretical and teaching interests. From a theoretical point of view, it provides specific information concerning the use of inductive reasoning in solving combinatorial problems. Concerning representational formats, we confirm that the use of tree diagrams allows students to abandon manipulative strategies and move towards generalisation. Furthermore, we have analysed how such diagrams are used to construct the multiplication principle, in some cases extending recent research findings. In particular, we have seen that the diagrams have been used by students at different stages of cognitive processing spontaneously, using them to represent particular cases from which the multiplication principle is derived efficiently without specific instruction.

We highlight the importance of using materials in problem combinatorial problems. As we observe in the analysed cases, students can use them till they feel comfortable with written representation. This has been observed in different stages of the inductive procedure, which lead students to generalise the multiplication principle and justify their conjectures.

This study support that experience with two-dimensional combinatorial problems help students to adopt more efficient strategies for three-dimensional combinatorial problems, as English suggested in previous studies.

The findings have a direct impact on instruction in the area of combinatorics. Although we do not question the importance of instructional programs including the use of tree diagrams to generate algorithms for enumeration and counting, we suggest that they should not be the first approach to combinatorial problems. We base this recommendation not only on the fact that some students are able to produce for themselves this type of representation, but on the fact that some students face basic combinatorial problems using inductive reasoning and not tree diagrams. In these cases, although the process to generate the scheme of the multiplication principle is slower, students advance significantly towards generalisation from particular cases.

Acknowledgements

References

Castro, E. and E. Castro. 1997. Representaciones y modelización. In *La educación matemática en la enseñanza secundaria*, ed. L. Rico, 95-124. Barcelona: Horsori.

Empson, S. B. and E. Turner. 2006. The emergence of multiplicative thinking in children's solutions to paper folding tasks. *Journal of Mathematical Behavior* 25: 46-56.

English, L. D. 1991. Young children's combinatoric strategies. *Educational Studies in Mathematics*, 22(5): 451-474.

———1993. Children strategies for solving two- and three- dimensional combinatorial problems. *Journal for Research in Mathematics Education*, 24(3): 255-273.

———2007. Children´s strategies for solving two- and three-dimensional combinatorial problems. In *Stepping stones for the 21st century: Australasian mathematics education research*, eds. G. C. Leder, and H. J. Forgasz, 139-156. The Netherlands: Sense Publishers.

Fischbein, E. and A. Gazit. 1988. The combinatorial solving capacity in children and adolescents. *Zentralblatt für Didaktik der Mathematik*, 5: 193-197.

Fischbein, E. and A. Grossman 1997. Schemata and intuitions in combinatorial reasoning. *Educational Studies in Mathematics*, 34: 27-47.

Hernández, E. and M. Sánchez. 2008. ESTALMAT: un programa para estimular el talento matemático precoz. Unión, 16: 113-122.

Hiebert, J. and T. Carpenter. 1992. Learning and teaching with understanding. In *Handbook of research on mathematics teaching and learning*, ed. D. Grows, 65-97. New York: MacMillan.

Holyoak, K. J. and R.G. Morrison. 2005 . *The Cambridge Handbook of Thinking and Reasoning*. Cambridge: Cambridge University Press.

Jones, G. A. 2005. *Exploring probability in school: Challenges for teaching and learning*. Dordrecht: Kluwer.

Kolloffel, B. 2008. The effects of representational format on learning combinatorics from an interactive computer simulation. *Instructional Science*. Open access DOI 10.1007/s11251-008-9056-7

Rico, L. 2009. Sobre las Nociones de Representación y Comprensión en la *Investigación en Educación Matemática*. PNA, 4(1): 1-14

Steel, S. and E. Funnell, E. 2001. Learning multiplication facts: A study of children taught by discovery methods in England. *Journal of Experimental Child Psychology* 79: 37-55.

Identifying and Developing Strategies: Beyond Achievement

Dr. Haneet Gandhi

Central Institute of Education, University of Delhi, Delhi 110007, India.

To put into praxis the theoretical assumptions that self-regulation skills are teachable (Schunk and Zimmerman, 1998), this paper describes a research on the efficacy of an instructional approach, the Strategic Content Learning approach adopted to promote self-regulated learning in average mathematics performers of grade VIII of Indian schools. These students with poor metacognitive skills, who lacked productive approaches in implementing or adapting learning strategies, were helped to manage their cognitive, volitional and motivational skills. Concomitantly, the students were helped in identifying and developing strategies while solving problems in mathematics. They also developed their personalised strategies that they could transfer across problems and time, thus becoming better self regulators.

Keywords: Self-regulation, metacognition, Strategic Content Learning

Naïve self-regulators need support and assistance to engage flexibly in the sequence of cognitive processes that comprise self-regulated learning (Schunk and Zimmerman, 1998). Drawing from a socio-cultural perspective social dialogue between the more learned peer and/or adult conducted within the context of meaningful tasks in students' 'Zone of Proximal Development (ZPD)' (Vygotsky 1978) promotes self-regulation. This general description of scaffold instructions though agreeable leaves the exact nature of adult guidance unspecified. Stone (1993) further articulates the nature of interactive instructions and suggests that support must be provided during scaffold instruction in a form of 'prolepsis' i.e. instructors make comments or statements that students strive to interpret, given their current incomplete understandings (optimally in their ZPD). It is this quest to make sense of the adult communications that promotes the active construction of knowledge and that spurs students' development of self-regulation. Butler (1995, 1998a, b) proposed an instructional approach, 'Strategic Content Learning (SCL)' to implement the theoretical assumptions of Stone (1993).

In Strategic Content Learning Approach, both instructors and students are equally charged with interpreting each other's comments, as a way of establishing a shared communicative context within which interactive discussions will be meaningful to students (Butler 1998a, b; Gandhi 2009).

This paper summarises the instructional dynamics and research findings of a study advocating the efficacy of SCL approach in promoting self-regulated learning in average mathematics performers of grade VIII of Indian origin. The study helped these students to remediate their performance by reflecting on their inaccurate understanding of the mathematical tasks, unproductive metacognitive knowledge, negative motivational beliefs, interfering external causal attributes such as frustration and anxiety, and faulty self-regulated skills.

An Overview of Strategic Content Learning (SCL) Model

In Strategic Content Learning approach, students are supported to engage in the cycle of self-regulated activities associated with successful learning. These activities include analysing task demands, selecting, adapting, or even inventing personalised or task specific strategies, implementing and monitoring strategy effectiveness, self-evaluating performance, and revising goals or strategies adaptively (Figure 1).
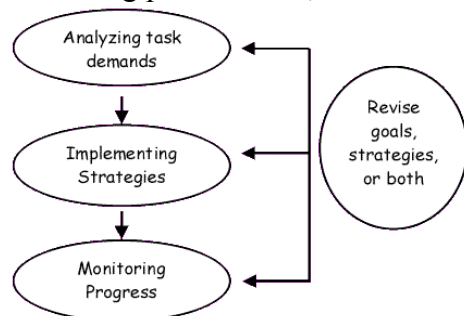
Figure 1: Strategic Content Learning Approach (Butler 1995, 1998a, b)

SCL is based on an analysis of self-regulated or strategic performance. Key instructional goals are defined, including students' construction of metacognitive knowledge, motivational beliefs, and self-regulated approaches to learning. In making students self-regulated a central instructional guideline is for teachers to support students' reflective engagement in cycles of SRL (i.e., task analysis, strategy implementation, self-monitoring). For example, to support the sampled group of students in solving problems in mathematics, the teacher started by helping them analyse the common task selected (problems in mathematics). They were asked to interpret available information (e.g. information given in the problem). They were guided to identify and implement strategies for meeting task requirements (e.g. organising the given information, finding relationship between the given information and what has been asked for in the problem). Finally, the students were supported to self-evaluate outcomes in light of task criteria (e.g. Are they happy with the solution strategy, can there be any other method of doing the same problem) and to refine their task-specific strategies so as to redress problems or challenges encountered (e.g. is their chosen method an elegant one, to compare and judge the most appropriate solution strategy for solving the problem).

A primary emphasis was not on teaching predefined strategies for completing academic tasks but to think about what the students would have done on their own if the teacher weren't there. The teacher guides students in their cognitive processing so that they become successful, intervening only when required. No direct explanations of the concepts are given. From a theoretical perspective, it could be argued that, if instruction focuses primarily on the direct explanation of predefined strategies, students may be inadvertently excluded from the problem-solving process central to self-regulation (Butler, 1993, 1995). If it were the teacher who has analysed a task, anticipated problems, and defined useful strategies, then students would have little opportunity to solve problems themselves and arrive at strategies. To avoid this problem, the teacher co-constructed strategies with students, bridging from task analysis. The teacher and the students worked collaboratively to find "solutions" (i.e., strategies) to the given problem. So, for example, when defining strategies for solving a problem, the students were to consider strategy alternatives in light of task demands (e.g., what strategies will they adopt to solve the problem, will they make a table, draw a diagram or do guess and check, etc.). Then, while working through the task

collaboratively, the students were supported to try out strategy alternatives (e.g., to apply different problem solving strategies to solve the same problem), judge strategy effectiveness (e.g., whether they found the ideal and an appropriate strategy, how do they know), and modify strategies adaptively. Over time, through these iterative processes, the students (ideally) learnt how to construct personally effective strategies for meeting varying problems in mathematics.

Implications for instruction based on this integrated view are that teachers should (a) Collaborate with students to complete meaningful work (to generate a context for communication), (b) Diagnose students' strengths and challenges by listening carefully to students' sense making as they grapple with meaningful work, (c) Engage students in collaborative problem solving while working towards achieving task goals, (d) Provide calibrated support in given students' areas of need to cue more effective cognitive processing, (e) Use language in interactive discussions that students might employ to make sense of experience, and (f) Ask students to articulate ideas (e.g., about task criteria, productive strategies) in their own words to promote distillation of new knowledge.

For instance, to support average mathematics performers with their math problem solving, each group and the teacher worked collaboratively on the mathematical problems to set a context for communication (collaborating to complete meaningful work). The teacher began by observing students solve one or two problems, asking them to think aloud and discuss with their peers as they worked (diagnosing students' strengths and challenges). Attention focused on how they interpreted their task (the given problem), interpreted or understood mathematical concepts, represented problems, identified solution strategies and implemented procedures, and monitored their work collaboratively. Then, as described earlier, the teacher assisted the students to work recursively through cycles of task analysis, strategy use, and self-monitoring (collaborative problem solving while working towards finding a suitable strategy). When the group did well, the teacher supported them to recognise their success and reflect on the strategies they just used that worked (articulating ideas). The students documented these strategies in their personal math journals that they could review, test, and refine over time. When they encountered difficulties, the teacher assisted them to solve more effectively (calibrated support). For example, sometimes the teacher directed their attention to a sample problem and supported them to interpret that information. The students were helped to verbalise new insights and to try out new ideas (articulating ideas). Note that depending on the whole group's areas of difficulty discussion focused on problem-specific strategies (e.g., how to solve an algebraic equation), strategies usefulness for solving math problems in general (e.g., always checking your work in-between the steps, seeing patterns), and/or strategies focusing on learning math more independently (e.g., working through simpler examples if stuck, breaking the problem in parts, plan sub problems while working, computing on smaller numbers instead of large numbers).

Methodology

The study was carried out with 5 small-groups (5-8 students in each group) of average mathematics achievers of class VIII from 5 different English medium schools of India. Both boys and girls were selected.

An essential component of the study was students' voluntary participation; therefore in the first stage of multi-purposive sampling only those students were selected who voluntarily felt the need of assistance in learning mathematics. Subsequently, average achievers in mathematics (60-65% in last two years annual

exams in mathematics) with average intelligence (45-55 percentiles on Cattle's Culture Fair Intelligence Test: Scale II) were finally selected from 5 different schools.

With each group the sessions were conducted for 15 days (of 1-1½ hour duration) wherein students worked in groups (peers and teacher) within the instructional dynamics of SCL. In a session 3-4 problems were taken and after completing the discussion on a problem the students were given private space and time to write their personal reflections in their respective journals.

In the pre-post research design the indicators of self-regulation were assessed both qualitatively and quantitatively. Qualitative data that assessed students' gradual success in becoming self-regulated by gauging their areas of problems and their approaches to overcome them was obtained through observations, researcher's field notes and comments, transcripts of audio tapes, students' self-reflections taken out from their math journals and the changes in initiation roles of students and instructor as the instructional sessions progressed. The quantitative inputs were attained through questionnaires on metacognition, self-efficacy, and perceptions of causal attributions that students generally state for their good or poor performance in mathematical tasks. The same questionnaires were administered during the pre and post-test situations. In addition to above, two parallel sets (one each for pre and post test) of non-routine mathematical problems were made to assess students' accomplishment in solving mathematical problems. This coalescence of inputs from qualitative and quantitative sources provided an in-depth view of each student's progress and a record of process of the instructional intervention. It also allowed for an explicit tracing of the relationships between instructions and outcomes.

Results and Discussions

One of the most consistent findings across the five groups was a positive shift in student's knowledge and beliefs related to the process of learning. In each group it was observed that students had developed a focused understanding about mathematical problems and problem solving strategies. For instance, a shift in comments of one of the students can be observed through his annotations taken from his math journal from three phases: initial, middle and last. There were instances of positive shifts in student's metacognitive understanding revealing his better understanding about mathematical problems, problem solving strategies and management of learning.

> In the initial phase (on one of the problems): I didn't find this problem interesting. My friends could do it, but somehow I did not understand anything. It was stupid of me to think in those terms… I felt a little awkward when the problem was over, because it was stressing me… I did not understand a bit what I am supposed to do…

> By the middle phase his reflections were more insightful: I needed help from my group members… It is good to write and think… I underlined the important words…it was also fun to draw diagrams…

> In the latter sessions: It was an easy problem. I could break the problem into sub parts. …solving smaller problems was easy… though lengthy but it leads to the answer. I understood the method to solve it. It was great. I liked this problem.

A noticeable increase was also observed in improved confidence level in students' work and in their learning styles. This can substantiated from the transcripts of the audiotapes extracted from early, middle and last phases of the interactions.

Students: Ma'am we are really bad in mathematics

Teacher: What makes you think that?

Student1: I study a lot but when I see the paper I go blank

Student2: I don't think I can do well in math…. I can't remember the formula

Student3: I do lot of careless mistakes. I just can't get the answer.

Teacher: What do you think would help you to improve?

Student1: I don't think I can ever like the subject

Student2: We always get low marks.

A conversation of the same group, taken from middle phase:

Students: Ok, if we would have thought of it a little more we would have definitely done it.

Teacher: How do you feel after doing this question?

Student1: Good, … I think I can now do better in my exams. Ah! (Relaxed)

Latter phase transcript:

Students: I think we know how to do this. Please don't help us. Give us time we will show you.

(They worked in the group for 5-10 min)

Students: We have done and checked the answer too.

Teacher: Have you verified?

Student: I think we don't need to… well… it's done and …correct!.

As the intervention progressed, students had developed personalised strategies that targeted their solution to the tasks. To trace changes in student's strategic approaches to problems in mathematics, their strategy descriptions were chronicled over time and related to their specific difficulties with tasks. Analyses also depicted that most of the students had independently transferred strategies across contexts, time and problems. Transcripts of one such observation from one of the groups:

Early Phase: We don't know how to solve it we have never seen such problems. … How can we do this… any hints? (looks at the teacher)

Middle phase: O.k. now let's think. First, let's break the problem and understand what it says…Maybe converting it into algebraic form may help.

Last phase: (Each student was equally participating in the solution. Each one of them gave a suggestion and after much discussion, grappling and convincing each other they came to a common consensus): This could be done by drawing a table as I see this value is increasing as that. (After a pause).. can you see this relation… well let's start… (there is lot of discussion and they stated scribbling in the sheets).

Comparing the verbatim from the phased transcripts provide evidence of students becoming strategic performers as the intervention sessions progressed. They could understand problems more quickly, think of the problem solving strategies that were useful in solving problems and could justify and verify their strategies. Initially students didn't have a faith in their working but gradually they became active participants who could control their thinking. They had a better understanding of their strategies and could judge their own working.

Early phase interactions:

Students: What should we do?

Teacher: Think…

(Long silence, no progress)

Teacher: It is better if you first rethink on what you have already done. (No response)

Teacher: let's try to rethink on our previous steps. Ok let's see what is given. (No action taken)

Teacher: Ok let's do it altogether again. Let's try to put whatever has been asked for

(The students and the teacher re-look at the given problem and the teacher rereads the problem several times, underlining some key words that might help them)

Teacher: So… should we begin?

Student: Ah… (Pauses and hesitant of replying)…ah… does it mean we take the first value as "x"

Middle phase interactions:

Teacher: Why did you work backward?

Students: because the problem is like that. (Students work on the problem)

Students: See it becomes easier like this…

Last phase interactions:

Students: This was a good problem. We enjoyed it.

Teacher: What did you learn?

Students: That this problem was not as difficult as we thought. We solved each subpart in an order. We just thought if we draw the diagram this might help… Ayush could see it after we drew this. Now he also agrees…

As the sessions progressed it was observed that students had started taking lot of interest in their work. They had all, eventually, learned to think about the task, devise or plan the strategies that would be most appropriate in solving a problem. They often provided convincing logical arguments in support of their selecting or choosing the problem solving strategy. It was observed that in initial sessions students took lot of time to solve a problem. They usually took 10-15 min in only grappling with the problem with no concrete idea of initiating the work. Peer discussions were also limited and not all the students were involved. Students generally relied on the teacher to begin and provide the solution. In gradual sessions it was observed that students took less time to solve the problems and finally in the latter sessions it was observed that students could efficiently organise their strategies and hence could come to the solution in much less time and with fewer instructions from the teacher. This account of time and instructions corroborate students as becoming better mathematics problem solvers. The collective data advocates improvements in students' increased awareness of selection, adaptation or invention of personalised or task specific strategies.

Strategies that students developed included steps focusing on each of the cognitive processes central to self-regulation. For example, students' strategies included steps related to problem analysis (e.g. " find out what the problem is asking for"); strategy selection based on problem requirements (e.g. "I think we need to make a table"); strategy use (e.g. "most of the times it helps to break the problem"); self-evaluation (e.g. "reread and think about how my equation relates with given information of the problem") and strategic adjustments (e.g. "if confused, take a break and rethink about the underlined words").

As suggested earlier, strategic learning may be best evidenced when students responsively adapt strategic approaches based on task demand. Thus, a good measure of shifts in self-regulated approaches would be student's independent development of strategies. In this study, students were observed to add steps to their developing strategies that targeted activities such as task analysis, strategy selection, self-evaluation, and self-monitoring. Evidence for changes in student's self-regulated approaches was also provided by student's descriptions of how they transferred strategic approaches for use across problems. In many cases, students described adapting specific strategy steps from their previously attempted strategies. For instance, while attempting a problem they commented:

> "Oh! This is similar to the one we had done before; the only difference is… This should be simple to solve."

Another example can be drawn from the math journal of one of the students:

> "I knew what she (teacher) wants us to do, so, I translated (from English to Hindi) it (problem) for my friends. Heena did not understand it. When Pinky and I were talking about it, Heena got it … the way we do by putting arrows between the variables helped in making connections…. I was happy that we were right. I knew we should work systematically because if we do not plan we can't get it right…"

Consistent shifts in student's attributions indicated that students were more likely to attribute successful performance to internal factors like ability, effort, strategy use, motivation rather than the external factors like luck, help from others or conditions in the environment.

Conclusion

Through these qualitatively detailed and cumulative analysis it was ascertained that the participants not only developed and mastered task-specific strategies, but also learned how to self regulate more effectively. The research had served to introduce the Strategic Content Learning as a successful instructional approach for promoting Self-Regulated Learning in average performers of mathematics of class eight in small group situation. Notable results were consistent gains in task performance and metacognitive awareness about mathematical tasks and strategies. Also important were the findings that all the students were actively involved in developing strategies for themselves, and that the majority of students reported adapting productive approaches for use across the problems.

In short, the SCL approach seems to be a workable instructional approach wherein students and teachers work collaboratively in a shared communicative context and in the process of striving to understand each others comments become better performers and reflectors.

Notes

The names are fictional.

References

Butler, D. L.1995. Promoting strategic learning by post-secondary students with learning disabilities. *Journal of Learning Disabilities*, 28:170-190.

_____ 1998-a. The Strategic Content Learning approach to promoting self-regulated learning: A summary of three studies. *Journal of Educational Psychology,* 90(4):682-697.

_____1998-b. A Strategic Content Learning Approach to promoting Self-regulated learning by students with learning disabilities. In *Self Regulated*

Learning. From Teaching to Self-Reflective Practice, ed. D. H. Schunk and B. J. Zimmerman, 160-183. New York: Guildford Publications.

Butler, D.L. and Winne. P.H. 1995. Feedback and self-regulated learning: A theoretical synthesis. *Review of Educational Research*, 65:245-281.

Gandhi, H. 2009-a. Problem solving in mathematics through Strategic Content Learning approach: Promoting self-regulated learning. *Unpublished doctoral dissertation.* Department of Education, University of Lucknow, Lucknow, India.

Gandhi, H., and M. Varma. 2009-b. Strategic Content Learning approach in promoting self- regulated learning. Paper presented in epiSTEME-03, An International Conference to Review Research on Science, Technology and Mathematics Education: India. 119-124

MacLeod, W.B. Butler, D.L. and Syer. K.D., 1996. Beyond achievement data: Assessing changes in metacognition and strategic learning. Paper presented at the annual meeting of American Educational Research Association: New York.

Schunk, D. H. and Zimmerman, B. J. 1998 *Self Regulated Learning. From Teaching to Self-Reflective Practice*. New York: Guildford Publications.

Stone, C.A. 1993. What's missing in the metaphor of scaffolding? In *Contexts for Learning: Soci-cultural Dynamics in Children's Development,* ed. E. Forman, N. Minick and C.A. Stone. 169-183. New York: Oxford University Press.

Vygotsky. L. 1978. *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press.

Zimmerman, B.J. (1989). A social cognitive view of self-regulated academic learning. *Journal of Educational Psychology*, 81, 329-339.

Developing early algebraic reasoning through exploration of the commutative principle

Jodie Hunter

University of Plymouth

Student transition from arithmetical understandings to algebraic reasoning is recognised as an important but complex process. An essential element of the transition is the opportunity for students to make conjectures, justify, and generalise mathematical ideas concerning number properties. Drawing on findings from a classroom-based study, this paper outlines how the commutative principle provided an appropriate context for young students to learn to make conjectures and generalisations. Tasks, concrete material and specific pedagogical actions were important factors in students' development of algebraic reasoning.

Introduction

For those students who complete their schooling with inadequate algebraic understandings access to further education and employment opportunities is limited. An ongoing concern for these students in New Zealand and internationally, has resulted in increased research and curricula attention of the teaching and learning of algebraic reasoning. To address the problem one response has been to integrate teaching and learning of arithmetic and algebra as a unified curriculum strand in policy documents (e.g., Ministry of Education 2007, National Council of Teachers of Mathematics 2000). Within the unification of arithmetic and algebra, students' intuitive knowledge of patterns and numerical reasoning are used to provide a foundation for transition to early algebraic thinking (Carpenter, Franke, & Levi 2003). Importantly, this approach requires the provision of opportunities for students to make conjectures, justify, and generalise their mathematical reasoning about the properties of numbers.

Carpenter and his colleagues explain that deep conceptual algebraic reasoning is reached when students engage in "generating [mathematical] ideas, deciding how to express them …justifying that they are true" (2003, 6). We know, however, from exploratory studies (e.g., Anthony & Walshaw 2002, Warren 2001) that currently many primary age students have limited classroom experiences in exploring the properties of numbers. These studies illustrated that more typically students experience arithmetic as a procedural process. This works as a cognitive obstacle for students when later they need to abstract the properties of numbers and operations. These studies also investigated student application of the commutative principle and illustrated that many students lack understanding of the operational laws. Both Anthony and Walshaw's study of Year 4 and Year 8 students and Warren's studies involving Year 3, Year 7 and Year 8 students demonstrated that many students failed to reach correct generalisations regarding commutativity. The students recognised the commutative nature of addition and multiplication; but also thought that subtraction and division were commutative. Anthony and Walshaw showed that although students offered some explanation of the commutative property none offered generalised statements nor were many students able to use materials to model conjectures related

to arithmetic properties. These researchers concluded that very few students were able to draw upon learning experiences which bridged number and algebra.

Nevertheless studies (e.g., Blanton & Kaput 2003, Carpenter et al. 2003) which involved teaching experiments provided clear evidence that young children are capable of reasoning in generalised terms. These studies illustrated that they can learn to construct and justify generalisations about the fundamental structure and properties of numbers and arithmetic. Importantly, they demonstrated that when instruction is targeted to build on students' numerical reasoning they can successfully construct and test mathematical conjectures using appropriate generalisations and justifications.

Theoretical framework

The theoretical framework of this study draws on the emergent perspective promoted by Cobb (1995). From this socio-constructivist learning perspective, Piagetian and Vygotskian notions of cognitive development connect the person, cultural, and social factors. Therefore, the learning of mathematics is considered as both an individual constructive process and also a social process involving the social negotiation of meaning.

I draw also on the body of research that suggests that making and representing conjectures, generalising, and justifying are fundamental to the development of algebraic reasoning (Kaput 1999). For young children the development of early algebraic thinking needs to go beyond simply making conjectures. Children need to gain experience in using mathematical reasoning to make explicit justifications and generalisations. Constructing notations for representing generalisations is also an important part of the generalising process (Carpenter et al. 2005). Carpenter and Levi (2000) promote using number sentences to provide students with access to a notational system for expressing generalisations precisely. Also, number sentences provide a context whereby students' implicit knowledge becomes explicit.

Whilst students' propensity to offer justifications can be encouraged by classroom norms that reinforce the expectation that justifications are required, providing adequate mathematical explanations requires appropriate scaffolding, modelling and teacher intervention (Carpenter et al. 2005). Studies (e.g., Carpenter et al., Lannin 2005) which have examined the forms of arguments that elementary students use to justify generalisations classify students' justification as either empirical or generic examples. In the first instance, most students view specific examples, or trying a number of cases, as valid justification. These and other studies (e.g., Kaput 1999) have shown that using concrete material can support young students to develop their justification skills. Therefore the purpose of this paper is to report on how an examination of the commutative principle offered young students a valuable context in which to learn how to make conjectures and construct generalisations. A particular focus is placed on the role of mathematical inquiry, concrete materials, and teacher interventions which scaffold the students to use arithmetic understandings as a basis for early algebraic reasoning. The specific question addressed in the paper asks: How can the exploration of the commutative property of numbers support students to use arithmetic understandings as a basis for early algebraic reasoning?

Methodology

This research reports on episodes drawn from a larger study which involved a 3-month classroom teaching experiment (Cobb 2000). The larger study focused on

building on numerical understandings to develop algebraic reasoning with young students. It was conducted at a New Zealand urban primary school and involved 25 students aged 9-11 years. The students were from predominantly middle socio-economic home environments and represented a range of ethnic backgrounds. The teacher was an experienced teacher who was interested in strengthening her ability to develop early algebraic reasoning within her classroom. Each lesson followed a similar approach. They began with a short whole class introduction, then the students worked in pairs and the lesson concluded with a lengthy whole class discussion.

At the beginning of the study student data on their existing numerical understandings was used to develop a hypothetical learning trajectory. Instructional tasks were collaboratively designed and closely monitored on the trajectory. The trajectory was designed to develop and extend the students' numerical knowledge as a foundation for them developing early algebraic understandings. This paper reports on the tasks on a section of the trajectory which built on student understanding of commutativity as a context which supported their algebraic reasoning. The students were individually pre and post interviewed using a range of tasks drawn from the work of other researchers (e.g., Anthony & Walshaw 2002, Warren 2001). The rationale for selecting these questions was to replicate and build on the previous findings of these researchers. Other forms of data collected included classroom artefacts, detailed field notes, and video recorded observations.

The findings of the classroom case study were developed through on-going and retrospective collaborative teacher-researcher data analysis. In the first instance, data analysis was used to examine the students' responses to the mathematical activity, and shape and modify the instructional sequence within the learning trajectory. At completion of the classroom observations the video records were wholly transcribed and through iterative viewing using a grounded approach, patterns, and themes were identified. The developing algebraic reasoning of individuals and small groups of students was analysed in direct relationship to their responses to the classroom mathematical activity. These included the use of concrete materials, the classroom climate of inquiry, and the pedagogical actions of the teacher.

Results and discussion

I begin by providing evidence of the initial understandings of the students. I then explain the starting point for the section of the trajectory related to the commutative principle. The initial starting point for classroom activity is outlined and I explain how this was used to press student reasoning towards richer understandings using concrete representations. Explanations are then offered of how the press toward deeper student reasoning was maintained through the introduction of symbolic notation. I conclude with evidence of the effect of the classroom activities using post student interview data.

Interview data of the initial concepts of the commutativity principle of the students

This section presents the initial task interview results. True and false number sentences (see Figure 1) were used to explore student understanding of the commutative principle.

$15 + 6 = 6 + 15$	$15 - 6 = 6 - 15$
$15 \times 6 = 6 \times 15$	$15 \div 6 = 6 \div 15$

Figure 1: True and false number sentences

Twenty of the twenty five students participating in the study could not identify which number sentences were true or false. Many considered that they were all true which confirmed that they had limited understanding of the commutative property of addition and multiplication. Five of the twenty five students could identify which sentences were true. However none of these students were able to provide further explanation or justification for their reasoning.

The stepping off point on the trajectory

In order to focus student attention on the correct application of the commutative principle, in an initial activity the students worked in pairs to identify true and false number sentences. The data illustrates that the students readily recognised that addition number sentences were true (e.g., 15 + 3 = 3 + 15 and 5 + 6 = 6 + 5).

> Hamish: It's just the same equation spelt backwards.

> Matthew: Three plus fifteen is just fifteen plus three twisted around so it is exactly the same.

However the commutative principle of multiplication posed more challenges. As an example, in an initial lesson, one group of students concluded that the commutative law only applied to addition. During a discussion a student stated:

> Ruby: Six times five equals more than five times [six] so it wouldn't work in that way.

Another student supported her argument with an erroneous example:

> Hamish: One times zero is zero and zero times one is one.

Although the students had begun to justify their reasoning using additional examples it was evident to the teacher and I that they needed to extend and deepen their reasoning. This was particularly so if they were to learn generalise the numerical relationship between addition and multiplication.

Using representational material to press the reasoning

Collaborative discussion led to revision to the trajectory and the insertion of other mathematical activities. It was evident to us that the students did not have access to representations on which they could base their mathematical explanations of the commutative law. Therefore, we placed an explicit focus on the use of a range of different equipment. This offered the students ways to justify their conjectures, and shift their arguments into more generalised terms.

The students working in pairs, using the true and false number sentences as a basis for discussion, were asked to formulate conjectures about the commutative properties of addition and multiplication. Equipment (popsicle sticks, and counters) was introduced. The students were required develop explanations but also to represent and justify their conjectures using the concrete materials. The teacher carefully scaffolded the explanations so that the students integrated their verbal statements and justifications using concrete materials. But she also ensured that the students were pressed beyond the use of concrete materials, or further verbal examples to more generalised reasoning. For example in the following whole class discussion the teacher selected students to model how they represented and explained their conjectures for 3 + 15 = 15 + 3 using popsicle sticks.

> Hannah: [swapped the pile of three sticks with the pile of fifteen sticks] Three plus fifteen equals eighteen but you could just swap the other ones like the fifteen with the three and the three with the fifteen so it does equal eighteen.

The teacher then revoiced Hannah's statement to develop a more specific explanation of the commutative nature of addition:

> Teacher: So you are saying that if you just swap them around it will still be exactly the same amount?

She then shifted the discussion into general terms and facilitated all the students to consider a more generalised understanding:

> Teacher: Would that work for any set of numbers then when you are adding?

On the trajectory we had considered the need to consider deepening the students' understanding of multiplication specifically. Other forms of equipment were introduced to support the students to visualise the commutative property of multiplication. These included the use of animal arrays (pictures of animals) and counters used in an array. As a result the students became facile in the use of counters to justify the commutative nature of multiplication as evident in the explanation:

> Sabrina: [builds an array with counters] We put four down there and then we did five across…we thought that if you turn it around and put them down here, it is the same five rows of four.

Evidence is provided in the data that mid way in the study the students were now able to provide appropriate mathematical explanations for the commutative property for addition and multiplication and justify their reasoning using concrete representations.

At the same time we were aware from the data collected in the initial interviews that the students over-generalised commutativity to include subtraction and division. At this mid-point in the study specific true and false number sentences were designed and used which extended beyond addition and multiplication to include subtraction and division. These were used with materials to prompt student exploration of the commutative principle with other operations. The teacher closely observed student activity and probed student reasoning:

> Teacher: Does it work with other things like division or subtraction?

Through lengthy discussion which caused conflict for many students they began to provide clear explanations of the non-commutative nature of subtraction and division. Justifications were provided most often as a counter-example as illustrated in the following student's response:

> Gareth: Seven minus four doesn't equal four minus seven… because seven minus four equals three and four minus seven equals minus three.

Constructing clear understandings of the commutative properties of addition and multiplication formed a foundation for their explanations of why the commutative property did not operate for subtraction and division.

Shifting the press to representing conjectures symbolically

In line with the progression on the trajectory we now analysed that the students were ready to use symbolic notation to further press towards generalised reasoning. The teacher scaffolded the use of symbolic notation during a whole class discussion after the students had used both materials and notation to represent their conjectures. She asked them to refer to the conjectures they had recorded while using the material to

explore the commutative property and to use an algebraic number sentence to represent these:

> Teacher: Can you write this as a number sentence that would be true for any number?

Analysis of the data illustrates that in response, many students readily provided a range of algebraic number sentences. For example, the following students said:

> Susan: Z times Y equals Y times Z.

> Steve: A B equals B A

> Gareth: We did rectangle plus B equals B plus rectangle.

In accord with the trajectory, in the following lesson the teacher further promoted generalisation of the commutative principle through a discussion of symbolically represented conjectures. She recorded symbolised conjectures (see Figure 3) on the white-board.

$$B + \blacksquare = \blacksquare + B$$
$$J + T = T + L$$
$$Q \times R = R \times Q$$

Figure 3: Symbolised conjectures

Students were then asked to discuss the symbolised conjectures:

> Teacher: Can you look for the ones which are always true…think about why it is always true as well?

In response to the teacher prompt the students illustrated their knowledge that addition was always commutative:

> Heath: They always will be true... because they are just a reflection…

> Ruby: It's just swapped… it is just the same numbers the opposite way.

Another student used the symbolised conjecture to make a general statement about the commutative law:

> Sangeeta: If you use two sets of numbers which are the same the statement will always be true… with addition or multiplication.

In this way, number sentences both provided notation for the students to represent their conjectures and facilitated them to develop more proficient generalisations about the commutative property.

The combination of extended exploration, the requirement that students explain and justify their reasoning with materials and the press for more generalised reasoning using symbolic notation, supported the students to provide more proficient explanations of the commutative principle. This is evident in the following statement:

> Josie: If you get two numbers and you times them by each other…and then if you times them by each other the other way around it will always be the same answer.

Interview data of understandings of the commutativity principle post-study

At the concluding interview nineteen of the twenty-five students correctly applied the commutative principle to addition and multiplication. Appropriate justification was provided using concrete models. For example, one student using an array explained:

> Josie: Because if you swap it around it is like 12 groups of 4 or 4 groups of 12 which is the same.

She then extended the explanation and drew an array of six times five (see Figure 2):

> Josie: Because you can put it into groups that way or that way and it always works.
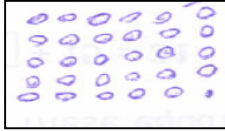
Figure 2: Josie's array

Other students constructed number sentences which generalised the relationship using symbolic notation.

> Steve: J plus C equals C plus J … because you can always reverse stuff in adding.

Six of the twenty five students over-generalised the commutative principle to include subtraction and division. These students had participated in the whole class activities and evidence is provided in the data that during collaborative group work activities they were able to recognise the non-commutative nature of subtraction and division. However without the scaffolding of group activity, in the interview process they reverted to over-generalising the commutative properties.

In the final interview, the way in which most students represented the commutative principle using mathematical explanations, representations, and justification confirmed that the tasks, pedagogical actions, and the classroom environment had scaffolded student understanding of the commutative property.

Conclusion and implications

This study sought to explore how student exploration of the commutative principle deepened their understanding of arithmetic properties whilst also supporting their construction of conjectures, justification and generalisations. Similar to the findings of Anthony & Walshaw (2002) and Warren (2001), many of these students initially failed to reach correct generalisations regarding commutativity. Extending the task beyond true and false number sentences and the introduction of equipment led to student modelling of conjectures and provision of concrete forms of explanatory justification. Importantly, teacher interventions were required to shift students to make generalised statements about the commutative principle.

Results of this study support Carpenter and Levi's (2000) contention that use of number sentences provides students with access to a notational system for expressing generalisations precisely. The symbolic representation of their conjectures coupled with the use of equipment and teacher press for generalisation led to more specific student generated generalisations.

Many of the students in this study deepened their understanding of arithmetic properties. However, the small proportion of students who continued to over-generalise to include subtraction and division indicate the need for multiple opportunities over an extended period of time for students' to develop deep understanding of operational laws.

Findings of this study affirm that the context of the commutative principle can provide students with effective opportunities to make and represent conjectures, justify and generalise. Appropriate tasks, concrete material and teacher intervention supported students to develop their understanding of the commutative principle. Opportunities to develop explanations with concrete material and use notation to

represent conjectures led to students developing further generalisations. Due to the small size of this sample further research is required to validate the findings of this study.

References

Anthony, G., & Walshaw, M. 2002. *Students' conjectures and justifications: Report of a probe study carried for the National Education Monitoring Project*. Palmerston North: Massey University

Blanton, M., & Kaput, J. 2003. Developing elementary teachers "algebra eyes and ears". *Teaching Children Mathematics*, 10(2). 70-77.

Carpenter, T., Franke, M., & Levi, L. 2003. *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

Carpenter, T., & Levi, L. 2000. Building a foundation for algebra in the elementary Grades. *In Brief*, 1(2), 1-6.

Carpenter, T., Levi, L., Berman, P., & Pligge, M. 2005. Developing algebraic reasoning in elementary school. In T. Romberg, T. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 81-98). Mahwah, NJ: Lawrence Erlbaum.

Cobb, P. 1995. Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education*, 26(4), 362-385.

Cobb, P. 2000. Conducting teaching experiments in collaboration with teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, NJ: Lawrence Erlbaum.

Kaput, J. 1999. Teaching and learning a new algebra. In E. Fennema & T. Romberg (Eds.). *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum.

Lannin, J. 2005. Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning*, 7(3), 231-258.

Ministry of Education. 2007. *The New Zealand curriculum*. Wellington: Learning Media.

National Council of Teachers of Mathematics. 2000. *Principles and standards for school mathematics*. Reston, VA: Author.

Warren, E. 2001. Algebraic understanding and the importance of number sense. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 399-406). Utrecht: PME.

The Transition to Advanced Mathematical Thinking: Socio-cultural and Cognitive perspectives

Kamilah Jooganah and Julian Williams

University of Manchester, U.K.

This case study of the first, 'transitional' year of a mathematics programme at a research intensive university aims to deepen understandings of the transition to 'advanced mathematical thinking', or in effect, 'rigour and proof'. The case draws on ethnographic data that includes: interviews with lecturers and students; observations of tutorial/teaching sessions; a video-stimulated recall interview with a lecturer; and documents from relevant A-level and university programmes. We consider insights into transition using Activity Theory (after Leontiev, Cole, Engeström etc.) and draw on selected cognitive perspectives to Advanced Mathematical Thinking (after Fishbein, Tall, Harel, etc.). We conclude that the different activity systems of school and university involve contradictory mathematical practices and hence can result in cognitive conflicts, including those well documented by the cognitivist 'psychology of mathematics education' tradition. Drawing on these perspectives and capturing the voices of students and lecturers may be important to understanding identity, motivation, and student engagement.

Keywords: Advanced mathematical thinking, transition, identity

Introduction

This paper draws from our ongoing case study work as part of the Transmaths project, and from one case in particular. The aim of the project is to understand the transition from school/college to university study. Here we seek to explore a case of transition to 'Advanced Mathematical Thinking' from the point of view of (i) socio cultural and identity perspectives, and (ii) traditional cognitive perspectives. At the time of writing we are midway through case studies involving five universities with a focus on STEM 'programmes'.

The focus of this paper arises from one case study in a 'research university' mathematics programme, and its concern with rigour and proof especially. The School of Mathematics takes 'transition' seriously and aims to support first years in this. Yet we were told that the mathematical reasoning or 'proof' course taught throughout the first semester continues to be challenging for many of the students.

Socio-cultural and Cognitive perspectives

Harel and Sowder (2007) raise the 'socio-cultural' issues involved in the transition to Advanced Mathematical Thinking (AMT) and proof at university. However, by 'socio-cultural' they refer to curriculum and instruction, e.g. with the use of technology. This paper seeks to generalise and deepen the 'socio-cultural' to include considerations of mathematics as practice and as activity. Furthermore, we will draw

on cognitive approaches to AMT, specifically the cognitive conflict students may experience during transition.

We deepen the analysis by situating mathematical cognition within Activity Theory, in the tradition of Vygotsky, Leontiev and as more recently developed by Cole and Engeström (for a full account see Williams and Wake 2007). Activity Theory views cognition as distributed in practice across systems of activity mediated by tools, social norms and social relations including the division of labour in activity. Tools include languages and concepts as well as technologies, texts and tasks. These 'carry history' and can introduce contradictions into activity (typically for instance when the tools in use are no longer fit for the purpose of the activity).

For us, then, the cognitive conflicts that students may experience in transition must be viewed as a result of a contradiction between the typical cognitions associated with pre-university and university systems and practices. In particular, we might expect to find students attempting to work in the university system using conceptual tools developed in school practices, reflecting the school system. The intuitive 'cognition' of the function concept in mathematics for instance must be understood as situated within the particular school procedural practice of graphing. By contrast, the function concept used in the university proof course is conceptual, and draws specifically in its concept definition... But we are getting ahead of ourselves.

There is substantial literature that have provided important insights into the topic of 'rigour and proof' and student learning (see Harel and Sowder 2007 for a recent summary). Essentially, these works provide an analysis of students' cognitive deficits: according to Dreyfus (1999) the reason so many students have trouble with AMT in general, and proof in particular, is that they are simply not prepared – there is a cognitive 'gap'. Others have suggested this can result in a type of cognitive conflict. Tall (1991) drew on Piaget's idea of equilibrium, and noted that when students are confronted with new knowledge which conflicts with previous knowledge, the cognitive equilibrium maybe disturbed which could result in cognitive conflict. However little insight is provided by Tall (1991) and Dreyfus (1999), and in general cognitive research to AMT, concerning how cognitive conflict may relate to other factors located in the wider socio-cultural context of the learning-teaching environment.

Furthermore, it may be argued that many students at school are prepared to treat proof – and mathematics in general – procedurally and instrumentally rather then conceptually. This may reflect a more general *'surface'* approach to learning in school: students may learn to perform instrumentally for exams rather than to think critically and approach concepts using a *'deep'* approach (Marton and Booth 1997; Williams et al. 2010).

For our research we propose that the different activity systems and practices students are expected to engage with at university which emphasises proof and the need for rigour may cause a type of cognitive conflict in students (Tall 1991). This can be framed as a conflict between old ways of being and new demands made by university practices. This can manifest itself in a misalignment between student and lecturer perspectives in learning.

Furthermore, the literature on AMT tends to emphasise learning as an individualistic process that occurs mainly on an internal cognitive level, i.e. 'in the head'. We draw on Activity Theory and relate obstacles students encounter, as well as frustrations lecturers experience as part of a wider systematic context. This includes factors related to the life-world of students such as their previous experience of mathematical practices, including approaches and motivation to learn. We also highlight other socio-cultural forces such as how certain practices and activity

systems can position students/lecturers in terms of their identities in relation to mathematics. Thus, we argue that situating the literature on AMT within the framework of Activity Theory can provide insights about why students may struggle in making the transition into AMT, the effect this can have on student identity and their future trajectories in mathematics. We also consider how systems of pedagogy can be structured to help students in this transition process.

Methodology

The case study was conducted at the School of Mathematics of a research-intensive university. We employed a mixed-method qualitative ethnographic approach: this allowed for a holistic way of investigating student transition to AMT and complimented our use of Activity Theory to the research. Collecting data from a variety of perspectives allowed us to capture the different dynamics operating at various levels in learning contexts (see Jonassen and Rohrer-Murphy 1999 on methodological issues relating to Activity Theory).

Convenience sampling was used when selecting students to interview. Prospective undergraduates that intended to do a mathematics degree at our case study institution were contacted through the post and asked if they would take part in our research. Students that had agreed were contacted and interviewed over the phone or in person. These interviews involved a semi-structured format; this allowed us to question students on their first year experiences and to explore with them a number of other themes such as their reflections of how they learned and identified with mathematics prior to university. A similar format was applied when interviewing lecturers. In total six lecturers and seven students were interviewed. Each student was interviewed at three data points: (1) before or during induction week when students first joined university; (2) after students completed their first semester; and (3) when students started their second year. Observations were made during lectures, tutorials and other environments where learning took place. We video recorded six of these learning-teaching sessions, and in two cases conducted video-stimulated recall interviews with some of the lecturers taking the tutorials. Documents in the form of textbooks and worksheets were also collected. The analysis of the data collected followed a narrative and thematic approach.

Results

In the transition to university students experience a number of changes, e.g. the classrooms and learner-teacher relations provide a significant transitional 'gap' for many. Some lecturers are conscious of these factors, and structure support for learners early in the first year to get the students working together in groups. They also offer small tutorial sessions to support what students cover in lectures, including the course on proof. Interestingly, a mathematician (with a strong interest in mathematics education) wrote the 'proof' course with an awareness of much of the literature in mind. The text book illustrates a remarkable sensitivity to some of the transitional issues students encounter on a cognitive level, explicitly stating in the preface that it was designed to rectify this deficit or the 'gap' in their knowledge of advanced mathematics (indeed Tall's work is cited in the text).

In making the transition to AMT we noted conflict in the activities and practices students were expected to engage with at university. From student and lecturer interviews we were informed that the type of mathematics they generally had experienced in school could be characterised as procedural, requiring little deep

conceptual engagement. Thus, for many of the students interviewed, university required a different approach: one which placed emphasis on proof and rigour requiring them to engage with and critically think about concept definitions.

This different approach to mathematics resulted in a type of cognitive conflict in students, which made them question their identities as learners and influenced their future trajectories in mathematics. This cognitive conflict we found could be related to a range of factors located in the learning-teaching environment, that included a misalignment between lecturer and student meanings attributed to mathematical objects; in our case the need for proof and rigour in AMT.

Vignette 1. John's experiences with AMT

This vignette is constructed from two interviews with John: the first over the summer holiday before he started his course at university, the second after his first semester exams. The third is yet to come.

Over the summer before starting at university John told us he was eager to get stuck in. John comes from a working class family in a relatively deprived area: his school was closed and reopened; he was in the first year in the sixth form of the re-opened school and got on really well with his mathematics teacher. He converged on mathematics as his degree choice largely because he was an outstanding student. He felt he learnt to become an independent learner through the Further Maths (FM) network: he experienced lectures on visiting days at his local university, and on-line support in between. He believed he was really 'up for uni'.

By the second interview, things looked very different: the transition was a 'bit of a shock', e.g. the pace: weeks of Further Maths study were covered in one lecture (e.g. complex numbers). In addition, the 'proof' course presented a completely new view of mathematics that he really struggled with (in fact he was relieved to pass). What John experienced could described as a type of cognitive conflict: he described the proof course as very 'different', requiring a rethink about such commonplace concepts as 'counting' and 'function' that he thought he knew so well or 'clicked' in his mind.

> All I can say with the proof module, it was so new and such a different way of, just a different sort of side and view of Maths that it's a bit of shock to the system, because it didn't click straight away and it was like the first thing we did when we came to university, it was a sort of, 'hold on, we understood everything at A-level, why are we not understanding this straight away', and it was just a bit of, I guess people sort of - I probably don't like it for that reason, because it didn't click.

He was eloquent in reflecting about how A-level mathematics was reduced to 'just procedures' – even proof by induction was procedural at A-level. Looking back, he now sees some things he did wrong: he initially did not work hard enough but he now feels he has the work-social balance right. Where next? He's not certain - he has survived the shock to the system, but his view of himself as a mathematician is shaken. For John his experiences with proof lessened his confidence with mathematics: indeed he reports that he will consider taking non-proof modules in his second year. This partial story raises the question: how important are his various experiences in the trajectory of his identity? Will he veer towards or away from the kind of mathematical identity on offer, and what practices make the difference?

Vignette 2. Misalignment between students' and the lecturer's mathematical activities and practices: the case of function and its different meanings

Small group tutorial sessions are provided twice a week to support students in what they cover in lectures. These sessions involve lecturers going through the problem sheets and working with students on the solution, ideally in the form of tutor-student discussions. For this second vignette we look closely at one 'enlightened' lecturer's engagement with students' difficulties with the 'function' concept, during a tutorial. We noted that the concept definition is included A-level texts, but close examination of a typical text revealed that the school mathematical practice was essentially procedural: as we will see, the procedural experience at school set the students' dispositions in conflict with the lecturer's conceptual approach. For this lecturer, as he informed us, 'proof and the need for rigor are what define maths'.

 The tutorial was spent discussing one question from the problem sheet: To find the inverse function of *f*: *R to R*, *f* (x) = 2x+3. Several of the class had answered this in a way that Robert, the lecturer, said *"Was not wrong, but likely to cause problems due to its imprecision"*. What the students had apparently done in the work they handed in was to rearrange the formula/relation y = 2x+3 to get x = ½ (y − 3), then rewritten this exchanging the x and the y to get something like y = *f* (x) = ½ (x-3).

 This is a correct but arguably incomplete answer, as the domain and co-domain are left implicit (in fact both are *R* for the inverse function too, as in fact *f* is in this case a bijection from *R to R*, and it is hardly surprising that the students might therefore leave this implicit even if they thought it was important). But right or wrong, why is this method such a powerful attractor for the students? In practice, in school activity systems – a function is always described as a 'rule' mapping from x-values to y-values, represented as real number lines horizontally and vertically, most often as a graph. In this sense it can be described as a powerful concept image, or better prototype (Fishbein 1990). However, a social practice perspective would assert merely that this is how the students are 'schooled': this is typically what they practice for school exams.

 Robert seems to have selected this example to discuss because he had marked their work beforehand and saw an opportunity to emphasise the need for complete, explicit writing out of the mathematical proof here (with explicit definition of the sets X and Y). He therefore came to the supervision knowing the students work and what he wanted to achieve in *'deconstructing and reconstructing'* their informal mathematics using his preferred degree of rigour and precision. Robert proceeded to write down how he would prove *f* is a bijection, starting with the definition of an inverse in which the domain and co-domain are explicitly included, thus:

> Definition: the inverse function of *f*: X to Y [that is, a function that maps any x in X to a single value y in Y given by a rule y= *f* (x)] is a function g that maps a value y in Y to a value x in X, (i.e. g(y) = x) such that g *f* (x) = x for all x in X.

 He began to work through the algebra but was interrupted by a student who said he had the answer right his own way (as mentioned above). Robert stated *"It is not wrong but why exchange the values x and y?"* which he claimed was likely to lead to confusion of the domain and co-domain. Robert encouraged the same student to illustrate the reasoning behind his answer on the board. Indeed, most of the class seemed to agree *"That is the way we were taught at A-level"*.

 The student then fortuitously chose to exemplify his approach using exponential functions (from a later worksheet question): he argued that the expression

y = exp(x) was the same as x = log(y), and that switching x and y was required to 'get' the inverse function y = log(x). Implicitly we assume the graph of y = exp(x) is the same as that of x = log(y), and so does not represent the inverse. However, Robert noted *"Oh yes it might be, if you consider the graph as a representation of the function mapping y to x"*. The idea that the same graph 'represented' two inverse functions however seemed counter-intuitive – but drawing on Activity Theory, with a focus on practice, we say rather it is counter to all school practices of mathematics the students had experienced hitherto.

Why is this serendipitous example pedagogically 'useful'? Because unlike the worksheet example the domain and co-domain of the inverse function log(x) is problematic: since the exponential function has no negative values, log(x) has no values for negative x (within the Reals anyway). Robert was therefore able to point out how important the specification of the domain and co-domain are to this work. An intuitive approach would have simply led one to say log and exp are inverse functions without specifying the functions rigorously (i.e. completely and precisely).

Analysis of two mathematical activity systems and practices at work

This case study highlighted a conflict between school activity systems and practices, and the 'rigour/precision' required in the university mathematics activity system and practice: it rests in one sense in the contrast between informal procedures in school and rigorous conceptual work of pure mathematics at university.

From the second vignette we can note that the students believe exp(x) and log(x) ARE inverse functions, whereas Robert pointed out that these expressions are NOT functions, but values. Students would have been introduced to the idea that the graphs of *f* and its inverse *g* would be reflections/images of each other in the line y=x. The suggestion that the one graph can be thought to represent two functions (y as a function of x and its inverse x as a function of y) would be counter to school practice – and what cognitivism might say is counter-intuitive. In Activity Theory terms, these 'cognitive intuitions' are embedded in unexamined 'operational conditions' that fall below the level of consciousness, and are all the more powerful for being unexamined and uncritically accepted. We can view school mathematics as a practice developed within the AS/A2 of the school/classroom/curriculum; what we are seeing is a conflict between the old ways of being and the new demands caused by Robert's subjective view of mathematical objects (drawing on university/academic practices). Objects, like 'functions', have contested meanings here because they are boundary objects used in two quite different practices and systems – that of the school and the university. This contradiction might suggest that the school activity system, or maybe the university activity system, should be modified or re-designed to help better align these practices: to shed light on the conflicts for the benefit of both learner and teacher. In a sense the dialogue between Robert and his students can be thought of as just such a re-design.

Analysis of the pedagogy/curriculum

The fact that the tutorial was based on the students' work (previously marked) and that Robert brought this prominently into the session may have triggered what happened: making the intuitive and implicit approaches taken by the students more explicit. We have rarely seen this Assessment-for-Learning style pedagogy in our other university case studies (usually the lecturer uses the postgraduate assistants to mark the work, and thus misses a great opportunity to learn about their students'

mathematical practices/errors). This is difficult to envisage in a lecture with large student numbers, and the ensuing pre-conceived 'coverage' and norms of interaction.

Even though Robert was aware of the vagueness and unhelpfulness of the informal school approach to learning AMT, he was unaware of the depth of the clash of values and practices involved in this one example. He might have said that he wanted them to *"write their solutions out properly"* with attention to precise definitions, etc, but had not explicitly realised that his students' practices with regard to function did not take account of its concept definition. Thus the most important element of learning derived from the interaction between Robert and his students might be his enhanced understanding of the implicit beliefs they brought to their learning. By confronting their mathematical practices he was able to impart both a deepened understanding of AMT and cause them to reflect on the cognitive conflicts which may occur when old knowledge conflicts with new ways of understanding mathematics (Tall 1991). This kind of learning is one that Robert endorses: he is a keen and enthusiastic teacher as well as a research academic. After our video-stimulated recall interview he says that he has learnt some things about the way students think and have been influenced by schooling, and sees its potential use for him as a lecturer.

Concluding Remarks

The cognitive perspective helps us understand the transition to Advanced Mathematical Thinking as a form of correction of a cognitive deficit. Furthermore, we argue that viewing the 'conceptual' transition to AMT as part of a shift from the practices and activities of one system to another raises a broader range of issues: these practices are shaped by more than only the 'thinking' that goes on in the head, but also by the learning approach, a set of priorities, expectations and dispositions, a set of tools, social relations and norms of behaviour too. These are all important aspects to consider. Thus, it could be argued that the predominant practice with graphs and functions in school mathematical practices (mainly focusing on graphing) leaves the specification of domain and co-domain irrelevant, even though A-level text books define them as essential to the concept. More broadly, proving is perceived by students as practising a set of instrumental procedures: it has never been practised as a social process required to persuade a critical audience beyond reasonable doubt. We have particularly seen how the teaching and learning practices may be an essential part of the transition problem/solution.

Finally we have alluded to the issue of identity. For the learner (especially noted with John above) transition may be a threat to an existing or emerging mathematical identity, or conversely it may be an opportunity to re-invent himself as a different kind of mathematician. The resolution of this will, in our view, be critical for the way John engages in his future studies (we currently await the third interview with interest). However, identity is a crucial issue for Robert also: 'rigour and proving' is for him what constitutes the essence (the pure episteme) of mathematics. Yet he also identified himself as a teacher: he wants to take the students with him across the gap (the threshold), while at the same time acknowledges that for many students this will not happen.

We can see the contradictions between school and AMT practices, or between schooling and university activity systems being lived out in the mutual engagement of these identities in the tutorial: it is manifested as a tension between the subjective 'ways of being' a mathematics student and a lecturer. These tensions can be constructive as they bring to the fore different discourses and what it means to be a

mathematician and a learner of mathematics. This could be incorporated into the design and pedagogy of the curriculum: but there are so many obstacles to this, not least the macro- and meso-economic and political structures in the systems of schools and universities, but also in the historically constituted professional identities of 'homo academicus'.

References

Fishbein, E. 1990. *Intuition in science and mathematics*. Dordrecht: Kluwer Academic Press.

Tall, O. D. 1991. *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers.

Marton, F., and S. Booth. 1997. *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum Associates Publishers.

Dreyfus, T. 1999. Why Johnny can't prove. *Educational Studies in Mathematics* 38: 85—109.

Jonassen, H. D., and L. Rohrer-Murphy. 1999. Activity theory as a framework for designing constructivist learning environment. *Educational Technology, Research and Development* 47 (1): 61-79.

Harel, G., and L. Sowder. 2007. Towards comprehensive perspectives on teaching and learning proof. In *Second handbook on mathematics teaching and learning*, ed. Frank Lester, 805-842. Charlotte, NC: Information Age Publishing.

Williams, J., and G. Wake. 2007. Black boxes in workplace mathematics, *Educational Studies in Mathematics* 64: 317–343.

Williams, J., L. Black, P. Davis, P. Hernandez-Martinez, G. Hutcheson, S. Nicholson, M. Pampaka, and G. Wake. 2010. Keeping open the door to mathematically demanding programmes in further and higher education: a cultural model of value. In *Improving learning by widening participation in Higher Education*, ed. Miriam David, 109-123. London: Routledge.

Promoting a Cross-curricular Pedagogy of Risk in Mathematics and Science Classrooms

Ramesh Kapadia and Phillip Kent, with Ralph Levinson, Dave Pratt, and Cristina Yogui

Institute of Education, University of London, WC1H 0AL, United Kingdom

This paper reports on a research project on the teaching of risk with teachers of mathematics and science in Key Stage 4. Software models of socio-scientific issues have been developed and tested to support teachers towards developing a pedagogy of risk in their teaching. Transcripts from a workshop with teachers are used to illustrate some key findings.

Background and Research Aims

This paper reports on the findings of a research project[1] whose aim has been to support and enhance the teaching of risk at Key Stage 4. An innovative and cross-curricular approach has been used, based on modelling socio-scientific issues using new technological tools, designed specifically to enable the consideration of ethical and social issues, whilst building personal value systems alongside quantifiable mathematical models. By interacting more deeply with interdisciplinary knowledge, the aim is that mathematics and science teachers can be empowered to develop meaningful activities around the concept of risk. A key component of the research has been that pairs of teachers (one mathematics, one science) from the same school work together to explore the possibilities of cross-curricular working in their school, and also become involved with the design of materials in the project.

The discourse of risk permeates public policy and, increasingly, the choices people face in everyday life - consider how current medical practice favours informed choices to be made by the patient where previously decisions were taken by the medical expert on behalf of the patient. In response to these societal changes, risk is now recognised as a school topic through its recent incorporation into the mathematics curriculum alongside probability, and it has been part of the science curriculum for some time, particularly through the "How Science Works" Programme of Study. However, we note that science teachers continue to be reluctant to teach risk, not only because of concerns about their own knowledge limitations, but also because they are unsure about appropriate pedagogy for engaging students in the topic. Risk as a science topic is intimately connected with "socio-scientific issues" (such as stem cell-based medicine and climate change) and requires discussion and evaluation of argument, a challenge since evidence shows that science teachers are not always confident in using teaching methods based around discussion (Bryce & Gray 2004). This difficulty is even more a problem for mathematics teachers where "social issues" are unfamiliar ground.

If the current generation of school students are to be sufficiently confident in grasping the idea of risk in varied contexts, a number of obstacles need to be overcome. Mathematics teachers do not feel confident in teaching topics on risk and

---

need to be able to have the resources and the experience to teach it well; this is also partially true of science teachers though they have more experience of social issues impinging into science lessons. Hence this project is helping teachers with resources and a pedagogy to reflect the new curricula they are teaching, which responds to their present level of knowledge and which is viable to use in the classroom.

The understanding of risk incorporates concepts which go beyond substantive science or probability into affective and social science domains. To address this problem, we have designed (in collaboration with teachers) computer-based modelling tools in which teachers can explore and interrogate their knowledge of risk. The tools are set within decision-making scenarios to help teachers increase their own confidence in teaching about risk, and be able to enthuse their students, particularly with regards to probability.

The idea of microworlds began with Seymour Papert and Logo, as described by Noss and Hoyles (1996). They linked direct manipulation of computational objects with the description of their arrangement in sequences as Logo programs. Their intention was to help students build links between seeing, doing and expressing through the common representational infrastructure of Logo, to investigate how mathematical meanings were structured by the tools available for expressing the relationships under study and to trace how far this mediational process assists in the construction of the mathematical meanings intended by their teachers or curriculum designers. The approach is based on a constructivist philosophy.

The Idea of Risk

The judgement of risk is important in the public understanding of science in modern society (Rees 2006). Many socio-scientific and financial issues in the media can only be rationalised through an appreciation of risk. The confusion about the nature of risk evident in the media is reinforced by its ambiguous use in text books and academic articles. Often, risk is expressed as likelihood, equivalent to probability. We might then ask if the risk of an aeroplane crashing is 1 in 10 000 and the risk of catching measles is also 1 in 10 000, do we wish to equate the two risks? There is the underlying question of how such probabilities are estimated. However, the *impact* of these two events is very different. We adopt the view that risk is best considered to be a combination of both the impact of the hazard (also called the utility or dis-utility) and its likelihood of occurrence. This is the standard concept of risk as used in statistical decision theory (Edwards & Tversky 1967); however we do not go along with the standard view that risk is always a simple arithmetic product of the likelihood and utility values.

Although our approach emphasises how subjective values can be marshalled through an objective framework, risk does not have an established and agreed epistemology. Wilensky (1997), in relation to probability, has discussed how such uncertainty about the foundations of a concept can lead to "epistemological anxiety".

Factors other than epistemological anxiety also affect how risk is perceived (Kahneman & Tversky 1979, Campbell 2006). Dying from an aeroplane crash is a concentrated risk in so far as many people simultaneously would suffer the consequences should the accident occur. In comparison, dying from a road accident is a diffuse risk. Fear of flying is not uncommon and can be seen as an example of how concentrated risk is often regarded as worse than diffuse risk, even when, as in this case, the diffuse risk is in fact much greater in terms of actual deaths. Risks that are voluntary or associated with benefit from the observer's point of view may often appear lower, so that risks from alcohol or smoking may be perceived lower than they

really are. The problem in understanding risk is accentuated by a corresponding difficulty in appreciating its mathematical core of probability. The literature contains many studies that demonstrate that, without a secure understanding of (probabilistic) situations, people's reactions are often guided by misleading intuitions, which are rooted in inadequate cognitive biases (Kahneman et al. 1982, Kapadia & Borovcnik 1991, Lecoutre 1992).

Nevertheless, research shows that participation is best enhanced by engaging students with topics about which they are deeply concerned (Ainley & Pratt 2006). Children and adults routinely encounter science in the media: climate change and global warming, smoking and cancer, alcohol consumption and heart disease, nutrition and obesity to name just a few. It is exactly these issues that students, find engaging, so they are an important starting point for increasing involvement (Haste 2004).

Teachers can be assisted to find ways of helping students to recognise the unpredictability of chance at the individual level as well as its apparently contradictory predictability at the aggregated level and to assess the reliability of the source of data on which a judgement is based (Kolstø 2001). They can also be enabled to explain the relationship between deterministic and stochastic thinking (Prodromou & Pratt 2006). There is now an expectation that teachers should respond to this challenge. For example, the *21st Century Science* curriculum scheme[1] (used by about one-third of school students in England) requires that students "can explain why a correlation between a factor and an outcome does not necessarily mean that one causes the other", and "can suggest reasons for given examples of differences between actual and perceived risks" (OCR 2009, Appendix F.5).

There are few materials which have addressed these issues for mathematics teachers. Early work by a Schools Council Project on Statistical Education (1980) developed materials, relating to smoking and other areas where risk can be explored. More recently, the Bowland Maths Project[2] produced several modules for Key Stage 3 called 'How risky is life' and 'Reducing road accidents'. These modules are exploratory and open-ended, but not explicitly cross-curricular. They certainly advance the debate about how to make mathematics engaging but from our perspective they present a limited view of risk: there is a focus on probabilities of hazards to the exclusion of considerations of impact. The modules are clearly developed with the mathematics classroom in mind, and do not explicitly draw students into consideration of ethical and moral values in decision-making.

Our perception of practice in the science classroom (based on literature and discussion with teachers) is that the topic of risk is usually handled with a focus on the "social" dimension of contemporary socio-scientific issues, such as how the popular media mis-represents scientific knowledge and practice for its own ends (for example, the risks involved in children having or not having combined MMR vaccination). What is largely absent is a quantified approach involving numerical probabilistic information or mathematical modelling of risk. We believe that it is essential for teachers and learners to be able to build and explore their own personal models in decision-making situations, which combine quantitative, semi-quantitative and unquantified factors. Hence in our research we have developed scenarios, one of which is described below, which have a strong focus on trying to put numerical data into models, using both probability and impact as dimensions of risk. This also offers the opportunity for mathematics teachers to make probability (which can sometimes

---

[1]   www.21stcenturyscience.org
[2]   www.bowlandmaths.org.uk

be seen as a quite dull topic with artificial situations) more interesting by linking to real world situations and contexts.

Deborah's Dilemma: Some evidence about teachers' thinking in a risk scenario

A software environment has been developed called Deborah's Dilemma. Teachers are invited to advise a fictitious person, Deborah, on whether to have an operation that could cure a spinal condition that is causing her considerable pain. The operation would entail certain hazards with risks that need to be inferred from various sources of information. Choosing not to have the operation would entail a lifestyle choice to manage the ongoing pain resulting from the spinal condition. Two software tools accompany the information about the condition. The first ("Operation Outcomes") is a probability simulator in which users model the possible consequences of having the operation. The likelihoods for various complications (i.e. side effects of the surgery, ranging from minor to serious, such as super bug infection, or death through general anaesthetic) are quoted in differing forms. Ambiguities in the data are deliberately set up in order to provoke discussion and debate; for example a "spine expert in London" and "a surgeon in a regional hospital", give conflicting opinions – who should one trust?

The second tool, the "Painometer", is a less conventional attempt to give a quantified experience of Deborah's pain and how different activities may cause it to increase and decrease, relative to a "tolerable" level. Pain is an experience that defies objective measurement; hence the personal perception of pain is a potentially interesting context for probing people's personal models of risk. Users are required to decide what everyday and leisure activities Deborah should or should not do, and in what amounts, and to infer from the information the effect on Deborah's pain level of those activities (as expressed through the dynamically-varying height of a vertical bar as a "painometer"). The intention is for the tools to promote quantification of risk in a real context, whilst allowing for personal interpretation, and not constrained by formal models of risk such as exist in statistical decision theory, and which are conventionally used in risk assessment.

We now present some data from workshops that we have run with a small number of teachers exploring Deborah's Dilemma. Data were collected in the form of audio recordings of group discussions, and screen capture with audio of teachers working with the software in pairs; also researchers observed and questioned the teachers as they worked. We have assessed the teachers' attempts to compare and weigh up information about the pain against the risks of the operation, using ideas of impact such as minor or major complications, and what these data suggest about the teachers' thinking. Below, some results with one pair of teachers are presented: A is the mathematics teacher and L the science teacher. The transcript is given before the discussion.

Episode 1

- *[The surgical operation]*
- L: It must be pretty bad pain to consider…
- A: If you're going to consider the surgery it's got to be
- A: 'Unwanted after effects' (laughs). So 95 to 98% successful. So that's…is that 95 to 98% of the time the pain is relieved? Or 95 to 98% of the time there's no complications?
- L: That's a good question

- ...
- L: Complication three…oh ok so you don't need to…more risky than what's going on. 'This was helpful…I asked my doctor for a second consultation…' Yeah, major surgery; general anaesthetic is another risk, and the superbug
- A: 'Currently about 0.00025%' 25 in 100,000
- L: See that's what we were thinking; '…developing my lifestyle to support long term management of the pain'. '…Yes, I can live with the pain and I have adjusted my lifestyle successfully'. It's more of a lifestyle operation; it's not something that's going to save her life is it?
- A: Well she'll still be alive but it's got implications right through doesn't it. I suppose if it does deteriorate you set yourself up for…
- L: Well if anything did deteriorate you'd be in a different position but if you lived slightly different then… Do you want to look at the other two options?
- A: One in 1000 for general anaesthetic
- L: That's one in 10,000
- A: One in 4000 for the superbug

An unusual event happens in lines 8-14. A probability of 0.00025% is mis-translated to be 25 in 100,000, or 1 in 4000. This is taken to be the probability of contracting a super-bug. In fact this is far too high for the current prevalence of super-bug infections in UK hospitals. It is even the sort of mistake for which a teacher may admonish a pupil: getting an answer which cannot really be correct. It would mean that the super bug is contracted weekly in hospitals. This illustrates a problem which is known to arise: that people find large numbers and low probabilities very hard to comprehend and deal with. It is likely that, with more time and thought, these teachers would have realised their error. However, it is salutary to remember such issues in the design of materials about risk.

Episode 2

The second episode conveys a positive outcome from the exploration of Deborah's dilemma by teachers from different disciplines. A teaches mathematics and is positive about recommending the operation, while L teaches science and has some doubts.

- A: I'd have the operation, give it a go. That's my lifestyle.
- L: If someone said you couldn't play football that would be important? You wouldn't be prepared.
- A: Not only that, it's self-sufficiency as well. You need someone to do your shopping for you, already at 38.
- L: Well she can use a trolley.
- A: No, she can't lift a shopping bag – doing shopping would make you worse. Everything you chose to do for yourself would make you worse, leading to a more serious condition.
- L: If you're recommending it purely on the odds, right...
- A: No, not purely, I'm recommending it possibly against the odds, you know 3... over here [looking at written notes??] complications or failure is 5%.
- L: If you were saying it for yourself, you would be prepared, but say you were making it for her – then you'd be saying, well actually.
- A: Right, we have to come down on one side or the other, we can't just re-

present the evidence, we've got to say 'we recommend you do this', we can't sit on the fence.

- L: Do you think if you were the doctor, if you were someone close to Deborah...
- A: You might have a different opinion.
- L: ...it's going to come down to the relationship with the patient as well. What are we, we're just people looking at statistics, if the computer says no, because at the moment to me the computer is saying no.
- A: 3 in 1000 serious complications.
- L: That's serious, you are 38 your life could-
- A: Yeh!
- L: Because the computer... [laughs] Who are we??
- A: That's the question isn't it?
- L: All these things, is lack of sleep, nausea, even depression
- A: That can ruin your life.
- L: [reads] I will experience high level of ... I've had enough, I've had several years of it It's kind of difficult to see Deborah on paper...
- A: Do you think she should shut up and stop moaning?

A key overall observation about A and L's exploration is that there was a regular switching of position which depended on the information immediately in front of them (the 'availability heuristic), and their shifting estimations of impact. Initial exploration of the "outcomes" tool suggested a likelihood of serious complication arising from the surgery as the order of several cases in a 1000, and they initially talked about this as a "reasonable risk". However as they explored with the Painometer tool they began to favour a non-surgical approach and to re-assess the reasonableness of the surgical risks. Our interpretation is that they were left in an ambiguous position between the two "sets of evidence" and in the end rationalised their position by appealing to the authority of the "specialist in London": L comments:

> I think the fact she's gone to the person who knows more about it, spends his life looking at risk, he says no there are better things out there for you, other options … As a scientist I would go with the specialist spine doctor who knows more about it, I would go with what he says.

There is an interesting – if speculative – positioning here, worthy of more systematic exploration, where L talks "as a scientist" explicitly about authority of a person, whereas at other moments with A leading they look to the evidence coming from "the numbers".

It would seem that A (the mathematics teacher) brings in the probabilities to make the choice; this is mediated by L (the science teacher) in bringing in social issues, placing stress on impact (minor and major complications); A changes his mind, as illustrated in the final recommendation that the pair produce, presented (interestingly) as a personal letter, reproduced below. It should be noted that this happened with one pair of teachers and illustrates the importance of collaboration and discussion between teachers from (different) disciplines. The approach taken by the project has enabled effective cross-curricular working in an aspect of pedagogy which is often absent from classrooms, the discussion of issues beyond the quantitative or scientific evidence.

Dear Deborah, our recommendation is that you do not have the operation and further investigate pain relief and other options for support and managing your lifestyle. We have looked at the probabilities of failure and complications during surgery and whilst the likelihood of severe complications is around 0.4% - possibly quite low – we are unsure as to the exact chance of success as the study quoted by your first doctor referred to arm pain only and we are inclined to take the opinion of the spine specialist that managing your condition is the best course of action. We feel the probabilities are too high against you if the surgery is not entirely necessary. This is obviously balanced against how your condition is affecting your daily life and how often the pain is above your tolerance level. If the condition worsens then surgery is not a prohibitively dangerous option in that case.

Conclusions

The research is continuing with a wider evaluation of our decision-making scenarios with mathematics and science teachers, and advisers and educational researchers. However, a number of interesting ideas have already emerged.

Deborah's Dilemma encourages teachers to introduce quantitative and numerical elements via a decision problem. It also encourages them to recognise and introduce personal and social elements into decision-making through ideas of impact and utility of different choices. The data suggests that the cross-curricular dialogue approach does produce a rich exploration for both, and leads to views about risk being shared. This is certainly important for mathematics learning as social issues rarely feature in mathematics classrooms and yet are obviously important in studying uncertainty.

The risk context also offers a fresh perspective on research into perceptions of uncertainty. For example, it has been known for some time that people find it hard to manipulate and use very low (or very high) probabilities (Kapadia & Borovcnik 1991). This research has found some evidence of this problem (cf. episode 1) (confirming the need for more study), which may need to be taken into account in the pedagogy of teaching risk, particularly as the spectrum of risks commonly discussed range over many orders of magnitude.

References

Ainley, J., D. Pratt and A. Hansen. 2006. Connecting engagement and focus in pedagogic task design. *British Educational Research Journal* 32(1): 21-36.

Bryce, T. and D. Gray. 2004. Tough acts to follow: the challenges to science teachers presented by biotechnological progress. *International Journal of Science Education* 26(6): 717-733.

Campbell, S. 2006. Risk and the Subjectivity of Preference. *Journal of Risk Research* 9(3): 225-242.

Edwards, W. and A. Tversky, eds. 1967. *Decision making: Selected readings* Harmondsworth: Penguin Books.

Haste, H. 2004. *Science in my future, Nestle Social Research Programme report No. 1*. Accessed January 8th, 2010 from http://www.spreckley.co.uk/nestle/reports.htm.

Kahneman, D. and A. Tversky. 1979. Prospect theory: An analysis of decision under risk. *Econometrica* 47(2): 263-292.

Kahneman, D., P. Slovic and A. Tversky. 1982. *Judgment under uncertainty: Heuristics and biases*. Cambridge: Cambridge University Press.

Kapadia, R. and M. Borovcnik, eds. 1991. *Chance encounters: Probability in education.* Dordrecht: Kluwer Academic.

Kolstø, S. D. 2001. 'To trust or not to trust, . . .' - Pupils' ways of judging information encountered in a socio-scientific issue. *International Journal of Science Education* 23(9): 877-901.

Lecoutre, M. P. 1992. Cognitive models and problem spaces in "purely random" situations. *Educational Studies in Mathematics* 23: 589-593.

Noss, R., and C. Hoyles. 1996. *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht: Kluwer Academic.

OCR. 2009. *GCSE in Science A Twenty First Century Science suite specification*. Cambridge: Oxford Cambridge and RSA Examinations.

Prodromou, T. and D. Pratt. 2006. The role of causality in the coordination of two perspectives on distribution within a virtual simulation. *Statistics Education Research Journal* 5(2): 69-88.

Rees, M. 2006. Address of the President, Lord Rees of Ludlow Kt FRS, given at the Anniversary Meeting on 30 November 2006. *Notes & Records of the Royal Society* (22 January 2007) 61(1): 75-83. (Accessed January 4th, 2009 from http://rsnr.royalsocietypublishing.org/content/61/1/75.full)

Schools Council Project on Statistical Education. 1980. *Statistics in your world*. Slough: Foulsham.

Wilensky, U. 1997. What is normal anyway? Therapy for epistemological anxiety. *Educational Studies in Mathematics* 33: 171-202.

Corresponding author: R.Kapadia@ioe.ac.uk

Perceived parental influence on students' mathematical achievement, inclination to mathematics and dispositions to study further mathematics

Irene Kleanthous and Julian Williams

University of Manchester

This paper explores perceived parental influence on students' achievement in mathematics, inclination to mathematics and dispositions to study further mathematics among 563 students in Cyprus. The reliability of the scale designed to measure perceived parental influence was investigated using the Rasch model. It was found that perceived parental influence had a statistically significant effect on students' inclination to mathematics, but it did not have a statistically significant effect on students' mathematics achievement and dispositions to study further mathematics in Higher Education. The effect of parental influence on students' dispositions towards mathematics is discussed.

parental influence, parental aspirations, mathematics achievement, dispositions

Introduction

A considerable number of studies have investigated the role that parents play in their children's mathematical learning. Previous research results suggest that parental involvement has a significant impact on students' mathematics achievement and attitudes towards mathematics (Fan and Chen 2001; Aunola et al. 2003). Parental aspirations and parents' attitudes towards mathematics have been identified as having a significant impact on students' participation in advanced level mathematics and students' achievement in mathematics (Ma 2001).

Despite the vast research on parental involvement in primary mathematics (Campbell and Mandel 1990) there is a scarcity of research on parental aspirations and their impact on adolescent students' dispositions towards mathematics. Notably, although parental influence has been investigated to some extent in relation to mathematics education, there is still some inconsistency in the terminology used in the literature to define parental aspirations. Jacobs and Harvey (2005) define parental aspirations as the amount of education that parents would like for their children to have, ranging from secondary education to postgraduate university degree.

It is generally well documented that higher family socioeconomic status (SES) is related to higher educational expectations for their youths (Wentzel 1998). The concepts of habitus and capital, suggested by Bourdieu and Passeron (1990) have become prominent for investigating and understanding social inequalities among parent groups and crucial for understanding parents' choices and involvement in educational contexts. We conceptualise students' dispositions to study further mathematics in the field of Higher Education as part of their habitus. Students' dispositions towards mathematics might partly be a result of the habitus their families have inculcated.

> "Pedagogic work accomplished by the family is a function of the distance between the habitus it tends to inculcate, and the habitus inculcated by all previous forms of pedagogic work" (Bourdieu and Passeron 1990: p.72).

Undoubtedly, cultural beliefs have an influence on the value parents place on their children's education. Mau (1997) remarks that the degree of parental expectation which is perceived by students differs between cultural/ethnic groups and has a direct impact on children's academic performance. Cross-cultural research on perceived parental influence has attracted the interest of many researchers. Specifically Cao, Bishop and Forgasz (2006) compared Chinese and Australian students' perceptions of parental influence, and concluded that Chinese students have stronger perceived parental influence than Australian students.

In a comparative study of Asian and Caucasian Americans parents, Campbell and Mandel (1990) divided parental influence into four elements: parental pressure, psychological support, parental help and parental monitoring. Furthermore Cao, Bishop and Forgasz (2006) distinguish between direct and indirect parental influence. They argue that direct parental influence, such as helping children with mathematics difficulties, has a less important impact on students' mathematics performance. Indirect parental influence such as parental encouragement, parental expectation and parents' attitudes towards mathematics have been identified as having a significant impact on students' attitudes towards mathematics.

Due to the complexity of parental influence, there is not a standardised scale in the literature for measuring perceived parental aspirations. The aim of the present study is to develop and validate a scale for measuring perceived parental aspirations and to investigate their impact on students' dispositions towards mathematics. A working definition of parental aspirations which underpins the development of the items for this scale, is the extent to which parents value education and urge their children to do well in school and the strength of the parents' expression of the importance of their children's social advancement through education (in this case as perceived by their child). We hypothesise the impact of parental aspirations on students' formation of habitus and decision making to study further mathematics in Higher Education (HE) is crucial.

Methodology

A questionnaire was designed to measure students' perceived parental aspirations (PAR), prior maths achievement (MACH), inclination to mathematics (MINC), maths self-efficacy (MSE) and dispositions to study further mathematics (DISP.MATHS). The questionnaire was distributed to 563 students in Cyprus, aged 16-17 attending four different upper secondary schools (*lyceums*).

The PAR scale consisted of 14 items, seven items measuring perceived parental aspirations for school mathematics (PAR-SC) and seven items measuring perceived parental aspirations for studying mathematics in HE (PAR-HE). Some items for the PAR-SC scale were adopted from Marchant et al. (2001) i.e. *"My parents encourage me to do my best at school"* and were paraphrased in order to be included in the PAR-HE scale i.e. *"My parents encourage me to study at university"*.

Students self-reported their prior maths achievement (MACH) and inclination to mathematics (MINC). An indicative item for measuring students' inclination (MINC) to mathematics is *"Mathematics is my favourite subject"*. The DISP.MATHS scale was adopted from the ESRC-TLRP project "Keeping open the door to mathematically demanding programmes in further and higher education" (Williams et al. 2008). A four point Likert-type response format was used for the scales. For each item students were asked to indicate whether they Strongly Agree=4, Agree=3, Disagree=2 or Strongly Disagree=1.

The items for the maths self-efficacy (MSE) scale were designed according to the mathematics curriculum for each year group. These were contextualised questions with mathematics problems drawn from the national textbooks of mathematics and from the ESRC-TLRP project (Williams et al. 2008). Students were asked to rate their confidence in solving each maths problem ranging from 1=Not confident at all to 4=Very confident.

Results

The reliability of the two subscales of PAR, PAR-SC and PAR-HE was investigated separately using the Rasch model for Rating scales. According to Bond and Fox (2007) Rasch analysis provides indicators of how well each item fits within the underlying construct, in this case perceived parental aspirations. Model fit statistics and item analysis was carried out for each subscale using Winsteps. There were no statistically significant misfitting items in any of the subscales. The Cronbach alpha for the PAR-SC scale was $\alpha=0.54$ and for the PAR-HE scale $\alpha=0.58$.

The two subscales were subsequently merged into one scale measuring perceived parental educational aspirations (PAR). The new PAR scale's validity was also investigated using the Rasch model. Table 1 provides information on the INFIT MSNQ and OUTFIT MNSQ which are expressions of the fit of the item to the model. Infit MSNQ values that range from 0.7 to 1.3 are generally deemed to be acceptable infit statistics. As with the previous scales, none of the fit values posed a serious threat to construct validity. The Cronbach alpha for the PAR scale was $\alpha=0.72$.

Table 1. PAR scale items fit statistics

```
                    ITEM STATISTICS:  MISFIT ORDER
-------------------------------------------------------------------------------
|  INFIT  |  OUTFIT |
|MNSQ ZSTD|MNSQ ZSTD| ITEM    ITEM description
------------------------------------+---------+---------+-----------+-----------+--
|1.14  1.7|1.36  3.6|PAR-SC1  My parents encourage me to do my best at school.
|1.16  2.3|1.23  3.2|PAR-HE6  My parents allow me to decide on my own what courses
                              to choose at university.
|1.09  1.3|1.2   2.9|PAR-SC6  My parents allow me to decide on my own what subjects
                              to choose at school.
|1.12  1.7|1.10  1.3|PAR-HE1  My parents encourage me to study at university.
|1.11  1.1|1.01   .2|PAR-HE2  My parents think that obtaining a degree is important.
|1.09  1.6|1.08  1.4|PAR-HE7  My parents stress the importance of mathematics more
                              comparing to other courses at university.
|1.07  1.3|1.06  1.1|PAR-SC7  My parents stress the importance of mathematics more
                              comparing to other subjects.
|1.00   .0|.94  -.6|PAR-SC2  My parents think that succeeding in school is important.
|.97  -.6|.97   -.5|PAR-SC4  My parents don't pressure me about my school work.
|.97  -.6|.97   -.5|PAR-SC5  My parents think it is more important to be happy than
                              worry about grades.
|.95  -.9|.95  -1.0|PAR-HE5  My parents think it is more important to be happy than
                              worry about studies.
|.85 -3.0|.86  -2.7|PAR-HE3  My parents are demanding to study at university.
|.86 -2.8|.86  -2.9|PAR-HE4  My parents don't pressure me to study at university.
|.84 -3.0|.84  -3.1|PAR-SC3  My parents are demanding about my school work.
------------------------------------+---------+---------+-----------+-----------+--
```

Figure 1 shows the item-person map. Bond and Fox (2007) note that the logit scale is the measurement unit common to both person ability and item difficulty. The Rasch analysis produces a single difficulty estimate for each item (δ) and an ability estimate for each student (α). Persons and items are located on the map according to their ability and difficulty estimates respectively. The numbers on the left represent the logit scale on which items and cases are calibrated. The symbol # presents the distributions of the cases (pupils) according to their 'ability' estimate.

Figure 1. Item-person map for the PAR scale

```
INPUT: 563 PERSONS 14 ITEMS  MEASURED: 562 PERSONS  14 ITEMS  56 CATS
---------------------------------------------------------------------

                  PERSONS - MAP - ITEMS
                     <more>|<rare>
                         +
                       . |
                         |
                      .# |
        2             .  +T
                         |
                       # |
                      ## T|   PAR-SC6
                     ### |    PAR-HE6          Parental permissiveness
                     .## |
                         |
                     ### |
        1          .#### +S
               .######## S|   PAR-HE7
                .####### |    PAR-SC5          Parents' attitudes to maths
                .####### |    PAR-SC7
                .######## |
             .########### |                    Parental support
               ########## M|   PAR-HE5
             .########## |    PAR-SC4           Parental pressure
        0    .########## +M
                      .  |    PAR-HE3  PAR-HE4
             .########### |    PAR-SC3          Parental demandingness
               .####### S|
               .###### S|
                 ##### |
                 ### |
                .### |    PAR-HE1
       -1        ### +S
                 .# |                           Parental encouragement
                 # T|    PAR-SC1
                 .  |
                 .  |    PAR-SC2
                    |    PAR-HE2                 Parental expectations
                 .  |
                    |
       -2           +T
                    |
                    |
                    |
                    |
                    +
               <less>|<frequ>

     EACH '#' IS 4.
```
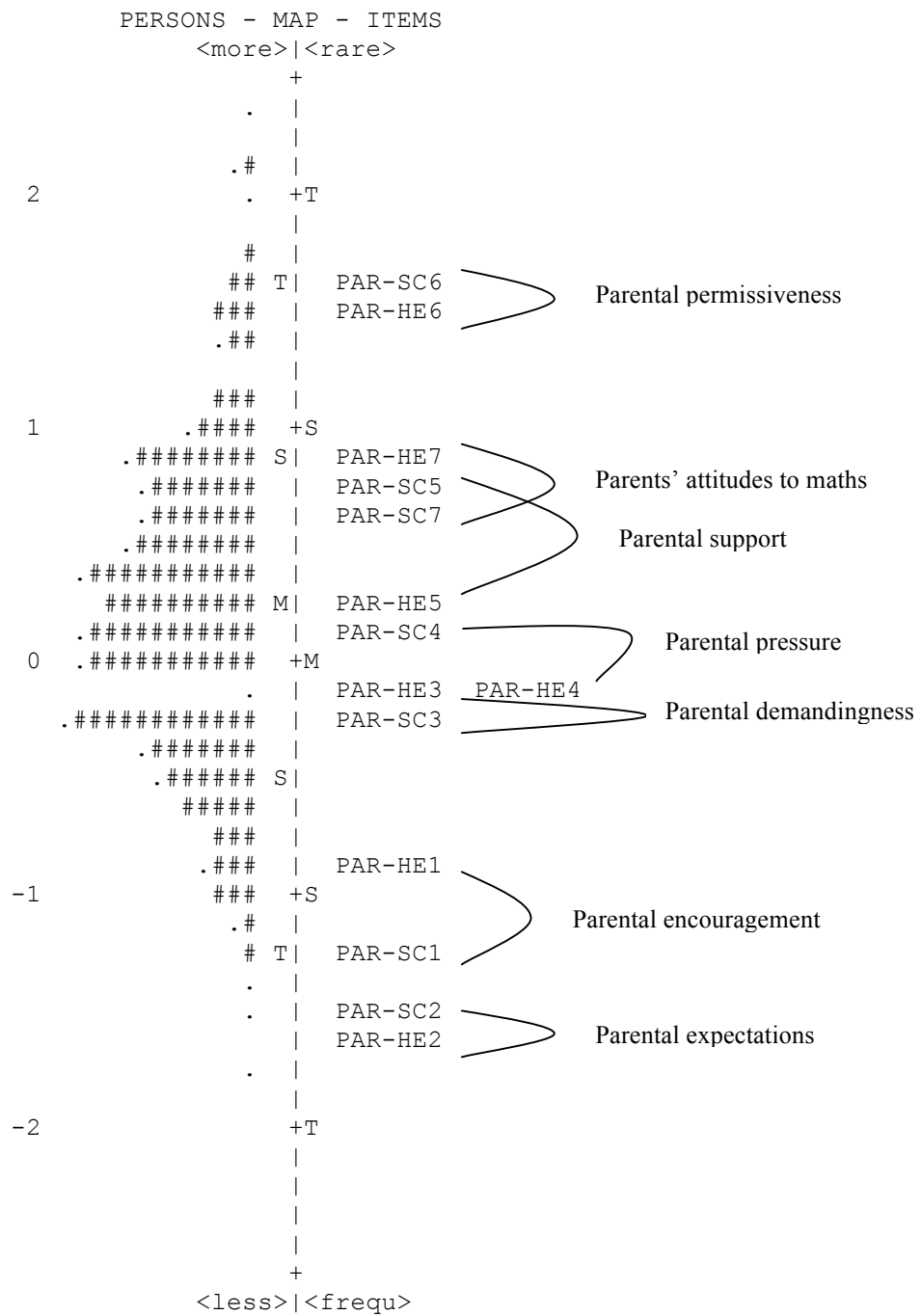
Figure 1 shows good spread of the items and how the items rank in terms of difficulty. Apparently item 6 is the hardest item and item 2 is the easiest item to agree with. A possible explanation why item 6 is the hardest item to agree with, could be students' reluctance to report their parents' influence on their decision making about school subjects and university courses. The items of the two subscales for each category of parental influence behaved similarly, as the grouping of the items indicates.

Regression models for MINC, DISP.MATHS and MACH

Once the Rasch analysis was conducted to check the validity of the new PAR scale, a step-wise model selection procedure was adopted to build generalised linear models (GLM) in the statistical software R. PAR was used as an explanatory variable to model students' inclination to mathematics (MINC), dispositions to study further mathematics (DISP.MATHS), and mathematical achievement (MACH).

Other variables which were used for these models were background variables such as gender, socio-economic status (SES), maths self-efficacy (MSE) and mathematics course (Advanced or Core mathematics). Perceived parental aspirations (PAR) had a statistically significant effect on students' MINC as can been seen in table 2, but it was not statistically significantly related to students' DISP.MATHS and MACH as can be seen in table 3 and 4.

Table 2. The GLM model for students' inclination to mathematics (MINC)

MINC ~ MACH + Gender + Maths.course + SES + MSE + PAR

	Estimate	Std.Error	t-value	p-value	
(Intercept)	-3.65224	0.47560	-7.679	7.95e-14	***
MACH	0.28492	0.02318	12.294	< 2e-16	***
Gender[T.Male]	0.51347	0.14531	3.534	0.000446	***
Maths.course[T.Core]	-1.04515	0.15312	-6.826	2.44e-11	***
SES[T.low]	-0.25459	0.23453	-1.086	0.278186	
SES[T.medium]	-0.10846	0.22084	-0.491	0.623545	
MSE	0.10594	0.05423	1.953	0.051304	.
PAR	0.16539	0.07881	2.099	0.036336	*

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Multiple R-Squared: 0.3706,   Adjusted R-squared: 0.3621
F-statistic: 43.9 on 7 and 522 DF, p-value: < 2.2e-16

Apparently the F-value (F=43.9, p=0.00) shows the model is statistically significant and it can predict 36% of the variance of MINC (R^2= 0.36). Students' mathematics achievement (MACH), gender, mathematics course and perceived parental aspirations (PAR) are statistically significant explanatory variables for this model, whereas MSE and SES are not statistically significant.

Table 3. The GLM model for students' dispositions to study further mathematics (DISP.MATHS)

DISP.MATHS ~ MACH + PAR + Maths.course + MINC

Coefficients:

	Estimate	Std.Error	t-value	p-value	
(Intercept)	0.97657	0.34252	2.851	0.00452	**
MACH	-0.06132	0.02148	-2.855	0.00447	**
PAR	0.02024	0.08567	0.236	0.81327	
Maths.course[T.Core]	-1.53449	0.13250	-11.581	< 2e-16	***
MINC	0.50507	0.03754	13.454	< 2e-16	***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Multiple R-Squared: 0.508,   Adjusted R-squared: 0.5044
F-statistic: 139.2 on 4 and 539 DF, p-value: < 2.2e-16

Clearly the F-value (F=139.2, p=0.00) shows the model is statistically significant and it can predict 50% of the variance of DISP.MATHS (R^2= 0.50). Students' inclination to mathematics (MINC), Mathematics course and MACH are statistically significant explanatory variables for this model. PAR is not statistically significant for predicting students' dispositions to study further mathematics (DISP.MATHS), but since PAR influences MINC it could be argued that it influences DISP.MATHS indirectly.

Table 4. The GLM model for students' maths achievement (MACH)

```
      MACH ~ Maths.course + MINC + MSE + PAR + Gender + SES
```

Coefficients:	Estimate	Std.Error	t-value	p-value	
(Intercept)	16.44156	0.41508	39.610	< 2e-16	***
Maths.course[T.Core]	-0.01069	0.26837	-0.040	0.9682	
MINC	0.78390	0.06435	12.182	< 2e-16	***
MSE	0.07055	0.09067	0.778	0.4368	
PAR	-0.14396	0.16549	-0.870	0.3847	
Gender[T.Male]	-1.71009	0.23368	-7.318	9.55e-13	***
SES[T.low]	-0.78448	0.38955	-2.014	0.0445	*
SES[T.medium]	-0.14056	0.36803	-0.382	0.7027	

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Multiple R-Squared: 0.325,    Adjusted R-squared: 0.3159
F-statistic:  35.9 on 7 and 522 DF,  p-value: < 2.2e-16
```

This model is also statistically significant (F-value=35.9, p=0.00) although it can only predict 32% of the variance of MACH (R^2= 0.3159). Students' inclination to mathematics (MINC), gender and SES were statistically significant predictors of MACH in this model. Although perceived parental aspirations (PAR) were not statistically significantly related to students' mathematics achievement (MACH), PAR might influence MACH indirectly through its influence on MINC.

Discussion

A surprising finding of the present study was that perceived parental aspirations did not have a statistically significant effect on students' achievement in mathematics. This finding contradicts other researchers (Fan and Chen 2001; Aunola et al. 2003) who found that parental aspirations have an impact on students' mathematics achievement. This contradictory result might be due to the fact that in the present study parental aspirations were reported by students. If parents had self-reported their aspirations for their children, parental aspirations could have been proven statistically significant as the literature suggests.

Another plausible explanation why students' perceived parental aspirations were not statistically significant can be attributed to the students' age. Bearing in mind that the sample consisted of adolescent students, it could be argued that parental influence was 'denied' and students reported low parental aspirations which might not represent their parents' actual aspirations. This finding adds additional support to Neuenschwander's et al. (2007) argument that "as students seek more autonomy from their parents in adolescence they begin to reflect upon and to disagree with their parents' attitudes and beliefs. We expect that parents' expectations may lose some of their predictive power across adolescence" (p.601).

Our statistical analysis showed that there are different ways in which students perceive parental influence which is consistent throughout their educational career (school or HE). The items of the two subscales of PAR, which measure the same category of parental influence, such as parental pressure, parental encouragement and parental support, appear to behave similarly (see figure 1). Other researchers have also noted different forms of parental influence such as behavioural control and pressure in contrast to psychological support (Campell and Mandel, 1990). The diverse ways students perceive parental influence might therefore explain why parental aspirations did not have a statistically significant effect on their mathematics achievement, as the statistical analysis showed.

On the other hand however, perceived parental aspirations were statistically significantly related to students' inclination to mathematics. Students who are positively disposed towards mathematics reported higher perceptions of parental aspirations. This finding further supports the notion that parental aspirations have a significant effect on students' attitudes towards mathematics (Ma, 2001). Surprisingly perceived parental aspirations were not statistically significantly related to students' dispositions to study further mathematics in HE. Previous research findings have also noted that students make their choices for future studies autonomously whilst they feel that their parents approve of their choices. Beyers and Goossens (2008) stress that "parents might react positively when the adolescent makes an autonomous choice of his or her study or career, rather than actively encouraging the adolescent to make such a choice" (p.169).

We suggest that the concept of 'symbolic violence' suggested by Bourdieu (1977) can be useful for understanding students' perceptions of parental influence. Bourdieu (1977) argues that symbolic violence is at the heart or every social relationship. He defines symbolic violence as 'the gentle invisible form of violence, which is never recognised as such, and is not so much undergone as chosen' (p.192). We argue that students' perceptions of parental influence might not correspond to their parents' actual aspirations or influence, which remain hidden but are all the more powerful because they are relatively invisible. If so parental influence is subconscious and therefore misrecognised by the students and perhaps even by their parents.

Conclusion

The present study has developed and validated a scale for measuring perceived parental aspirations (PAR) with the use of the Rasch model. The calibration of the two subscales of PAR indicates that there are different categories of parental influence which are perceived by students in similar ways. Whether students reported perceived parental aspirations for school mathematics or for studying mathematics at university, they perceived different forms of parental influence in the same manner. Thus we argue that parental educational aspirations are the same underlying construct cross educational levels (school or HE). The ways parents communicate their educational aspirations to their children and the influence of parental aspirations on students' dispositions towards mathematics still deserves further investigation.

References

Aunola, K., J.E. Nurmi, M.K. Lerkkanen, and H. Puttonen. 2003. The roles of achievement-related behaviours and parental beliefs in children's mathematical performance. *Educational Psychology* 23(4): 403-421

Beyers, W., and L. Goossens. 2008. Dynamics of perceived parenting and identity formation in late adolescence. *Journal of Adolescence* 31: 165-184

Bond, T. B., and C.M. Fox. 2007. *Applying the Rasch model: Fundamental Measurement in the Human Sciences* (2nd ed.). Mahwah, New Jersey Lawrence Erlbaum Associates.

Bourdieu, P. 1977. *Outline of a theory of practice*. Cambridge, Cambridge University Press

Bourdieu, P. and J.C. Passeron. 1990. *Reproduction in education, society and culture* (2nd ed), London, Sage.

Campbell, J., and F. Mandel. 1990. Connecting math achievement to parental influences. *Contemporary Educational Psychology* 15: 64-74

Cao, Z., H. Forgasz, and A. Bishop. 2006. Perceived parental influence on mathematics learning: A comparison among students in China and Australia. *Educational Studies in Mathematics* 64: 85-106

Fan, X., and M. Chen. 2001. Parental Involvement and Students' Academic Achievement: A Meta-Analysis. *Educational Psychology Review* 13(1): 1-22

Jacobs, N. and D. Harvey. 2005. Do parents make a difference to children's academic achievement? Differences between parents of higher and lower achieving students. *Educational Studies* 31(4): 431-448

Ma, X. 2001. Participation in Advanced Mathematics: Do expectation and influence of students, peers, teachers and parents matter? *Contemporary Educational Psychology* 26: 132-146

Marchant, G.J., S.E. Paulson, and B.A. Rothlisberg. 2001. Relations of middle school students' perceptions of family and school contexts with academic achievement. *Psychology in the Schools* 38(6): 505-519

Mau, W.C. 1997. Parental Influences on the High School Students' Academic Achievement: A comparison of Asian immigrants, Asian Americans, and White Americans. *Psychology in the Schools* 34(3): 267-277

Neuenschwander, M.P., M.Vida, L. Garrett, and J.S. Eccles. 2007. Parents' expectations and students' achievement in two western nations. *International Journal of Behavioral Development* 31: 594-602

Wentzel, K.R. 1998. Parents' Aspirations for Children's Educational Attainments: Relations to Parental Beliefs and Social Address Variables, *Merrill- Palmer Quarterly*. Retrieved October 30, 2008 from http://find.galegroup.com

Williams, J., L. Black, P. Davis, G. Hutcheson, S. Nicholson, and G. Wake, 2008. *Keeping open the door to mathematically demanding programmes in further and higher education*. ESRC-TLRP Research briefing, June 2008

The effect of using real world contexts in post-16 mathematics questions

Chris Little (ctl@soton.ac.uk) and Keith Jones

School of Education, University of Southampton, Southampton, UK.

This paper reports on a study into the effect of real-world contextual framing in sequence questions. Alternative versions of the same questions were presented in explicit, algebraic, word and pattern contexts, and set to a sample of 594 Year 13 students (aged 17-18) in a one-hour test. Facility levels of the questions were then compared. In addition, the paper presents results of a student questionnaire on real-world context which accompanied the test.

Keywords: Assessment, post-16 mathematics, real-world context, modelling

Introduction

A well known problem from the Rhind (or Ahmes) papyrus of 1650 BC (Boyer, 1985) may be stated in the form:

> 'Seven houses have seven cats per house. Each cat catches seven mice. Each mouse eats seven ears of spelt, each ear would have produced seven hekats of grain. What is the sum of the houses, cats, mice, ears and grain?'

'Word' problems, set in some context, have therefore been around a long time, and have been the subject of considerable research. While current mathematics curricula often frame mathematics questions in 'real-life' terms, the utility and realism of many of the contexts used in mathematics problems have been called into question (Wiliam, 1997, Boaler, 1993b). Some research has suggested that the perceived realism of contexts may be gender-dependent (Boaler 1994) Much research has focused on understanding the difficulties that many children seem to have in applying realistic constraints to 'real-world' problems (Verschaffel et al., 1994, Verschaffel et al., 1997, Cumming and Maxwell, 1999, Silver, 1993), or of transferring from one context to another (Boaler, 1993a). In terms of designing suitable examinations in mathematics, the validity of contexts deployed in summative assessment tasks has been identified as a particular issue (Ahmed and Pollitt, 2007, Pollitt and Ahmed, 2001, Pollitt et al., 2007). Some research has suggested that linguistic and cultural factors may affect the interpretation of real-world context in questions (Pollitt et al., 2000, Barwell, 2001). Cooper and Dunne (2000) have, in particular, provided a critique of the use of real-world contexts in national curriculum assessments on the grounds that they discriminate by social class.

Despite this work, little research has directly investigated the effect of real-world context on the facility of questions, or on the attitudes of learners to its usage in tasks. Moreover, most research has focussed on pre-16 mathematics. This paper describes a study conducted to investigate these issues, using 17-year old students and the AS-level topic of sequences.

The extent to which real-world contexts are used in UK public examination syllabuses varies substantially. For example, an analysis (carried out as part of the study reported in this paper) of pure mathematics papers from two current GCE A/AS specifications – OCR 'A' and 'B' (MEI) - suggests that they utilise real-world

contexts in questions contributing about 5% of the marks and 30% of the marks respectively. This difference of emphasis can be traced back to 'modern' and 'traditional' A-level syllabuses of the 1970s and 1980s (Little, 2010, in press).

The research indicated above suggests a fundamental dilemma concerning the use of real-world context, or, as we prefer to call it, real-world contextual framing (RWCF) (Little, 2008a, Little and Jones, 2007) in mathematics questions. On the one hand, by making a connection between the abstract world of mathematics and everyday, or scientific, contexts, we are reinforcing the utility of mathematics as a language for explaining the patterns and symmetries of the 'real' world. On the other hand, if we manipulate and 'sanitise' real-world experiences to enable them to be modelled by a pre-ordained set of mathematical techniques, then the result can appear to be artificial and contrived, or, in the words of Wiliam (1997) a 'con'-text, providing a deception that the activity is worthwhile.

Another dilemma concerns the effect of real-world contextual framing on questions. On the one hand, it may be said to help to solve mathematical tasks by providing the solver with a 'mental scaffolding' (Vappula and Clausen-May, 2006) for solving the problem; on the other, it may be seen to complicate the task by requiring the solver to identify and match mathematical concepts with elements of the real-world context (Pollitt and Ahmed, 2001). This may assume knowledge of the context as well as the mathematics, which may actually be disadvantageous in solving problems in the manner intended, or according to the artificial rules of the mathematics classroom (Boaler, 1994).

This paper considers the effect of RWCF on post-16 examination questions. The study forms part of doctoral research which also explores the wider historical, philosophical and pedagogical roots of real-world context (Little 2010, in press).

Theoretical Framework

Some theories of teaching and learning, such as Realistic Mathematics Education (see, for example, Freudenthal, 1991, Treffers, 1987, Dickinson and Eade, 2005) espouse the view that 'realistic' contexts, or contexts that are 'real' to learners, play a coordinating role in the development of mathematical concepts. When used in public examinations, however, assessment theory requires that questions are evaluated according to their construct validity (Wiliam, 2007). Ahmed and Pollitt (2007) use the term *construct fidelity* to assess the extent to which a real-world context faithfully tests what it intends to assess. They see context as a threat to validity, as it introduces variability to the question–setting and answering process. Additional demands of comprehension on candidates may be considered to introduce a *construct irrelevant variance* (Wiliam, 2007) to the assessment.

On the other hand, the subject criteria for GCE A/AS Mathematics (Qualifications and Curriculum Authority, 2002) require schemes of assessment to:

> 'Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinements of such models.'

> 'Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.'

These assessment objectives would appear to require a degree of real-world contextual framing in test items.

How do questions involving real-world context differ from those deploying language from the mathematics register (Pimm, 1987)? They require the solver to match elements of the real-world context with mathematical concepts and variables. The extent to which this matching process is explicit or implicit in the question, or overt or camouflaged, would appear to be relevant to the facility of the question.

Methodology

In order to investigate the effect of RWCF in post-16 mathematics, the study selected a topic from A/AS level which appeared to be amenable to a variety of approaches, both with and without RWCF. An analysis of past paper suggested that sequences and series was such a topic, and that questions on this topic may be categorised as follows:

- Explicit (e) questions, which contained no real-world context, and explicitly define the sequence to be used in the question;
- Algebraic (a) questions, which used mathematical notation, such as u_n or sigma notation;
- Word (w) questions, which used a real-world context defined in words;
- Pattern (p) questions, in which a sequence was defined using a pattern context.

The study therefore constructs e, a, w and p versions questions of with similar or identical solutions, in order to compare their facility.

The study comprised a one-hour test and a short student questionnaire. The test consisted of four questions on arithmetic sequences (AI to AIV) and four on geometric sequences (GI to GIV), and was given to a sample of 594 year 13 students (aged 17-18) from four centres. Students were randomly allocated to one of four tests (A, B, C or D), each of which contained one of four versions (e, a, w and p) of the questions. The solutions to each version were, except in a few small details, the same.

The total mean scores for the 'e', 'a', 'w' and 'p' versions were calculated and compared, to establish an overall measure of difficulty for each type of question. Each part question was then considered, using four difference of two means test to compare the 'e' version with the 'a', 'w' and 'p' versions, and the 'w' version with the 'p' version. The question versions, and their solutions, were then analysed in greater detail, in order to conjecture reasons for significant differences in response.

The questionnaire invited students to consider six statements on pure and applied mathematics and real-world context in questions, and register their level of agreement, from 'strongly disagree' to 'strongly agree'. These questions are listed in Fig. 2 in the 'results' section. They were then given space to make further comments. The results of the questionnaire were compared by gender and by whether students declared English as their first language. Open comments were classified into broad similarities of opinion.

Results

The mean total number of marks out of 40 for each version were as follow:

Explicit: 28.17, Algebraic: 20.17, Word: 22.31, Pattern: 23.37

Thus, the 'e' versions (explicit), as might be expected, gained higher marks overall, followed by the 'w' and 'p' versions, with the 'a' questions proving to be the most difficult. However, this pattern of results did not hold for all of the questions

Fig. 1 gives the versions of question AI, together with the mean scores for each part question, and is followed by a sample analysis.

A1e (A1)
An arithmetic progression has first term 7 and common difference 3.

(i)   Which term of the progression equals 73? [3] [2.60]

(ii)   Find the sum of the first 30 terms of the progression. [2] [1.76]

A1a (D8)
The nth term of an arithmetic progression is denoted by u_n. $u_1 = 7$, $u_2 = 10$ and $u_3 = 13$.
(i)   If $u_n = 73$, find n. [3] [2.35]
(ii) Find $\sum_{r=1}^{30} u_r$. [2] [0.99]

A1w (C5)
Chris saves money regularly each week. In the first week, he saves £7. Each week after that, he saves £3 more than the previous week.

(i) In which week does he save £73? [3] [2.19]

(ii) Find his total savings after 30 weeks. [2] [1.18]

A1p (B4)
A spiral is formed with sides of lengths 7 cm, 10 cm, 13 cm, … which are in arithmetic progression.

(i) How many sides does the spiral have if its longest side is 73 cm? [3] [2.52]   7 cm   10 cm

(ii) Find the total length of the spiral with 30 sides. [2] [1.52]   13 cm

Fig. 1 Versions of question AI with mean scores.

For this question, both parts of the 'a' version were significantly more demanding than the 'e' version, and the 'w' version proved more difficult than the 'e' version, with a more significant difference in part (ii). Comparing the 'e' with the 'p' versions, there was no significant difference in scores for part (i), but part (ii) was significantly more difficult in the 'p' version. Finally, comparison of the 'w' and 'p' versions shows the 'w' version to be significantly more difficult than the 'p' version.

The 'e' v 'a' results are easily explained in terms of additional demand of algebraic notation, in particular sigma notation. In the 'e' version, the explicit reference to 'arithmetic progression', 'term' and 'sum' leads the solver directly to the appropriate formulae. These cues were not present in the 'w' version, which requires solvers to translate elements from the real-world context to the algebraic model ('first week £7' = a, '£3 more' = d). This would account for the extra difficulty of the 'w' version compared to the 'e' version.

On the other hand, the 'p' version of part (i) proved to be no harder than the 'e' version. This might be because the first three terms of the sequence are stated explicitly (7 cm, 10 cm, 13 cm, ..), thus making the match to an arithmetic model easier than in the 'w' version. Indeed, it is possible to think within the context to derive the number of terms (length increased by $67 = 3 \times 29$, so 30 sides). This type of 'first principles' thinking is not available in part (ii), which perhaps explains why this proved harder than the 'e' version.

The explicit statement of the first three terms in the 'p' version, thus hinting at an arithmetic sequence model, might also explain why this version proved to be easier than the 'w' version, where this cue was not given

Similar analyses of all eight questions suggest a number of factors which affect the facility of questions, as follows.

- The use of algebraic language such as sigma notation and iteration formulae added substantially to the difficulty of questions.
- The requirement to identify the type of sequences and to use the appropriate term or sum formula from data given in a real-world context also adds to the demand.
- Some real-world contextual framing requires solvers to interpret text carefully in order to select the appropriate match between context and model. Semantic ambiguities may also cause unintended errors.
- In contrast, questions which can be solved by 'thinking within the context' can be as easy as explicit formulations.

The results of the questionnaire, together with the questions asked, are shown in Figure 2. Two thirds of the students believed that questions set in real-world context are harder than those without context. 33% agreed and 30% disagreed that RWCF made questions more interesting, whereas 55% agreed and 30% disagreed, with the statement that real-world context shows how mathematics is useful. Over half of the students preferred pure mathematics to applied mathematics, and felt that pure mathematics is interesting in its own right.
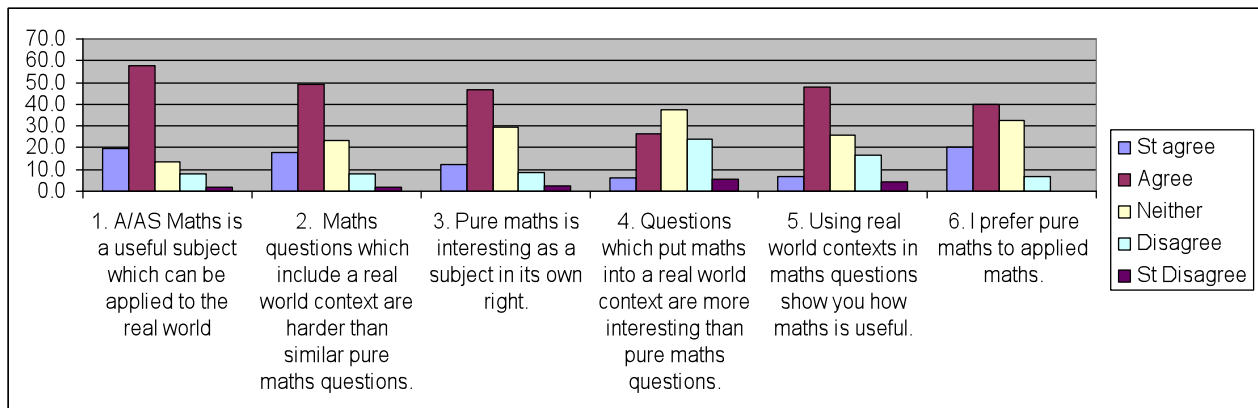
Fig. 2: questionnaire results

The results suggested that more girls prefer pure to applied mathematics. Girls agreed more with the statements that real-world contexts make questions harder and that pure maths is interesting as a subject in its own right, whereas boys agreed more with the statement that questions with real-world context are more interesting. Overall, these results show a consistent pattern of girls preferring pure mathematics questions without real world contextual framing.

Students for whom English was their first language agreed more strongly with the statements that A/AS mathematics is a useful subject which can be applied to the real world and that maths questions set in real-world context are harder.

The first result might be interpreted as showing cultural differences concerning the nature of mathematics. The second result is perhaps surprising, as one might have expected non-native speakers to find contextualised questions harder to comprehend. However, as the number of students in the first category was relatively small (49), the sample may not be large enough to be representative.

The variety of opinions which students hold on the use of real-world context is indicated by the following selection of comments:

> "Applying maths to real situations is certainly more difficult but a lot more interesting and satisfying to complete rather than straight pure maths questions."

"I strongly dislike real world context questions as they turn maths that I can do into something I can barely understand."

"I prefer the questions which are worded (without context). I believe this is due to dyslexia, which means I find (contextualised questions) harder to understand."

"Sometimes (contexts) help if you don't know terms or to get an idea of what the question requires, but otherwise they are just plain patronising!"

"Real world questions don't show you how maths is useful, because questions in context, such as question 2, are not useful. It is about a beetle, not useful in normal everyday life."

Conclusions

How do these results inform research on real-world contextual framing? It should be acknowledged first that these results apply to post-16 students of mathematics only, and test results are based on one only one topic. Further research using different topics would be needed to test the generality of the conclusions. However, they provide strong evidence that setting sequence questions in real-world contexts does indeed add to the overall demand, though a context can on occasions provide 'mental scaffolding' to help the solver to use context-specific heuristic strategies. For example, questions involving the 'term' formula from an arithmetic progression [$u_n = a + (n - 1)d$] are amenable to calculations using first principles (e.g. nth term = 1st term + $(n - 1)$ × the 'step'). On the other hand, using the 'sum' formulae for both APs and GPs is a pre-requisite to the efficient solutions of problems involving summation.

One could argue that such solutions effectively side-track the application of standard algebraic formulae to model realistic situations. The potency of algebraic formulae lies in their universality and blindness to individual contexts (Little, 2008b), and, in resorting to context-specific thinking to solve these questions, students are avoiding the necessity to transfer and abstract from context to mathematical model, which is, arguably, the heuristic strategy intended by the questions.

However, questions with RWCF need to be carefully constructed to avoid unwanted distracters and ambiguities. They can require greater interpretative acuity from the solver, in order to correctly match the context with the mathematical model. They may therefore disadvantage students with dyslexia, or non-native language speakers. Contextualised questions therefore require careful revision to ensure that the language used is clear and unequivocal. It is also important to consider the overall length of questions in relation to the time allowed to answer them: asking students to read and comprehend complex, novel contexts in a timed written examination clearly adds to the stress of the experience, and, by placing too much emphasis on comprehension skills, compromise the mathematical goals of the assessment.

Students in this survey generally see real-world context as reinforcing the perception that mathematics is useful, although contexts can be perceived as artificial. Some comments from students resonate with researchers (e.g. Boaler, 1994, Wiliam, 1997) who have criticised real world contexts on the grounds of artificiality. The gender differences identified above are also of interest, with girls preferring pure mathematics and non-contextualised questions, but boys finding such questions more interesting.

What is the function of real-world context in these sequence questions? An earlier paper on linear equation questions argued that, notwithstanding that students link real-world context to applicability of mathematics, most contextualised questions have little or no practical utilitarian value (Little, 2008a). They are, rather, embryonic, albeit contrived, exercises in modelling, which force the student to make connections

between the world of algebra and mathematics and real-world concepts such as finance, percentages, and physical patterns and shapes.

It is perhaps unrealistic, however, to expect short, closed examination questions to do more than this, since genuine mathematical modelling requires strategic thinking that is not possible to test in a timed written examination. Such questions might, as the current subject criteria for A/AS Mathematics require, test students' ability to 'recall, select and use their knowledge of standard mathematical models to represent situations in the real world and present and interpret results from such models in terms of the original situation'. However, they manifestly do not 'include discussion of the assumptions made and refinements of such models'. Neither do they require students to 'read critically and comprehend longer mathematical arguments', which a dedicated comprehension paper might do. It is therefore difficult to see how A/AS level syllabuses that rely wholly on short written examination papers can be said to satisfy such requirements.

References

Ahmed, A., and A. Pollitt, 2007. Improving the quality of contextualised questions: an experimental investigation of focus. *Assessment in Education: Principles, Policy and Practice* 4: 201-232.

Barwell, R. 2001. Difficulties with mathematical word problems. *Proceedings of the British Society for Research in Learning Mathematics* 21(2): 63–74.

Boaler J. 1993a. Encouraging the transfer of "school" mathematics to the "real world" through the integration of pocess and content, context and culture. *Educational Studies in Mathematics* 25: 341-73.

Boaler, J. 1993b. The role of contexts in the mathematics classroom: do they make mathematics more real? *For the Learning of Mathematics* 13: 12-17.

Boaler, J. 1994. When do girls prefer football to fashion? An analysis of female under-achievement in relation to realistic mathematics contexts. *British Educational Research Journal* 20: 551-564.

Boyer, C. 1985. *A history of mathematics*. Princeton. Princeton University Press.

Cooper, B., and M. Dunne. 2000. *Assessing children's mathematical knowledge*, Buckingham, Open University Press.

Cumming, J., and G.Maxwell. 1999 Contextualising authentic assessment. *Assessment in Education* 6: 177-194.

Dickinson, P., and F. Eade. 2005. Trialling realistic mathematics education. *Proceedings of the British Society for Research in Learning Mathematics* 25(3), 1–13.

Freudenthal, H. 1991. *Revisiting mathematics education*. Dordrecht, Kluwer.

Little, C. 2008a. The functions and effects of real-world contextual framing in A/AS mathematics questions: developing an evaluative framework. *Proceedings of the British Society for Research in Learning Mathematics* 28(3), 72-77.

Little, C. 2008b. The role of context in linear equation questions. *Proceedings of the British Society for Research in Learning Mathematics* 28(2), 55–60.

Little, C. 2010. (in press). *The Use of Real-World Contextual Framing in UK University Entrance Level Mathematics Examinations*, PhD Thesis, University of Southampton.

Little, C., and K. Jones. 2007. Contexts for pure mathematics: an analysis of A-level mathematics papers. *Proceedings of the British Society for Research in Learning Mathematics* 27(1), 48–53.

Pimm, D. 1987. *Speaking mathematically*. London, Routledge.

Pollitt, A., and A. Ahmed. 2001. Understanding students' minds: the key to writing more valid questions. *National Center for University Entrance Examinations*, Japan.

Pollitt, A., A. Ahmed, and V. Crisp. 2007. The demands of examination syllabuses and examination papers. In *Techniques for monitoring the comparability of*

examination standards.ed. P. Newton, 112-130. London, Qualifications and Curriculum Authority.

Pollitt, A., C. Marriott and A. Ahmed. 2000. Language, contextual and cultural constraints on examination performance. *International Association for Educational Assessment*.

Qualifications and Curriculum Authority. 2002. *GCE AS/A Level Subject Criteria for Mathematics*.London, QCDA.

Quinlan, M. 1995. A comparability study in Advanced level mathematics. *University of London Examinations and Assessment Council*. London.

Silver, E. A. 1993. Sense making and the solution of division problems involving remainders: an examination of middle school students' solution processes and their interpretations of solutions. *Journal for Research in Mathematics Education* 24: 117-35.

Treffers, A. 1987, *Three dimensions*. Dordrecht, Reidel.

Vappula, H., and T. Clausen-May. 2006. Context in maths test questions: does it make a difference? *Research in Mathematics Education* 8: 99-115.

Verschaffel, L., E. De Corte, and I. Borghart 1997. Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modelling of school word problems. *Learning and Instruction* 7: 339-359.

Verschaffel, L., E. De Corte, and S. Lasure. 1994. Realistic considerations in mathematical modelling of school arithmetic word problems. *Learning and Instruction* 4: 273-94.

Wiliam, D. 1997. Relevance as MacGuffin in mathematics education. *British Educational Research Association*. York.

Wiliam, D. 2007. Quality in assessment. In *Unlocking Assessment*, ed.S Swaffield,123-137. London: David Fulton.

Analysing children's calculations: the role of process and object

Carol Murphy

University of Exeter

This paper reviews the role of process and object in young children's calculation strategies. By drawing on the Action, Process, Object, Schema (APOS) framework (Dubinsky and McDonald, 2001) children's calculation strategies are analysed. It is suggested that the opportunity for children to reflect on the actions they perform and also to reason about them is important in developing a coherent framework and hence a deep understanding of the calculation strategies they are using.

Introduction

In relation to the current concern to support understanding in mathematics (Williams, 2008) this study examines children's understanding of the calculation strategies that they use. It explores the strategies that children use in single digit addition and subtraction for example $(7 + 8; 9 - 7)$ as they move to the use of single to multi-digit addition and subtraction (for example $5 + 13; 16 - 9$).

The children studied were from KS1 classes. Whilst not identified as needing intensive support, they were identified by their teachers as lower attaining. It is recognised that these children are often seen as 'passive' learners. They are often tentative about their own understanding in mathematics (DCSF, 2007) and may not make the expected progress in KS2.

The role of processes and objects

It has been suggested that lower attaining children rely on counting procedures in addition and subtraction (Gray, 1991). Such strategies are seen as primitive and inefficient. A reliance on such strategies could explain the lack of progress in KS2. Whilst it is possible to be successful in calculations up to 20 or so using such strategies, progression to calculations with other 2 digit numbers becomes less successful.

The National Strategies in England have encouraged the use of flexible strategies that provide a more efficient method for calculations. It is now recognised (Gray, 1991, Steffe, 1983) that there is a progression from simple counting strategies ('count-all' and 'count-on' strategies) and the use of commutativity ('counting-on from the larger number') to the use of number facts (additive components) and place value. Initiatives such as the National Strategies in England have proposed the explicit teaching of calculation strategies but limited understanding of the key ideas underlying the progression may result in children becoming passive recipients of the calculation strategies that are presented to them in the classroom (Murphy, 2004). Such flexible strategies require more active ways of using number (Plunket, 1979) and their use goes beyond the counting procedures that lower attaining children often rely on.

Sfard (1991) recognised a reliance on 'count all' where children count out each set (for example $3 + 4$ becomes 1,2,3 and 1,2,3,4) and described this as the child's view of numbers as processes, rather than as objects. In this way the 3 has to

be counted out. This is distinct from seeing 3 as a cardinal number or object that can be 'counted on' from. Sfard's work looked at the dual nature of processes and objects, where "the ability of seeing a function or a number both as a process and as an object is indispensable for a deep understanding of mathematics…" (p. 5). In order to work with calculations more flexibly children need to see the dual nature. Gray and Tall (1994) have also referred to this as a 'procept' where children see the symbol for an operation as both a process and as a concept.

Dubinsky and McDonald's (2001) work has been used to study undergraduate level mathematics but their theoretical framework Action-Process-Object-Schema (APOS) is drawn on here as a possible way to provide insight into children's calculations as they move from counting procedures to more flexible strategies. As an *action* addition or subtraction are said to be carried out as transformations of objects. Understanding is limited to the performance in each situation. By repeating the action internal reflection is possible and the *process* can be described and the steps reflected on, even without performance. By reasoning about the properties of the processes ideas are treated as *objects*. The eventual aim is that these actions, processes and objects become a coherent frame or *schema*. It could be argued that in order to be flexible in the use of calculation strategies children need to be able to reason about the processes and even to be aware of the overall frame or schema.

In this study examples from a study into children's calculation strategies are presented. The children's strategies are analysed according to their use as actions, processes, objects or schemas. Can the children's use of counting procedures be interpreted as actions or is it possible to identify reflection in relation to a process or reasoning in relation to an object? Are the children also able to see the overall coherent framework or schema? For example a child may use a direct modelling strategy for 13 + 8 by counting out one set of 13 toys and another set of 8 toys. These two sets can then be physically pushed together and the total number of toys counted altogether. It would seem possible to interpret this as an action. However if the child was able to reflect on the action they had taken without performing it again would this be seen as a shift to a process? Would the child also be able to reason why they had used the process? By interviewing children it may be possible to see if the children see a strategy as an action that is performed or whether they are able to reflect on the strategy or even reason it.

The use of the Clinical Interview

Clinical interview refers to a class of interview methods that typically involve an extended dialogue between adult and child (Ginsburg, 1997). This provides an intensive interaction with an individual child and allows for the observation of a child's work with 'concrete' intellectual objects. As a research method it is deliberately non-standardised. It provides an interpretive orientation where the examiner is engaged in interpreting each child's responses. Interviews may start with some common questions but the examiner will react to responses and tailor questions to the individual child. In this way it can provide a way to explore "deep insights into children's thinking" and go "beneath the surface" (Ginsburg, 1997, p.2).

The interview schedule was based on an adaptation of the diagnostic tasks developed by the Shropshire Mathematics Centre (1996) and introduces children to addition and subtraction problems starting with single digit and moving to multi digit problems. The main study is with 72 children aged between 6 and 7 years old from twelve classes. All of the 72 children have been assessed as average to low attainers by their teachers and represent children who use both spontaneous and taught

procedures but may not always use these correctly. Four children from one class were selected from the early stages to pilot the analysis.

Results

In this paper results are presented from four children, Helen, Cassie, Bill and Jim who were in a Year 2 class (6 to 7 years old). The class teacher had recently modelled the use of the hundred square for the addition of multi-digit numbers and in particular she had encouraged the children to count on in tens rather than counting on in ones.

In the interviews the children were shown calculations one at a time. They were asked what the answer was and then asked to explain how they had worked the answer out. Observations were made of the children's work and the resources they used were also noted. It was anticipated that the children would use a mix of spontaneous and taught strategies.

Helen

Helen chose to draw figures to help her calculate 6+9 (figure 1) and 13 − 6 (figure 2). It would seem that she was engaged with addition and subtraction as an action and she was able to talk through the steps as she took them. There is little evidence to suggest that she was reflecting on the actions as processes or even reasoning about them as objects.

Figure 1: 6+9

Figure 2: 13 - 6
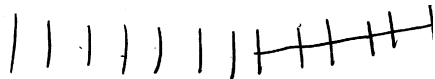
However Helen's solution to the subtraction problem 7-6 was different:

> H: You have six and then you only need to add one. So if you take away 6 there will only be one left. 'Cos it is the other way round.

In this solution Helen was able to describe the action she had taken and reflect on this as a process without performance. She used her knowledge of counting in ones 'You only need to add one' and she used addition and subtraction as inverse operations. Was she able to reason this or was she using this implicitly? Her use of the next word 'So…' might suggest that there was an element of reasoning about the process as an object.

Cassie

Cassie used her fingers and a hundred square to count on for the addition 5+7:

> C: I can't figure it out.. I think it's 12 or 11. Not sure. My fingers got 12 but I got 11 on the hundred square.

> Ex: When you did it with your fingers what did you start with?

C: I started with five and then I went 1,2,3,4,5,6,7 – went on 12 (Counts on again on hundred sq). Oh - I made it.

This could suggest Cassie's use of actions in performing the calculation. When she arrived at two different answers there was a need for her to review the steps she had taken in the actions. Although she did arrive at the correct solution there is little evidence that she reflected on the action as a process. She repeated the performance rather than reflected on it.

Cassie also used a 'count on' procedure to add 7+8 and got the answer 15. She then used the procedure to add 6+9.

C: Mmmm… 15. I got 15 again.

Ex: Did you expect it to be the same?

C: No. I didn't even know it would be the same

Cassie checks her calculation for 7 + 8.

C: Both answers the same.

Again Cassie was relying on actions. The equivalent solutions to the two additions did prompt her to review the steps she had taken but, again, she repeated the performance rather than reflected on it.

Bill

Bill was less reliant on counting strategies for his solution for 10+8.

B: Well I just add 10 and 10 is 20 and I took two away.

Bill was able to draw on known facts and to carry out an adjustment to find the solution. He was able to describe or reflect on the process without performance. However in the account that he gave it was less evident that he was reasoning about the process, he does not explain why he carried this out.

Bill used a counting on strategy for 7+8. He made an error in counting and gave an incorrect response 14. However for the next addition 6+ 9 he responded:

B: 14 again.

Ex: How did you know it was 14 again?

B: 'Cos you lose one from that one and you add that one there.

Although Bill's strategy for 7+8 provided the wrong answer he used the solution to inform the next calculation. He did not have to repeat the action for 6 + 9 but was able to reflect on the action as a process. He was also able to reason why he had used the process ('Cos if you lose one from that one…) and it is possible that he was treating the ideas as objects.

Bill also showed evidence of reflecting on his actions as a process for the addition 8+13. Bill first gave the answer as 23, and then he went on to explain his strategy.

B: Well I just… It's not 23.'

Ex. How did you know it wasn't 23?

B: Cos it's 8, not 10. If it was 23 you would have to have a ten there' (points to the 8)

Ex. So what is it?

B: 21.

Here Bill described the action as a process without performance and was also reasoning the solution.

Jim

Jim used counting strategies for single digit additions. However for 10+8 he asked to try another way using the hundred square as he had been shown by his teacher:

> Jim: You could go back to 8 and then count on 10, then you do down one and you're answer would be 18.

Although not relying on counting strategies Jim's account suggests that he was carrying out an action. He did describe and reflect on the action as a process but there is little evidence that he was reasoning about the process.

His attempt to use this strategy was not always effective. For example 8 + 13:

> Jim: So I go all the way to 13 and then I jumped down one and add on 3. So it will be 26.
>
> Ex. Why were you adding on the 3?
>
> Jim: You go down 10 and you go across 3.
>
> Ex. If you started on 13 and then added ten and then another 3 you would be adding what?
>
> Jim: You would be adding tens and units. That's what we call it.

Here Jim did reflect on the action as a process but he was unable to reason why he had used the action and referred to this as a strategy that he had been taught.

Concluding Remarks

These short accounts have attempted to illustrate how the APOS framework might be used to analyse children's calculation strategies. It is recognised that further studies are needed to refine the use of this framework but even with these short accounts the results show possible insights into children's approach to calculations.

The children were often reliant on counting procedures and on occasions would make errors in their use. However they would often be able to reflect on the action as a process and to describe this without performing it. There was also evidence that children could reason about the processes using properties of the numbers. In these examples it is not the case that the children would consistently work in one area of APOS. Their ability to reflect or reason was often reliant on the problem presented to them. Further studies would be needed to determine if other children also moved within the APOS framework.

These insights also have possible implications for the teaching of calculations. If the aim is to support children's understanding and their use of flexible strategies it would seem that we should be supporting children in working towards the use of objects, in that they can reflect on their actions, and even reason about the processes they use. Cassie's focus on the actions caused her to be surprised regarding the equivalent solutions. In order to support her to reflect more on the process or to reason about the number properties, further work on equivalence using visual images or objects would seem appropriate.

Jim's reflection on his action as a process would initially suggest that he is developing a use of flexible strategies as he uses 'jumps' of ten rather than 'counting on' in ones. He is also able to reflect on and describe the action without performance. However his lack of reasoning would suggest that he does not see this process as an

object. If the eventual stage is for the actions, processes and objects to become part of a coherent framework or schema, it would seem important that Jim is supported in being able to reason the use of the jumps of ten so that there is a shift from process to object.

In order for children to become active learners the need to be able to reflect on and reason about the actions and processes they use would seem paramount. Without the opportunity to do this it would not seem possible for children to progress to the development of a coherent framework or schema that would support them in using calculations flexibly.

References

DCSF: 2007, *Keeping up - Pupils who fall behind in Key Stage 2, Making good progress series*, Nottingham: DCSF Publications.

Dubinsky, E. & McDonald, M.: 2001, APOS: A constructivist theory of learning. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 275–282). Dordrecht: Kluwer Academic Publishers.

Gray, E. And Tall, D.: 1994, Duality, Ambiguity and Flexiblity: A proceptual view of simple arithmetic, *Journal for Research in Mathematics Education*, 26(2), 115 – 141

Gray, E.: 1991, An Analysis of Diverging Approaches to Simple Arithmetic: Preference and its Consequences, *Educational Studies in Mathematics*, 22, 551-574

Ginsburg, H.: 1997, *Entering the Child's Mind*, Cambridge: Cambridge University Press

Murphy, C.: 2004, How do children come to use a taught mental calculation strategy? *Educational Studies in Mathematics*, 56(1), 3-18

Plunkett, S.: 1979, Decomposition and all that rot, *Mathematics in School*, 8(3), 2-5.

Sfard: A. 1991, On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin, *Educational Studies in Mathematics*, 22, 1-36

Shropshire Mathematics Centre: 1996, *Teaching Numeracy in a Shropshire Primary School*, Shropshire Education Services

Steffe, L: 1983, Children's algorithms as schemes, *Educational Studies in Mathematics*, 14, 109-125

Williams, P.: 2008, *Independent Review of Mathematics Teaching in Early Years Settings and Primary Schools*, Nottingham: DCFS Publications

Exploring the role of confidence, theory of intelligence and goal orientation in determining a student's persistence on mathematical tasks

Ann O'Shea, Joan Cleary and Sinead Breen

NUI Maynooth, IT Tralee and St Patrick's College, Dublin

We consider Dweck's (1986) theory on the relationships between students' beliefs concerning the nature of intelligence, their learning goal orientation, their confidence, and their willingness to seek challenges and to persist when faced with difficulties. Dweck's theories have been studied for the past 20 years, for example by Stipek and Gralinski (1996) among many others. In this study the beliefs and behaviour of 182 third level students were investigated. These students had all chosen to pursue an undergraduate course in a numerate subject. It was found that the relationships between theories of intelligence and goal orientations were more complicated than those postulated by Dweck, and in particular seem to differ between the male and female students. We also found that a student's theory of intelligence, goal orientation, and confidence in his mathematical ability influenced his persistence at difficult mathematical tasks. However, once again, differences were found between the male and female groups.

Keywords: Confidence, goal orientation, theory of intelligence, persistence

Introduction

Dweck (1986) conjectured that a student's theory of intelligence and confidence in his/her present ability combine to influence the student's behaviour when presented with a mathematical task, particularly a challenging or unfamiliar task. A preliminary study by the authors (Breen, Cleary and O'Shea 2007) failed to endorse this theory and found that an individual's level of confidence but not his theory of intelligence played an important role in how he approached, persevered with and performed on a task. A more comprehensive study, using an instrument specially constructed for this purpose (Breen, Cleary and O'Shea 2009), was undertaken and initial findings are presented here.

Figure 1 overleaf summarises and illustrates Dweck's theory. The theory asserts that children's theories of intelligence seem to orient them towards different goals, which then appear to set up different behaviour patterns. We will focus, like Dweck, on performance and learning goals. Students who display performance goals wish to receive positive feedback on their abilities from themselves or others, and to avoid demonstrating a lack of ability. Students with learning goals however, wish to increase their competence and acquire new understanding. Those who believe intelligence is a fixed trait tend to display performance goal behaviour, with their entire process of task choice and pursuit built around their concerns about their ability. If their perceptions of their own ability are low, they tend towards defensive strategies, avoiding and withdrawing from challenge. On the other hand, children who believe intelligence is malleable focus on progress and mastery through effort in both their choice and pursuit of tasks. Even children whose assessment of their present ability is low will choose challenging tasks that foster learning.

Theory of Intelligence	Goal Orientation	Confidence in present Ability		Behaviour Pattern
Entity Theory → (Intelligence is fixed)	Performance Goal	If high	→	Seeks challenge High persistence
		If low	→	Avoids challenge Low persistence
Incremental Theory → (Intelligence is malleable)	Learning Goal	If high	→	Seeks challenge High persistence
		If low	↗	

Figure 1: Achievement goals and behaviour patterns (following Dweck 1986)

Dweck and her colleagues have continued to study and refine her theories. For example, Grant and Dweck (2003) describe 5 studies concerned with the impact of achievement goals. They found evidence that learning goals have a positive influence on performance and motivation in challenging situations. Performance goals were also found to have positive effects when major difficulties were not encountered but they were found to have negative effects in the face of challenges. Mangels, Butterfield, Lamb, Good and Dweck (2006) studied the manner in which theory of intelligence beliefs influence how students respond to negative feedback and their ability to learn from mistakes. Students were asked general knowledge questions, they were told whether their answer was right or not and were immediately shown the correct answer. The study found that entity theorists were less likely to focus their attention on the correct answer and were less likely to recall it later. The authors postulate that their results may explain how theory of intelligence beliefs influence learning and achievement.

Middleton and Spanias, in their 1999 review of mathematics education research on motivation, contend that students who are intrinsically motivated engage in academic tasks because they enjoy them and their motivation tends to focus on learning goals, whereas students who are extrinsically motivated engage in academic tasks to obtain rewards and their motivation is centred on performance goals. They support Dweck's belief that those who are intrinsically motivated exhibit adaptive or pedagogically desirable behaviour such as selection of more difficult tasks, persistence in the face of failure and selection of deeper learning strategies. They also observed goal orientation to be a strong predictor of achievement, with students with an orientation toward learning (or mastery) goals tending to perform better than those with performance (or ego) goals regardless of the learning situation. Middleton and Spanias (1999) explain how, by the middle grades, many students begin to believe that success and failure are attributable to ability and that effort seldom results in a change in their success patterns. Those who continue to conceive of ability as amenable to augmentation through effort tend to expend more effort and thus are better achievers than those who believe ability is fixed. Findings show that a belief in effort as a mediator of ability and failure as an acceptable stage in the learning of mathematics also increases students' confidence in relation to the subject.

Stipek and Gralinski (1996) also studied the relationships between children's goal orientations, theories of intelligence, and achievement. They found than an entity theory of intelligence was associated with performance goals and that an incremental

theory of intelligence was associated with learning goals, although they note that the relationships are not strong.

Yet other studies have not fully supported Dweck's theory. Dupeyrat and Mariné (2005) found that theories of intelligence did not seem to influence goal orientation. But they did find that students with learning goals were more likely to put effort into their studies and that this had a positive impact on learning and achievement, whereas performance goals seemed to have a negative impact on effort and achievement. Carmichael and Taylor (2005) found, in a study of 129 students (with median age 29) enrolled in a tertiary preparatory mathematics course, that most subscribed to an incremental theory of intelligence, and concluded that the issue of theory of intelligence is probably not relevant in an adult education context.

Dweck (1986) also states that a number of previous studies have found that girls, and particularly bright girls, display greater tendency towards challenge avoidance and debilitation in the face of obstacles. Dweck and Leggett (1988) found that girls were more likely than boys to subscribe to an entity theory of intelligence. However, in a study of secondary school pupils in England, Ahmavaara and Houston (2007) did not find a link between theory of intelligence beliefs and gender. Middleton and Spanias (1999) found that boys tend to be more confident in learning mathematics than girls and while boys' confidence is robust to failure, girls' insecurities tend to be resilient in the face of success.

In this study, our aim is to search for evidence for Dweck's theory as outlined in Figure 1. We will consider the theory of intelligence beliefs, goal orientation and confidence of a group of third level students enrolled in mathematics modules. We will look for relationships between these variables and study their influence on persistence on difficult mathematical tasks. We will also investigate the role of gender in Dweck's model.

Survey instrument & administration

The study was conducted in the second semester of the 2007/2008 academic year and the participants were all in the first year of their respective programmes at one of three third level institutions: namely St Patrick's College, Drumcondra (BEd or BA (Humanities) programme), the National University of Ireland, Maynooth (BA or BA (Finance)) and the Institute of Technology, Tralee (Higher Certificate in Engineering or BSc). All students were enrolled in mathematics modules taught by the authors. The survey was administered during class and students were invited to participate in the study. 182 students agreed to participate and of these 73 (43.2%) were male.

The students completed a 20-minute questionnaire in which they were asked to respond to sets of rating scale items addressing Confidence, Theory of Intelligence, Goal Orientation (Learning or Performance) and Persistence. The items used a 5-point Likert scale (with 1 representing 'disagree strongly', 2 representing 'disagree', 3 'not sure', 4 'agree' and 5 'agree strongly') and were gathered from a number of sources but modified to render them relevant to third-level students in Ireland (for full details see Breen, et al 2009). The Learning Goal rating-scale items are displayed in figure 2 for illustrative purposes. Personal information (including gender, age, level of mathematics achievement at post-primary school) was also collected from the participants.

Learning Goals
1. I work at maths because I like finding new ways of doing things.
2. I work at maths because I like learning new things.
3. I work at maths because I like figuring things out.
4. I work at maths because I want to learn as much as possible.
5. I work at maths because it is important for me that I understand the ideas.

Figure 2: Rating-scale items for Learning Goal trait

When undertaking a study of attitudes and opinions, it is important to ascertain that the instrument of inquiry used provides valid, reliable and interpretable information that addresses the specific question of interest. The validity and reliability of the survey instrument used here were determined using Rasch analysis (Bond and Fox 2007) by means of the computer software Winsteps (Linacre 2009). Full details can be found in Breen et al (2009). Rasch analysis is a means of constructing an objective fundamental measurement scale from a set of observations of ordered categorical responses (to assessment items or rating-scale items). The scale produced is an interval one centred at 0. Following the assumption that useful measurement involves the consideration of a single trait or construct at a time, the Rasch model was applied to each set of rating-scale items separately. This gave rise to five measures, namely, Confidence, Theory of Intelligence (TOI), Learning Goal (LG), Performance Goal (PG) and Persistence measures. (The questions on the performance goal scale were reverse coded so that a high (or positive) score on this scale indicates a low level of performance goal orientation.) For statistically stable measures to be computed from the data, it is recommended that at least 10 observations per category of reasonably targeted items should be collected (Linacre 2009). Each item used here offered 5 categories of response (disagree strongly – agree strongly) and as the participants, despite their voluntary nature, were appropriate targets for the instrument (as evidenced by Item-Person maps produced by Winsteps (Breen et al 2009)), the size of the sample (182) was deemed sufficient.

Results

Participants were recategorised on the basis of being assigned positive or negative scores on each scale for an initial exploration of the data. Each trait was considered individually using these (crude) binary measures before the interactions were explored further using chi-squared tests (unless stated otherwise).

Theory of intelligence

Items here included "You have to be smart to do well in maths" and "You can succeed at anything if you put your mind to it". A positive score on the Theory of Intelligence (TOI) scale indicates that a student subscribes to an incremental view of intelligence, while a negative score means that the student has an entity view. More extreme scores in either case represent stronger views. Only 27.2% of the participants in this study attained a negative score on the scale (indicating an entity view) and their views on the nature of intelligence, using this binary measure, were independent of gender (p=0.862).

Confidence

Broadly speaking, positive scores on the Confidence scale are awarded to those who feel confident in relation to mathematics, while negative scores are awarded to those who do not. 66.3% of the respondents here achieved positive scores, and this classification of 'confident' or 'not confident' is independent of gender (p=0.871). Confidence items presented to the students included "I learn mathematics quickly" and "I have trouble understanding anything with mathematics in it".

Goal Orientation

Learning goal items were shown already in Figure 2. Performance Goal items included "I work at maths because it is important to me that the lecturer/tutor thinks I do a good job" and "I work at maths because it is important to me to do better than the other students".

The data here showed very few students reporting a tendency towards performance goals, with only 21.3% falling into this category (that is, exhibiting a negative score on the scale). On the other hand, 62.7% exhibited a tendency toward learning goals, attaining a positive score on the LG scale. Orientation towards learning goals is independent of gender (p=0.873 using the binary measure) with 61.6% of males and 63.5% of females displaying evidence of this orientation. However, there is some evidence that males are more inclined towards performance goals than females with 26% of males and only 17.7% of females displaying this tendency, though this is not statistically significant (p=0.255).

Dweck's theory postulates that a student's goal orientation follows from his theory of intelligence. Thus, it may seem reasonable to expect a strong inverse relationship between LG and PG. However, the classification of students as having positive or negative LG scores is independent of their classification as having positive or negative PG scores (p=0.244).

Persistence

Students were invited to respond to a number of statements relating to persistence including "when presented with a choice of mathematical tasks, my preference is for a challenging task" and "when presented with a mathematical task I cannot immediately complete, I increase my efforts". Only 44 students (26%) were awarded a negative score on the persistence scale. The allocation of a positive or negative Persistence score was independent of gender (p=1.000).

Relationships between confidence, theory of intelligence, goal orientation and persistence

Dweck (1986) asserts that a student's level of confidence, combined with his goal orientation, determines his behaviour pattern (either adaptive or maladaptive) and the level of persistence he will employ on mathematical tasks. The binary measure of Persistence created was found not to be independent of the binary measures of Confidence (p<0.001), TOI (p<0.001) or LG (p<0.001) but independent of PG (p=0.287) using chi-squared tests. Of 125 students with a positive score for persistence, 96 (76.8%) are confident, 102 (81.8%) subscribe to an incremental theory of intelligence, 94 (75.2%) hold learning goals, and 24 (19.2%) hold performance goals. This can be contrasted with the attitudes of the 44 students who are assigned a negative persistence score: 16 (36.4%) are confident, 21 (47.7%) subscribe to an

incremental theory of intelligence, 12 (27.3%) hold learning goals, and 12 (27.3%) hold performance goals.

In an effort to understand more clearly the interplay between these traits, further analysis was carried out using the original interval measures constructed using Rasch analysis. Pearson correlations between these measures were computed and are displayed in Table 1.

	TOI	LG	PG	Persistence
Confidence	0.352 (p<0.001)	0.587 (p<0.001)	0.222 (p=0.004)	0.628 (p<0.001)
TOI	1	0.383 (p<0.001)	0.274 (p<0.001)	0.452 (p<0.001)
LG		1	0.155 (p=0.044)	0.659 (p<0.001)
PG			1	0.249 (p=0.001)

Table 1: Correlations between the measures

Note that the correlation between LG and PG was computed to be 0.155 (significant at the 0.05 level). Moreover, the correlation between TOI and LG was found to be 0.383, while that between TOI and PG was computed as 0.274 (both significant at the 0.001 level). Also, the relatively strong correlations of 0.352 between Confidence and TOI and 0.587 between Confidence and LG indicate a more complex situation than that portrayed by Dweck (1986). As seen in Figure 1, Dweck suggests that orientation towards a particular type of goal follows from the theory of intelligence held by a subject, and that level of confidence contributes to the determination of behaviour patterns at a later stage.

In order to see how students' persistence on mathematical tasks depends on their confidence, theory of intelligence and goal orientation, a regression with the Persistence measure as the dependent variable and gender, Confidence, TOI, LG and PG measures as independent variables was performed. The regression analysis revealed that Confidence (p<0.001), TOI (p=0.005) and LG (p<0.001) were indeed significant predictors of Persistence. PG and gender were not. The model had an adjusted R^2 value of 0.543, which suggests that these three variables account for 54.3% of the variation in the persistence measure.

Gender Effects

Gender seems to play an important role in the relationships under investigation. For instance, consideration of male and female students separately shows the correlation between TOI and LG to be 0.197 for males (not significant at the 0.05 level) and 0.504 for females (significant at 0.01 level). Looking at this from another perspective, by means of chi-squared tests on the binary (positive/negative) measures of TOI and LG, shows them to be independent for males (p=0.696) but not for females (p=0.015). However, when the sample is split by gender, positive scores on PG are independent of positive scores on TOI for both males and females.

Using gender to subdivide the sample and performing a regression using the interval measures of Persistence, Confidence, TOI, LG and PG with Persistence as the dependent variable yields TOI (p=0.003) and Confidence (p<0.001) as significant predictors of Persistence for males, with an adjusted R^2 value of 0.505. The goal orientation variables were not significant predictors of persistence. However, for females, the significant predictors are LG (p<0.001), Confidence (p=0.008) and PG (p=0.015) but not TOI. The adjusted R^2 value of the latter model is 0.644.

Conclusions

The majority of the students surveyed here portrayed themselves as confident, as subscribing to an incremental theory of intelligence (in agreement with the findings of Carmichael and Taylor 2005), as aspiring to learning goals and as persisting on challenging or unfamiliar mathematical tasks, while only a minority reported a tendency towards performance goals. We did not see a strong inverse relationship between the goal orientation measures. Hannula (2006) comments that learning and performance goals should not be seen as mutually exclusive. The binary measures of confidence and persistence used failed to provide evidence to support Middleton and Spanias' (1999) comment that boys tend to be more confident than girls or Dweck's (1986) remarks that girls are less persistent than boys in relation to learning mathematics. We found no evidence that TOI beliefs are related to gender, mirroring Ahmavaara and Houston's (2007) results.

When considering the low levels of performance goals exhibited by our sample, as with any self-reporting measure, caution should be employed: it may be that students do not like to think of themselves in this way and so have not responded 'honestly'. Alternatively, it may be that third level students are mature enough not to have these kinds of goals. Corroborating evidence is needed. We do have corroborating evidence for one of our self-reporting measures however. This survey was administered at the same time as a PISA-type test of mathematical literacy. Thus we were able to consider students' persistence on unfamiliar mathematical tasks. Preliminary analysis showed that our persistence measure correlated well with the evidence provided by the PISA-type test.

Let's consider Dweck's assertion that TOI determines goal orientation. The correlations between TOI and LG and TOI and PG are both statistically significant but seem to be weaker than we might have expected following Dweck's theory. This finds resonance with Stipek and Gralinski's (1996) acknowledgement that the proposed relationships between TOI and goal orientation are not strong.

In our study, entity theories of intelligence were not associated with performance goals, echoing the findings of a study of adult learners by Dupeyrat and Mariné (2005). We did see that female students with incremental views of the TOI were likely to have learning goals but the same was not true for males. However Blackwell, Trzeniewski and Dweck (2007) found that teenagers who subscribed to an incremental theory of intelligence were more likely to exhibit learning goals and also employ more effort in studying.

The second part of Dweck's theory asserts that persistence is determined by confidence and theory of intelligence (through goal orientation). We found that the TOI, LG and confidence measures were significant predictors of the persistence measure for the group as a whole, providing support for the finding of Dupeyrat and Mariné (2005) in relation to the positive impact of learning goals on effort but not in relation to a negative impact of performance goals. Some differences were evident when gender was taken into account: for the female group TOI was not a significant predictor of persistence, while for the male group goal orientation was not a significant predictor of persistence. Grant and Dweck (2003) assert that it is only when faced with difficult situations that performance goals impact negatively on persisitence. In future work, we hope to investigate this phenomenon further by examining the behaviour of students on challenging PISA-type questions.

In summary, it seems that there is evidence for Dweck's theory in certain groups of students but not in others. In particular, it seems to be that gender has a role

to play in the relationship between TOI and LG and in how confidence, TOI and goal orientation influence persistence.

Information about students' previous mathematical achievements was also collected and further analysis will be carried out to determine if this affects the relationships studied here.

References

Ahmavaara, A. and Houston, D. 2007. The effects of selective schooling and self-concept on adolescents's academic aspiration: An examination of Dweck's self-theory. *British Journal of Educational Psychology* 77, 613-632.

Blackwell, L. S., Trzeniewski, K. H., and Dweck, C. S. 2007. Implicit theories of intelligence predict achievement across an adolescent transition: A longitudinal study and an intervention. *Child Development* 78:246-263.

Bond, T. G. & Fox, C. M. 2007. *Applying the Rasch model – Fundamental measurement in the human sciences* (Second edition). Lawrence Erlbaum Associates, New Jersey.

Breen, S., Cleary, J. and O'Shea, A. 2007. A study of third level students' beliefs about mathematics. In *Proceedings of the Second Conference on Research in Mathematics Education*, ed. S. Close, T. Dooley and D. Corcoran, 202-215, Dublin: St Patrick's College.

——— 2009. Constructing and validating an instrument to measure students' attitudes and beliefs about learning mathematics. In *Proceedings of the Third Conference on Research in Mathematics Education*, ed. D. Corcoran, T. Dooley, S. Close and R. Ward, 333-343, Dublin: St Patrick's College.

Carmichael, C. and Taylor, J.A. 2005. Analysis of student beliefs in a tertiary preparatory mathematics course. *International Journal of Mathematical Education in Science and Technology* 36(7): 713-719.

Dupeyrat, C. and Mariné, C. 2005. Implicit theories of intelligence, goal orientation, cognitive engagement, and achievement: A test of Dweck's model with returning to school adults. *Contemporary Educational Psychology* 30: 43-59.

Dweck, C. **1986**. Motivational processes affecting learning. *American Psychologist* 41: 1040-1048.

Dweck, C. and Leggett, E. 1988. A social-cognitive approach to motivation and personality. *Psychologial Review*, 95, 256-273.

Grant, H. and Dweck, C. 2003. Clarifying achievement goals and their impact. *Journal of Personality and Social Psychology*, 85, 541-553.

Hannula, M.S., 2006. Motivation in mathematics: Goals reflected in emotions. *Educational Studies in Mathematics*, 63, 165-178.

Linacre, J. M. 2009 *Winsteps (Version 3.68)* [Computer Software]. Chicago: Winsteps.com.

Mangels, J., Butterfield, B., Lamb, J, Good, C. and Dweck, C. 2006. Why do beliefs about intelligence influence learning success? A social cognitive neuroscience model. *SCAN*, 1, 75-86.

Middleton, J. A. and Spanias, P.A. 1999. Motivation for achievement in mathematics: Findings, generalisations and criticisms of the research. *Journal for Research in Mathematics Education* 30(1): 65-88.

Stipek, D. and Gralinski, J. 1996. Children's beliefs about intelligence and school performance. *Journal of Educational Psychology* 88(3): 397-407.

Measuring Mathematics Self Efficacy of students at the beginning of their Higher Education Studies

Maria Pampaka and Julian Williams[1]

The University of Manchester

We report on the construction and validation of a self-report 'Mathematics self-efficacy (MSE)' instrument designed to measure this construct as a learning outcome of students entering their Higher Education (HE) studies. The sample of 1630 students ranged across different programmes with different levels of mathematical demand. The validation of the measure was performed using the Rating Scale Rasch model. Results include measures and fit statistics illustrating the construct validity, and a comparative analysis of sub-groups in the sample (i.e. gender and courses) ensuring validity across different groups. The comparison between the courses indicated the possibility of a two dimensional structure of the construct, which is explored here by performing separated analyses. The paper concludes with methodological implications and substantial considerations regarding the use of this instrument.

Keywords: Mathematics self efficacy, Higher Education, Rasch Analysis

Background

This paper is concerned with the widely known 'mathematics problem' (Smith 2004), which sees very few students to be well prepared to continue their studies from schools and colleges into mathematically demanding courses in Higher Education (HE). These courses include Mathematics, Science, Technology and Engineering (hereafter STEM). We particularly report here on the preliminary results of our ESRC funded research project "Mathematics learning, identity and educational practice: the transition into Higher Education" regarding the developed measures of mathematics self-efficacy.

The self-efficacy construct was initially described and contextualised by Bandura who distinguished two cognitive dimensions in this construct, i.e. personal self-efficacy and outcome expectancy. Self-efficacy (SE) beliefs *"involve peoples' capabilities to organise and execute courses of action required to produce given attainments"* and perceived self-efficacy *"is a judgment of one's ability to organise and execute given types of performances…"* (Bandura 1997, p. 3). Perceived self-efficacy beliefs have been explored in a wide range of disciplines and settings including educational research where they have been investigated in relation to progression to further study and career choices and in relation to affective and motivational domains and their influence on students' performance and achievement. Most important and relevant to our study are research findings that suggest that

perceived self-efficacy in mathematics is more predictive of students' choices of mathematically related courses in programmes of further study than prior attainment or outcome expectations (e.g. Hackett and Betz 1989; Pajares and Miller 1994), hence the importance of the construct for the STEM agenda.

In previous work we have shown how this measure was developed and validated for post-compulsory, pre-university students. In particular we presented how we measured mathematics self-efficacy with an overall measure and two additional measures for two distinct aspects of mathematical task: the pure and more applied. Measurement results led to a hypothesis that there may need to be these two sub-dimensions in the construct of MSE at this level (Pampaka et al. under review; Pampaka et al. 2007). Our aim in this paper is to report on the extension of this work for use of a revised version of this instrument with HE students. Hence, some description of the revisions made to our earlier instruments is presented next.

Instrumentation and Analysis

The development of the instrument

Taking an individual's self-efficacy to be their belief in their capability to successfully complete an identified range of actions in a given field, during our earlier work (see above) we devised an instrument that measures students' self-efficacy in the use (or application) of AS level mathematics. 24 items (mathematical tasks) were constructed based on seven mathematical competences (Williams et al. 1999) including costing a project and handling experimental data graphically. These were complemented with six purely symbolic mathematical items (e.g. solving an equation in x). From this initial instrument (with 30 items) which was used with students at AS level (pre-university), the seven most difficult items (i.e. corresponding to post 'AS study level') were employed for the current version of the instrument for use with students beginning their HE studies. Three more advanced items were also added, to make the total 10 tasks that constitute the instrument measuring students' confidence in the following mathematical areas: (1) calculating/estimating, (2) using ratio and proportion, (3) manipulating algebraic expressions, (4) proofs/proving, (5) problem solving, (6) modelling real situations, (7) using basic calculus (differentiation/integration), (8) using complex calculus (differential equations / multiple integrals), (9) using statistics, and (10) using complex numbers. Items were chosen so as to be relevant not only to students studying for Mathematics programmes but for a wider range of subjects, hence the 'use of maths' elements.

As usual in SE studies (e.g. Zimmerman and Martinez-Pons 1990; Hackett and Betz 1989), items were presented in the form of a 4-point Likert type scale where students were asked to choose the level of their confidence in solving them (but it was stressed they were not to solve the problems).

Analytical Considerations

Validation refers to the accumulation of evidence to support validity arguments. Our psychometric analysis for this purpose will be conducted within the Rasch measurement framework and therefore we follow the guidelines summarised by Wolfe and Smith Jr, (2007) based on Messick's (1989) validity 'definitions'. The Rasch rating scale is the most appropriate for scaling problems with Likert type items like ours. Analysis was performed with the FACETS software (Linacre 2003; Bond and Fox 2001) and the following statistics will guide our exploration for this paper

under our validation framework: (a) **Item fit statistics** to check fulfilment of the unidimensionality assumption and ensure construct validity, (b) **Category Statistics** to justify communication validity (Lopez 1996), (c) **Person – item maps** and the item difficulty hierarchy to provide evidence for substantive, content and external validity, and (d) **Differential Item Functioning (DIF)** to suggest gender and course group differentiation of the constructed measures.

Results

Sample description

The preliminary results presented in this paper come from the analysis of the first data point out of three in our project. This happened before, and at the beginning of academic year 2008-2009, just when students were in their induction phase to university. The sample includes 1630 students, mainly coming from five UK HE institutions, split by gender and course as shown in Table 1:

Table 1: Sample Description (by gender and course)

	Gender		
Course Classification	Female	Male	Total (%)
Mathematically Demanding	311	731	1042 (64%)
Non Mathematically Demanding	340	248	588 (36%)
Total (%)	651(40%)	979(60%)	**1630**

It should be noted that under the "Mathematically demanding' course classification there were students from Mathematic courses (including combined degrees), Electrical and Electronic, and Mechanical Engineering courses, Physics and Chemistry. The Non Mathematically demanding courses include Medicine and some educationally-related degrees.

Construct Validity: Checking for Unidensionality

In the Rasch context fit statistics indicate how accurately the data fit the model. Fit statistics are local indicators of the degree to which the data is cooperating with the model's requirements. Inconsistent data (e.g. misfit items or persons, i.e. with infit and outfit meansquare departing from the ideal of 1) may become a source of further inquiry. Fit statistics may also flag items to which responses are overly predictable (overfits), an indication that, in some way, they are over-dependent on the other items and might be the first choices for deletion (Wright 1994). For the purposes of this paper we take any number above 1.2 (of infit MnSq) as possible cause of concern, whereas infit values below 1 are considered as overfits and are not discussed. The results for our MSE measure are shown in Table 2 and indicate acceptable fit of almost all the items suggesting that they could constitute a scale, i.e. they measure what we call 'students Mathematics Self efficacy at University' (MSE@Uni). The only exception to this is the 'statistics' item which presents an Infit meansquare of 1.3 (highlighted) which indicates a possible mis-behaviour of this area of mathematics under our constructed measure. Further exploration of this aspect of mathematics can be justified with interview data from students; however this goes beyond the scope of this paper. We will seek however, more psychometric justification and explanation of this misfit through further analysis (i.e. DIF).

Table 2: Measures and fit statistics for the items of the scale

```
-----------------------------------------------------------------------------
| Obsvd  Obsvd  Obsvd  Fair-M|       Model | Infit     Outfit    |       |           |
| Score  Count Average Avrage|Measure  S.E.|MnSq ZStd  MnSq ZStd | PtBis | Nu Items  |
-----------------------------------------------------------------------------
|  3182   1328    2.4   2.41|  1.28   .04 | 1.0   0   1.0   0   |  .53  |  1 modeling          |
|  4225   1331    3.2   3.25|  -.56   .05 | 0.9  -3   0.9  -3   |  .56  |  2 calculating       |
|  4055   1323    3.1   3.13|  -.26   .04 | 1.0  -1   1.0  -1   |  .54  |  3 ratio_proportion  |
|  4568   1326    3.4   3.54| -1.39   .05 | 1.0   0   0.9  -2   |  .69  |  4 algebra           |
|  3770   1320    2.9   2.91|   .25   .04 | 1.0   0   1.0   0   |  .65  |  5 proof             |
|  3691   1318    2.8   2.85|   .38   .04 | 0.9  -2   0.9  -2   |  .55  |  6 problem_solving   |
|  4239   1309    3.2   3.31|  -.74   .05 | 1.0   1   0.9  -1   |  .71  |  7 Basic_calculus    |
|  3633   1313    2.8   2.82|   .46   .04 | 0.9  -2   0.9  -2   |  .65  |  8 complex_calc      |
|  3683   1309    2.8   2.86|   .36   .04 | 1.3   7   1.4   8   |  .39  |  9 statistics        |
|  3764   1308    2.9   2.93|   .20   .04 | 1.2   4   1.1   3   |  .63  | 10 complex_num       |
-----------------------------------------------------------------------------
|  3881.0 1318.5  2.9   3.00|   .00   .04 | 1.0   0.3 1.0  -0.1 |  .59  | Mean (Count: 10)     |
|   374.2    8.0  0.3   0.30|   .71   .00 | 0.1   3.2 0.1   3.3 |  .09  | S.D.                 |
-----------------------------------------------------------------------------
RMSE (Model)  .04  Adj S.D.  .71  Separation 16.15  Reliability 1.00
Fixed (all same) chi-square: 2476.3  d.f.: 9  significance: .00
Random (normal) chi-square: 9.0  d.f.: 8  significance: .34
-----------------------------------------------------------------------------
```

Rating scales and their response formats serve as tools with which the researcher communicates with the respondents. Lopez (1996) defines as 'communication validity' the extent to which the rating scale's categories perform as intended. Thus, category statistics are also examined for the appropriateness of the Likert scale used and its interpretation by the respondents, with the aid of Rasch analysis which provides the means for these checks (see Table 3).

Table 3: Category statistics for MSE@Uni Measure

```
-------------------------------------------------------------------------------------------
|     DATA          | QUALITY CONTROL  |   STEP         | EXPECTATION   | MOST    |THURSTONE| Cat|
| Category Counts  Cum.| Avge  Exp.  OUTFIT|CALIBRATIONS | Measure at    |PROBABLE |THRESHOLD|PEAK|
|Score  Used   %    % | Meas  Meas   MnSq |Measure  S.E.|Category  -0.5 | from    |   at    |Prob|
-------------------------------------------------------------------------------------------
|  1     954   7%   7%|-1.31 -1.41   1.1  |             |( -2.94)       | low     |   low   |100%|
|  2    2634  20%  27%| -.04  -.08   1.0  | -1.71   .04 | -1.06  -2.11  | -1.71   |  -1.90  | 50%|
|  3    5800  44%  71%|  .92  1.01    .9  |  -.32   .02 |   .92   -.13  |  -.32   |   -.22  | 61%|
|  4    3797  29% 100%| 2.36  2.28    .9  |  2.04   .02 |( 3.18)  2.25  |  2.04   |   2.11  |100%|
-------------------------------------------------------------------------------------------
                                                        (Mean)---------(Modal)--(Median)------
```

The most often used indices for this check are the average measure and the threshold (or step calibration). The average measure is approximately the average ability of the respondents observed in a particular category, averaged across all occurrences of the students in the category, whereas the threshold is the location parameter of the boundary on the continuum between category k and category k-1 of a scale (Linacre, 2002). A well functioning scale should present ordered average measures, and ordered step calibrations, with acceptable fit statistics, as happens with our case.

So far results indicate a rather healthy measure of MSE@Uni. But what about its validity across different groups of students?

What does DIF analysis indicate? Validity across different groups

When a variable is used with different groups of persons, it is essential that the identity of the variable be maintained from group to group. Only if the item calibrations are invariant from group to group can meaningful comparisons of person measures be made (Wright and Masters 1982). The groups we are interested to check here are male and female students and more importantly students in various HE courses classified according to their mathematical demand. A statistical way to inform this process is to check for Differential Item Functioning (DIF). DIF describes a serious threat to the validity of items and tests used to measure an aptitude, ability or proficiency of members of different groups. DIF measurement may be used to reduce this source of test invalidity and allows researchers to concentrate on the other explanations for group differences in test scores (Thissen, Steinberg, and Wainer 1993).

There are different methods to check for DIF. In our case a t-test on the two estimates of difficulty parameters based on the two groups of students was performed (see Figure 1, with the lines indicating the 95% confidence intervals in item estimates). The points that are outside the confidence intervals in Figure 1 denote the items with high DIF when comparing students of mathematically demanding subjects with the rest of the students[1].
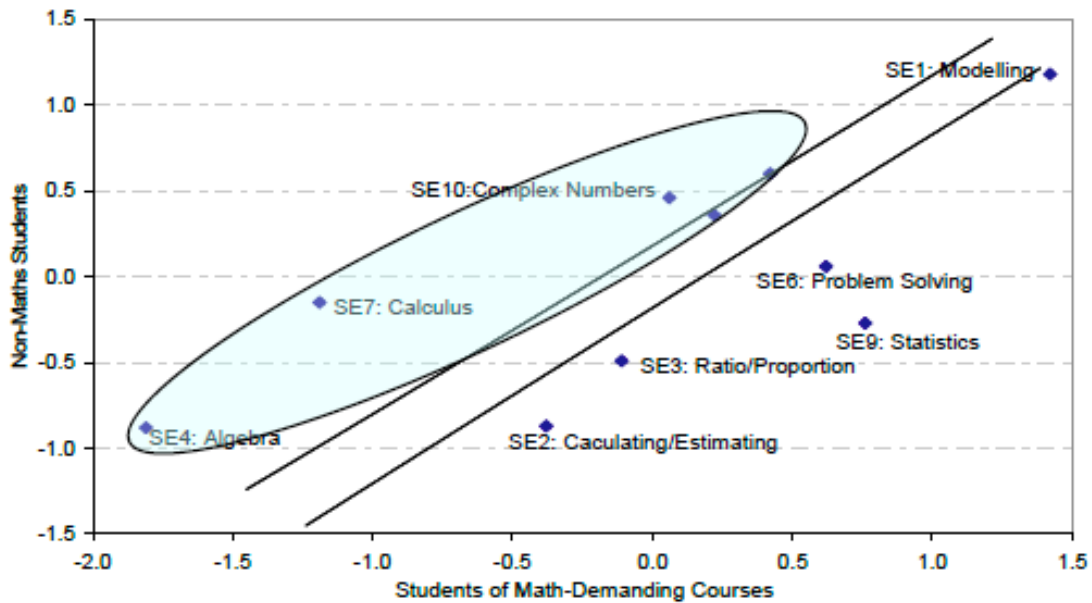
Figure 1: Comparison of item estimates for the two student groups

In this case, it appears that only two items are within the acceptable confidence intervals (these are advanced calculus and proof). Figure 1 indicates a 'bad' measure in psychometric terms, with the items on the top favoring the students of mathematically demanding subjects and those on the bottom the rest of the students. However, this picture should be given a closer and more careful insight, in respect to the underlying construct it measures (i.e. students' self efficacy) and its implications for mathematics education.

The 10 items of MSE@Uni seem to be clustering into two groups based on DIF results. The circled items denote areas of mathematics which form part of the AS/A2 mathematics syllabus; hence students who (successfully) completed this course will be at an advantage, in regards to their self efficacy. In contrast, the other items denote more applied mathematical areas. Hence the results point to the possibility of a second underlying dimension in the construct of MSE@Uni.

Two possible dimensions: Investigation of Subscales of MSE@Uni

In order to further explore the possibility of a two dimensional structure of the MSE@Uni measure, the two groups of items defined above based on the results of DIF analysis are analysed separately to check whether they could define two sub-measures. The results of this analysis are shown in Table 4 and indicate two measures

---

[1] Similar analysis for gender indicated smaller differences: the Statistics item is favoring female students, whereas modeling and ratio/proportion items are favoring male students. Due to limitations of the length of this paper, however, focus will be placed on the course classification.

of MSE with acceptable fit statistics and good reliability and separation indices. Category statistics are also acceptable but omitted from this presentation.

Table 4: Measures and fit statistics for the items of the two subscales of MSE@Uni construct (AS-related topics on the top, and Applied MSE at the bottom)

```
-------------------------------------------------------------------------------
| Obsvd   Obsvd  Obsvd  Fair-M|          Model | Infit      Outfit     |      |          |
| Score   Count Average Avrage|Measure   S.E.  |MnSq ZStd  MnSq ZStd   | PtBis| N Items  |
-------------------------------------------------------------------------------
| 3206    1192    2.7    2.71|   .93     .05  | 1.0   0   1.1    1    | .65  | 4 complex_calc    |
| 3339    1198    2.8    2.82|   .65     .05  | 1.0   1   1.0    1    | .67  | 2 proof           |
| 3337    1187    2.8    2.84|   .58     .05  | 1.2   4   1.2    3    | .65  | 5 complex_num     |
| 3812    1188    3.2    3.29|  -.65     .05  | 0.9  -3   0.8   -4    | .76  | 3 Basic_calculus  |
| 4127    1201    3.4    3.56| -1.51     .06  | 0.9  -1   0.8   -2    | .72  | 1 algebra         |
-------------------------------------------------------------------------------
| 3564.2  1193.2  3.0    3.05|   .00     .05  | 1.0   0.2  1.0  -0.1  | .69  | Mean (Count: 5)   |
|  348.9     5.5  0.3    0.33|   .93     .00  | 0.1   2.8  0.1   3.1  | .04  | S.D.              |
-------------------------------------------------------------------------------
RMSE (Model)  .05  Adj S.D.  .93  Separation 17.84  Reliability 1.00
Fixed (all same) chi-square: 1457.7  d.f.: 4  significance: .00
Random (normal) chi-square: 4.0  d.f.: 3  significance: .26
-------------------------------------------------------------------------------
| Obsvd   Obsvd  Obsvd  Fair-M|          Model | Infit      Outfit     |      |          |
| Score   Count Average Avrage|Measure   S.E.  |MnSq ZStd  MnSq ZStd   | PtBis| N Items  |
-------------------------------------------------------------------------------
| 3137    1313    2.4    2.44|  1.35     .05  | 1.1   2   1.1    2    | .46  | 1 model           |
| 4180    1316    3.2    3.20| -1.05     .05  | 0.8  -3   0.8   -4    | .57  | 2 calculating     |
| 4011    1309    3.1    3.09|  -.65     .05  | 1.0  -1   0.9   -1    | .55  | 3 ratio_proportion|
| 3646    1303    2.8    2.85|   .19     .05  | 0.9  -2   0.9   -2    | .54  | 4 problem_solving |
| 3638    1294    2.8    2.86|   .16     .05  | 1.2   4   1.2    4    | .44  | 5 statistics      |
-------------------------------------------------------------------------------
| 3722.4  1307.0  2.8    2.89|   .00     .05  | 1.0  -0.1  1.0  -0.2  | .51  | Mean (Count: 5)   |
|  360.1     7.8  0.3    0.26|   .83     .00  | 0.1   3.2  0.1   3.2  | .05  | S.D.              |
-------------------------------------------------------------------------------
RMSE (Model)  .05  Adj S.D.  .83  Separation 16.82  Reliability 1.00
Fixed (all same) chi-square: 1466.8  d.f.: 4  significance: .00
Random (normal) chi-square: 4.0  d.f.: 3  significance: .26
```

The results of DIF analysis of these two new measures are also shown in Figure 2. It is obvious that the differences between the two groups of students are now smaller; it should be noted that none of the DIF values is bigger than 0.5 logits, and hence they could be considered as a smaller problem (Linacre 1994) which may be ignored in some analyses.

2a. AS-related MSE@Uni   2b. Applied MSE@Uni
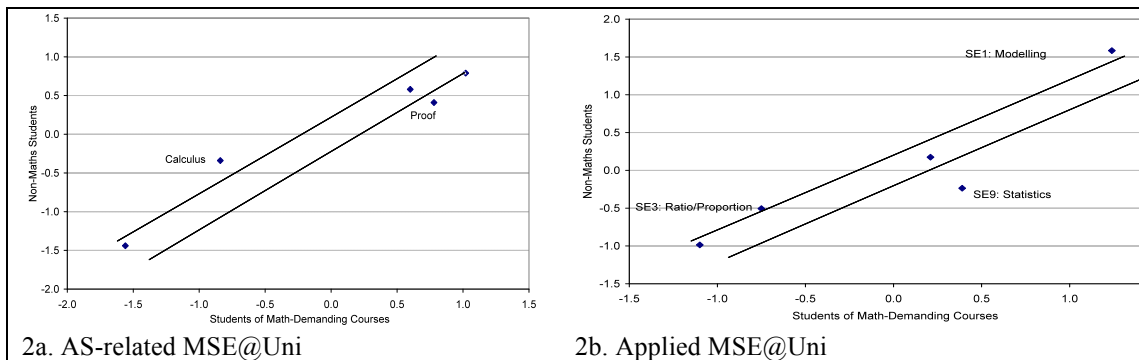
Figure 2: DIF Analysis for the two subscales of MSE@Uni

Figure 3 finally shows the three resulting measurement scales (MSE@Uni and the two subscales, namely MSE@Uni-AS_areas and MSE@Uni-Applied_areas). The right side of each scale shows the distribution of students (the higher the student's position the more self-efficacious they are); Numbers under the 'item' column indicate the item and its location on the same logit scale (For MSE@Uni these numbers are defined in Table 2). More difficult items are located higher on the scale. Observation of the scales, based on the spread of items and students can justify the split into the two subscales: It appears that items for the MSE@Uni are centred in a small area and do not cover the whole ability range of students, hence they do not discriminate enough for this group of students.

Figure 3: The Mathematics Self Efficacy Scales

Discussion - Conclusions

This paper presented some preliminary psychometric results for a constructed measure to capture students' mathematical self efficacy, from our research on students' transition to University. In sum we presented how a seemingly unidimensional measure of MSE was broken down into two sub-measures which may be more appropriate and productive for research in mathematics education. Two points should be emphasised here:
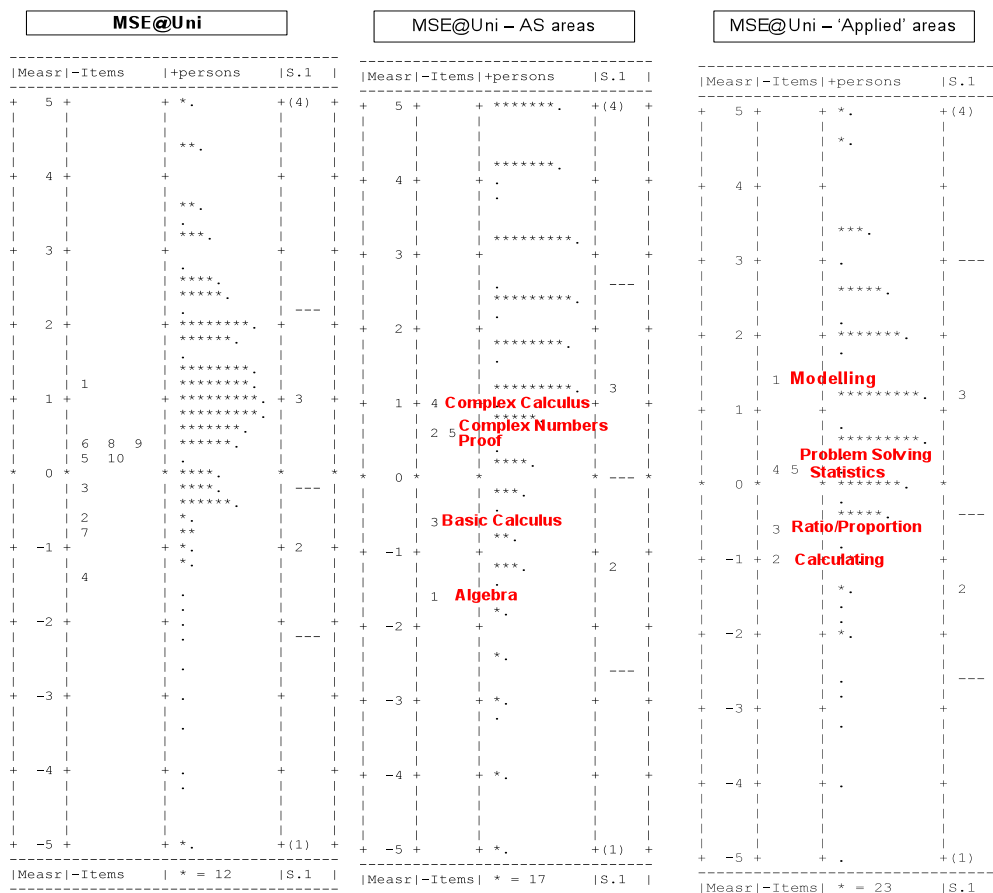
The first one is methodological and adds to current discussion about validation of measures. Our results indicate that even when a measure initially seems robust in regards to fit statistics and overall measures of reliability, care should be taken to consider how it can be used with different sub groups of the population. In our case DIF analysis flagged a possible extra dimension in our measure. This possibility has to be examined further by employing multidimensional models (Briggs and Wilson 2003).

The second remark is more substantive and regards the use of such measures in further statistical modeling. Given our psychometric results so far, it may be the case that some times two measures are more useful than one, to capture the desired relationships and consequently better inform research in mathematics education.

References

Bandura, A. 1997. Self-efficacy: the exercise of control. New York: Freeman.
Bond, T, and C. M. Fox. 2001. *Applying the Rasch Model: Fundamental Measurement in the Human Sciences*. Mahwah, NJ: Lawrence Erlbaum.

Briggs, D. C. , and M. Wilson. 2003. An Introduction to Multidimensional Measurement using Rasch Models. *Journal of Applied Measurement* 4 (1):87-100.

Hackett, G. , and N.E. Betz. 1989. An exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for Research in Mathematics Education* 20 (3):261-273.

Linacre, J. M. 1994. Sample Size and Item Calibration Stability. *Rasch Measurement Transactions* 7 (4):328.

———. 2002. Optimizing Rating Scale Category Effectiveness *Journal of Applied Measurement* 3 (1):85-106.

———. 2003. *A user's guide to FACETS: Rasch-Model Computer programs [software manual]*. Chicago: Winsteps.com.

Lopez, W. A. 1996. Communication Validity and Rating Scales. *Rasch Measurement Transactions* 10 (1):482-483.

Messick, S. 1989. Validity. In *Educational Measurement* edited by R. L. Linn. USA: American Council of Education and the Oryx Press.

Pajares, F., and M. D. Miller. 1994. Role of self-efficacy and self-concept beliefs in mathematical problem solving: A path analysis. *Journal of Educational Psychology* 86 (2):193-203.

Pampaka, M., L. Black, P. David, P. Hernandez-Martinez, G. Hutcheson, S. Nicholson, G. Wake, and J.S. Williams. 2007. Multi-dimensional structure of a (use of) Mathematics self efficacy instrument. In *American Educational Research Association annual conference (AERA)*. Chigago.

Pampaka, M., I. Kleanthous, J. Williams, Graeme Hutchenson, and G. Wake. under review. Measuring mathematics self efficacy as a learning outcome. *Research in Mathematics Education*.

Smith, A. 2004. Making mathematics count – the report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education. London: DfES.

Thissen, D., L. Steinberg, and H. Wainer. 1993. Detection of Differential Item Functioning Using the Parameters of Item Response Models. In *Differential Item Functioning*, edited by P. W. Holland and H. Wainer. London Lawrence Erlbaum Associates, Publishers.

Williams, J.S., G. Wake, and A. Jervis. 1999. General Mathematical Competence: A New Way of Describing and Assessing a Mathematics Curriculum. In *Rethinking the Mathematics Curriculum*, edited by C. Hoyles, C. Morgan and G. Woodhouse. London: Falmer Press.

Wolfe, E. W., and E. V. Smith Jr. 2007. Instrument Development Tools and Activities for Measure Validation Using Rasch Models: Part II - Validation Activities. *Journal of Applied Measurement* 8 (2):204-234.

Wright, B. D. 1994. Data Analysis and Fit. *Rasch Measurement Transactions* 7 (4):324.

Wright, B. D., and G. N. Masters. 1982. *Rating Scale Analysis*. Chicago: MESA Press.

Zimmerman, B. J., and M. Martinez-Pons. 1990. Student Differences in Self-Regulated learning: Relating Grade, Sex, and Giftedness to Self-Efficacy and Strategy Use. . *Journal of Educational Psychology* 82 (1):51-59.

Comparison at Two Levels of the Content Treatment of 'Early Algebra' in the Intended Curricula in South Africa and England

Nicky Roberts

University of Cambridge

This paper compares the treatment of algebra content in the intended curriculum of the early grades[1] in two countries: England and South Africa. Two levels of analysis are conducted. The first examines the content structure of each curriculum; and the second compares and contrasts the detailed 'learning objectives' in England to the 'assessment standards' in South Africa. The comparison reveals that a curriculum may include algebra by name, but may not deal with it in much substance (as in South Africa) and, a curriculum may include algebra in some substance, while not mentioning it by name (as in England).

Keywords: Early Algebra, Curricular Comparison, South Africa, England

Introduction

In the last few decades, there has been substantial mathematics educational research into precisely what is meant by 'early algebra', and a growing sophistication of its interpretation and use (see, for example: Davis 1985, Bodanskii 1991, Sasman et al 1998, Kaput et al. 1999, Bills et al. 2003, Amerom 2003, and Warren 2004). This has emerged from the identification of a 'cognitive gap' in the transition from arithmetic to algebra (Herscovics et al. 1994), which has resulted in research focusing on how the teaching of arithmetic at primary school level may better inculcate algebraic thinking. There seems, however, to be less published work on how this research has been interpreted and is manifest in particular curriculum frameworks. It therefore seemed opportune to compare particular intended curricula and contrast their treatment of 'early algebra', in order to garner how the research on this topic is being communicated to mathematics teachers in different national contexts. This paper limits its scope to only two countries: England and South Africa.[2] It is hoped that it provides sufficient detail to be of particular interest to mathematics educators from these countries, and simultaneously provides a basis for further comparison for the wider international mathematics education community interested in 'early algebra' or 'curriculum development'.

This paper begins by presenting Kaput's definition of 'early algebra', and explains how this is adapted for the purpose of this paper (2007). This is followed by an analysis of the curriculum policy documents in the two countries: the *Revised National Curriculum Statement for Mathematics (RNCS)* (Department of Education, DOE 2002) in South Africa, and the *Primary Framework for Mathematics* (Department for Children, Schools and Families, DCSF 2009d) in England. These

---

[1] For the purpose of this paper, 'early grades' is defined to be the first four years of compulsory primary education. In England this refers to Reception and Years 1 to 3, which are termed 'Key Stage One'. In South Africa this refers to Grade R and Grades 1 to 3, which are termed 'Foundation Phase'.

[2] These countries were selected as they are the national mathematics education environments with which the author is most familiar having conducted research and taught mathematics in both countries.

intended curricula are analysed at two levels: the 'macro level' which examines the content structure of each curriculum; and the 'micro level' which compares and contrasts the detailed 'learning objectives' in England to the 'assessment standards' in South Africa. A third 'meta level' of the published research on early algebra emerging from each country is also of interest, as this informs the other two levels. However this paper is limited to the micro and macro levels, as these are the standard national documents used to communicate the intended curricula to the majority of teachers in the early grades.

Analysing Early Algebra in Intended Primary Mathematics Curricula

Kaput provides a detailed framework for considering the various elements of algebra by referring to two core aspects and three strands, as represented in Figure 1.

Figure 1: [1] Kaput's definition of 'early algebra'

	Core aspect A: Using symbols to generalise	Core aspect B: Acting on symbols, following rules
Stand 1: Generalising from arithmetic and quantitative reasoning	**Generalising arithmetic:** properties of zero, commutativity, inverse relationships **Generalising about particular number properties and relationships:** equivalence of the equal sign	**Initial symbolisation that avoids numerals and uses symbols to then compare quantities:** Comparison of lengths, or heights or volumne Leads later to abstract algebra
		Leads later to calculus and analysis
Stand 2: Generalising towards the idea of a function	**Towards functions:** Recognising regularity through elementary patterning, ideas of change linearity, change, representation through tables, graphs and 'function machines'	
		Multiple representation types
Stand 3: Modeling as language	**Number-specific modeling:** Arithmetic problem where variable is unknown, Situation is modeled, more variable express a class of functions **Generalisation modeling**: variable as parameters to explore effects in pure arithmetic word problems, algebrafying, multiple representations	

The two core aspects are evident in Kaput's explanation that algebraic reasoning is taken to be "symbolisation activities that serve purposive generalisation" (or using symbols to generalise), and simultaneously to be "reasoning with symbolised generalisations" (or acting on symbols, following rules) (Kaput et al. 2007). Kaput explains that it is generally assumed in mathematics curricula that *Core Aspect A* usually precedes *Corey Aspect B*. The first two strands in this categorisation of early algebra consider the two types of generalising that are at the heart of algebraic thinking: generalising arithmetic; and generalising towards the idea of function. The third strand refers to modelling processes where situations are understood and interpreted using algebraic reasoning and language. Kaput explains that there are overlaps between the strands and provides useful illustrative examples relevant to the early grades for each strand, which are summarised in Figure 1.

For the purpose of this paper, 'early algebra' is taken to mean generalising in the early grades of primary school, which is as expressed in Strand 1 and Strand 2 of

---

[1] Source: Created be author, drawing on explanations provided by Kaput 2007.

Core Aspect A in Kaput's definition.[1] This generalising has two main manifestations: generalising from arithmetic and quantitative reasoning; and generalising towards the idea of a function. As such, one would expect that a primary mathematics curriculum which addresses 'early algebra' should include at least three elements. The first element is generalising arithmetic as the exploration of properties of numbers and operations. The second element is generalising about particular number properties and relationships. This is distinct from the first element, as it relates to properties and relationships for particular numbers and not to properties of numbers and operations in general. It is important to realise that arithmetic approaches which encourage 'partitioning' or 'breaking down' and 'building up' numbers draw on these properties of particular numbers and operations. The third element is generalising towards the idea of a function which includes recognising regularity in elementary patterns, ideas of change including linearity, and representation through tables, graphs and 'function machines'.

Comparison of Early Algebra in the Intended Curricula of England and South Africa

Comparison at the Macro Level: Content Structure of Each Curriculum

In both South Africa and England the mathematics content is expressed in the content focus areas and related assessment criteria in the mathematics curricula. The South African mathematics curriculum statement is structured using the following five learning outcomes: Numbers, Operations and Relationships (LO1); Patterns, Functions and Algebra (LO2); Space and Shape (Geometry) (LO3); Measurement (LO4); and Data Handling (LO5). For England the mathematics learning objectives are organised in relation to seven 'strands of learning' which give a broad overview of mathematics in the primary curriculum. The seven strands of learning in England are Using and Applying Mathematics (Ma1), Counting and Understanding Number (Ma2), Knowing and Using Number Facts (Ma3), Calculating (Ma4), Understanding Shape (Ma5), Measuring (Ma6), and Handling Data (Ma7).[2] Like South Africa, each learning strand in England is used to provide the structure for specifying the learning objectives for each year in the primary school.

In South Africa, learning outcomes one and two are most relevant to early algebra. Algebra could find expression in a variety of contexts including learning outcomes three, four and five, "however the core of the development of learner's knowledge and understanding of algebra will, by the nature of algebra, take place with the first two learning outcomes" (Vermeulen 2007, 21).

For 'numbers, operations and relationships' (LO1) learners are expected to "recognise, describe and represent numbers and their relationships, and to count, estimate, calculate and check with competence and confidence in solving problems" (DOE 2002). Although this does not explicitly mention algebra, this learning outcome could be expected to include statements that relate to algebra. For example the

---

[1] The motivation for this is twofold: *Strand 3* and *Core Aspect B* tend to emerge later in the primary curriculum as preparation for algebra in secondary schools and are considered by the author to be less relevant to the early grades; generalising is the core idea of early algebra as relevant to the early grades in primary mathematics curricula. There are other definitions of early algebra but in-depth discussion of these is beyond the scope of this paper.

2 Ma5 in England maps quite neatly to LO3 in South Africa. Similarly Ma6 maps to LO4 , and Ma7 to LO5. Ma1, Ma2, Ma3 and Ma4 in England can be broadly mapped to LO1 and LO2 in South Africa. Ma1 is a cross cutting objective, of which there is no equivalent LO in South Africa.

'Learning Outcome Focus' includes that this learning outcome "develops the learners understanding of how different numbers relate to one another; …how different numbers can be thought about and represented in various ways; and the effect of operating with numbers" (DOE 2002). This clearly shows that there is explicit intention to allow learners to explore the structural nature of arithmetic. Numbers are experienced in how they relate to each other, and can be thought about and represented in various ways, allowing for the exploration of the equivalence of the equal sign. With specific reference to the 'Foundation Phase Focus', the curriculum statement specifies that

> In this phase the number concept of the learner is developed through working with physical objects in order to count collections of objects, *partition and combine quantities*, *skip count* in various ways, solve contextual (word) problems and *build up and break down* numbers. (emphasis added) (DOE 2002)

Again, this appears consistent with the intention of early algebra to allow arithmetic to be seen in general terms: 'partitioning and combining quantities' or 'building up and breaking down numbers' promises the potential of exploring equivalence of numeric expressions; and the use of skip counting hints at the introduction of multiplication as repeated addition, and to the beginning of the concept of number sequences and the concept of a function.

'Patterns, functions and algebra' (LO2) requires learners to "recognise, describe and represent patterns and relationships, as well as to solve problems using algebraic language and skills" (DOE 2002). This 'Learning Outcome Focus' provides a detailed indication of the South African interpretation of what is meant by algebra, stating that algebra "can be seen as generalised arithmetic, and can be extended to the study of functions or relationships between variables" (DOE 2002). The description of the 'Foundation Phase Focus' for this learning outcome is that this learning outcome is intended to "lay the foundation for algebra in the Intermediate and Senior Phases" (DOE 2002). This reveals that although algebra is included in the learning outcome description, it does not, in fact, appear to be a major focus at the Foundation Phase.

In England, like in South Africa, early algebra may be found relevant to all of the learning objectives. However the core treatment of early algebra would be expected to be evident in learning objectives Ma1 to Ma4. The curriculum in England does not describe each of its learning objectives in any more detail nor in the same systematic way as the South African curriculum. A selection of 'Guidance Papers' accompany the Primary Framework have some relevance to the mathematics learning objectives. However, the term 'algebra' is not explicitly mentioned in any of the relevant Guidance Papers.[1] Nevertheless the concept of early algebra as generalising is evident in all three of these papers. To illustrate this implicit treatment of early algebra: the *Guidance Paper on Mathematics and the Primary Curriculum* states that "mathematics describes patterns, properties and general concepts" as one of its descriptors of what mathematics is, elaborating that "children's ability to extract the essential properties and generalise from particular cases is a key skill in mathematics" (DCSF 2009b). This is consistent with the notion of early algebra as generalised arithmetic. The *Guidance Paper on Calculating* also describes that children are expected to recognise how operations relate to one another and how the rules and laws of arithmetic are to be used and applied (DCSF 2009a). The *Guidance Paper on Using and Applying Mathematics* elaborates on five theme areas all of which have

---

[1] See the Guidance Paper on 'Mathematics and Primary Curriculum' on 'Calculation', or on 'Using and Applying Mathematics'.

relevance to early algebra. The following example of an extract from the reasoning theme illustrates this:

> Children need to be taught how to record their thinking and reasoning in mathematics as they describe, replicate and create patterns and explore properties and relationships. (DCSF 2009c)

Describing, replicating, and creating patterns and exploring properties and relationships are central ideas in generalising towards the idea of a function. So, although the electronic curriculum framework of England is rather fragmented and its links to multiple assessment frameworks seem inordinately complex, it does include two notions of early algebra as generalising, but views this as integrated into the treatment of number work more broadly.

In sum, in comparing and contrasting the curricula of the two countries at this macro level, the South African curriculum clearly has a more detailed and explicit focus on early algebra than the England curriculum does. However, when examining the underlying documentation provided in the *Guidance Papers* it is clear that England includes reference to the ideas underpinning early algebra as generalising, although this connection is not made explicit in the curriculum documents.

Comparison at the Micro Level: 'Learning Objectives' in England and 'Assessment Standards' in South Africa

At the micro level there is considerable similarity between the two curricula. Table 1 presents the similarities in the early grades, organised using the three elements that a curriculum which addresses early algebra should include (as outlined above).[1]

Table 1: Similarities in Assessment Standards in South Africa (SA) to Learning Objectives in England (relating to early algebra, in the early grades)

Table 1 shows that there are similarities in the two curricula in relation to all three aspects of early algebra as generalising. There are no assessment standards in the South African curriculum which are relevant to early algebra, which do not find expression, in an equivalent form, in the English curriculum. However, the curriculum in England does include several learning objectives that are relevant to early algebra, which are absent from the South African curriculum, as presented in Table 2.

Table 2: Learning Objectives in England where there are no equivalent Assessment Standards in SA

Aspect of Early Algebra	Example	ENGLAND	SOUTH AFRICA
Generalising arithmetic as the exploration of properties and number operations	Commutative, associative, and distributive properties	Ma4 Year 1: Relate addition to counting on; *recognise that addition can be done in any order* (emphasis added)	Does not make commutative property of addition explicit.
	Inverse relationships	Ma3 Year 2: Understand that halving is the inverse of doubling	Does not make explicit that halving is the inverse of doubling
		Ma3 Year 3: Use knowledge of number operations and corresponding inverses, including doubling and halving to	Does not make explicit use of inverse relationships for calculations

---

[1] See the last paragraph of the section on Analysing 'Early Algebra' in the Primary Mathematics Curricula

Aspect of Early Algebra	Example	ENGLAND	SOUTH AFRICA
		estimate and check calculations.	
		Ma4 Year 2: Understand that subtraction is the inverse of addition and vice versa; use this to derive and record related addition and subtraction number sentences.	Does not make explicit that addition is inverse of subtraction
	Properties of multiples, including odd and even numbers	Ma2 Year 2: Recognise odd and even numbers Ma3 Year 2: Recognise multiples of 2,5 and 10	Does not consider properties of multiples (for odd and even numbers, or other multiples)
Generalising about particular number properties and relationship	Using equivalence of '=' sign to find an unknown in an equation	Ma4 Year 2: Calculate the value of an unknown in a number sentence (e.g. +2=6; 30- =24)	Does not make explicit the equivalence of the equal sign, and its use beyond an instruction to compute

Table 2 demonstrates that the English curriculum makes the commutative property of addition explicit, while the South African curriculum does not. The English curriculum introduces the idea of an inverse and is explicit about both doubling and halving and addition and subtraction being inverse relationships. Also learners in England are expected to find an unknown in a simple linear equation. This shows that at the micro level of learning objectives and assessment standards, the English curriculum addresses early algebra concepts in more detail.

The South African curriculum includes examples of algebra as generalised arithmetic, but does not make this explicit through identifying general properties of numbers and relationships in arithmetic. Although doubling and halving are introduced, this is not explicitly taught as an example of an inverse relationship. There is no exploration of odd and even numbers, or properties of other multiples. Although multiplication is introduced as repeated addition, this is not done to explore the underlying general properties of association and commutativity.

Conclusion

This curriculum comparison focuses on how a particular conceptualisation of early algebra (an adaptation of Kaput's definition) is manifest in the intended curricula in the early grades in two countries. The analysis reveals that, the South African intended curriculum makes clear commitments to the teaching and learning of algebra, but it does not deal with it in any substance, and that, in contrast, the English intended curriculum includes more algebra content, although does not address 'early algebra' explicitly.

At the macro level of analysis, the content structure of the South African curriculum includes an explicit focus on early algebra from the first year of primary school as one of the five learning outcomes focuses on 'patterns, functions and algebra'. However at the first phase of primary school in South Africa the treatment of algebra is primarily a stepping stone towards algebra for subsequent phases in primary school. There appears to have been little reflection and exploration of how early algebra may manifest in the Foundation Phase. There is little focus on generalising, as having intrinsic mathematical value for this age group. England in contrast, does not include a specific content focus on early algebra in its primary

mathematics curriculum. However, some of its *Guideline Papers* refer to the ideas underpinning early algebra as generalised arithmetic, even though this connection is not made explicit. Making this connection explicit and enabling teachers to see how the curriculum in England supports a growing body of research work around the world on the topic of early algebra would be useful.

At the micro level of analysis, the level of learning objectives and assessment standards, clearly shows that the English curriculum addresses early algebra in more detail. While the South African curriculum includes some examples which can be recognised as having the potential for exploring generalised arithmetic, it does not expect the underlying algebraic reasoning to be explicitly recognised. Several examples of generalised arithmetic relevant for this age group are included in the curriculum in England but are absent from the South African curriculum.

This paper is not intended to advocate for a change in the curriculum framework in South Africa. Nor is it intended to advocate for yet another assessment framework in England. Rather, this paper strives to highlight the need for more detailed guidance for South African teachers on the treatment of early algebra, and the need for England to simplify and streamline the supporting documentation for its intended curriculum, which should make the treatment of early algebra explicit. It is hoped that this paper allows for reflection on how intended curricular may be used to better communicate research in early algebra, at an appropriate level of detail, to teachers in the early grades.

References

Amerom, B. A. V. 2003. Focusing on Informal Strategies When Linking Arithmetic to Early Algebra, *Educational Studies in Mathematics*, 54(1), pp. 63-75.

Bills, L., J. Ainley, and K. Wilson. 2003. Particular and General in Early Symbolic Manipulation, *27th International Group for the Psychology of Mathematics Education Conference Held Jointly with the 25th PME-NA Conference (Honolulu, HI, Jul 13-18, 2003), v2*, International Group for the Psychology of Mathematics Education, pp. 105-112.

Bodanskii, F. ed. 1991. *The Formation of an Algebraic Method of Problem Solving in Primary School Children*, (Reston, V.A, NCTM.), pp. 275-338

Davis, R. 1985. ICME-5 Report: Algebraic Thinking in the Early Grades, *Journal of mathematical behavior*, 4, pp. 195-208.

Department for Children, Schools and Families (DCSF). 2009a. *Guidance Paper: Calculation*. Available online at: http://nationalstrategies.standards.dcsf.gov.uk/node/47364 (accessed 2009).

___ 2009b. *Guidance Paper: Mathematics and the Primary Curriculum*. Available online at: http://nationalstrategies.standards.dcsf.gov.uk/primary/primaryframework/mathematics/Papers/mathematics_primary_curric (accessed 2009).

___ 2009c. *Guidance Paper: Using and Applying Mathematics*. Available online at: http://nationalstrategies.standards.dcsf.gov.uk/node/47324 (accessed 2009).

___ 2009d. The National Strategies: Primary Framework - Mathematics Framework, Crown Copyright (1997-2009).

Department of Education (DOE). 2002. *Revised National Curriculum Statement Grades R-9 (Schools) Mathematics*, (Pretoria, Department of Education, South Africa), pp. 1-120.

Herscovics, N. and L. Linchevski. 1994. A Cognitive Gap between Arithmetic and Algebra, *Educational Studies in Mathematics*, 27(1), pp. 59-78.

Kaput, J. and M. Blanton. 1999. Algebraic Reasoning in the Context of Elementary Mathematics, *Annual Meeting of the American Education Research Association*, (Montreal, Quebec, Canada).

Kaput, J., D. Carraher, and M. Banton. 2007. *Algebra in the Early Grades* (Routledge).

Sasman, M., L. Linchevski, A. Olivier and R. Liebenberg. 1998. Probing Children's Thinking in the Process of Generalisation, *Fourth Annual Congress of the Association for Mathematics Education of South Africa* (Pietersburg: University of the North, South Africa), pp. 210-218.

Vermeulen, N. 2007. Does Curriculum 2005 Promote Successful Learning of Elementary Algebra?, *Pythagoras*, 66 (December, 2007), pp. 14-33.

Warren, E. 2004. Generalising Arithmetic: Supporting the Process in the Early Years, *28th Conference of the International Group for the Psychology of Mathematics Education* (Bergen, Norway), pp. 417-424

Participation in mathematics post-18: Undergraduates' stories

Melissa Rodd, Tamjid Mujtaba and Michael Reiss

Institute of Education, University of London, 20 Bedford Way, London WC1H 0AL

Abstract: This paper reports on some of the social and emotional complexities young people negotiate, consciously or otherwise, when applying to study at university and presents reasons for why good candidates for mathematics degrees may not opt to study mathematics. The research comes from one strand of the UPMAP project which is seeking to understand profiles of participation in mathematics and physics. Data analysed come from narrative-style interviews which were conducted with first-year undergraduates who had A level mathematics and who were studying a range of subjects at university.

Keywords: higher education, affect, participation

Introduction

This paper reports on one of the three 'Strands' of the 'Understanding Participation in Mathematics and Physics' project (UPMAP) that is funded by the ESRC and other providers as part of their 2007 science and mathematics education call. The funders' intention was to commission research that could provide evidence for policy proposals that would encourage a technically proficient new generation given that post-16 participation rates in mathematics and physics were not considered sufficient from an economic-modelling perspective (e.g. HM Treasury, 2004) to sustain desired economic growth, (ESRC webref). In 2008 the Higher Education Funding Council for England injected a £350 million rescue plan to attempt to counteract the decline in the number of students taking science and mathematics. In recent years there has been a growing concern in the declining number of school-leavers continuing with mathematics and physics after the age of 16 (RSA, 2007). The gender gap in post-16 uptake of physics is larger in England than it is in Scotland, Ireland and Wales (RSA, 2008).

The UPMAP project

The principal focus of the UPMAP project is to understand better what determines student attitudes towards mathematics and physics and the reasons for student subject choice. We are particularly interested in making sense of the phenomenon in which many of those who do well in mathematics and the sciences do not opt to pursue science, technology, engineering or mathematics (STEM) - related study at higher education (e.g., Reiss, 2000). In this paper, we report on undergraduates' retrospective explanations of why they decided, or not, to study mathematics at university.

The findings presented in this paper sit within those of an overall larger project (i.e. UPMAP) of three 'Strands', where Strand 1 employs quantitative methods to map trajectories of engagement and disenchantment of high school students in mathematics and physics by analyses of large scale and longitudinal statistical data (N≅20,000 student datasets), Strand 2 investigates subjectivities and school cultures by working ethnographically in twelve high schools chosen for their

range of staying-on rates post-16, and Strand 3 investigates young peoples' higher education degree choices through analysis of individual narrative interviews. This paper focuses on our preliminary Strand 3 results; that is, we report on undergraduates' reasons for participation, or otherwise, in mathematics at university.

The outline of the rest of the paper is as follows: the body of this report starts with a brief explanation of the rationale and methodology of the higher education aspect of the UPMAP project then presents analyses and findings, concluding with the discussion where we draw out some points from the project as a whole.

Strand 3, rationale and methodology

UPMAP Strand 3 consists of 51 interviews from first year undergraduates under the age of 21, all of whom have mathematics or physics A level (or equivalent), about half of whom are reading STEM subjects. The interviews were narrative style and interpretation employed the psychoanalytic notion of the 'defended subject' (e.g. Waddell, 1998) as well as a form of grounded theorising we have called 'face value'.

The undergraduates we interviewed were recruited from four UK universities (referred to as 'University A', 'University B', etc.). The universities represent a range of institutions: two Russell group institutions in different regions of the country, one pre-1992 non-Russell group university and one post-1992 university. Recruiting the undergraduates for interview involved challenges in negotiating access, peculiar to each institution. In two of the four universities, we received an order of magnitude more applications from students than we were able to interview, so we had to have selection procedures. The main organising principle for the selection was that we should have comparable numbers in each cell of the table below. Numbers in cells give the breakdown from University A:

	STEM	Non-STEM
Female	8	5
Male	5	4

The ratios in the table above are representative of the responses overall with more female and more STEM undergraduates presenting for interview. The sample represents undergraduates with a range of stories, which was what was sought, but we do not claim that the sample is all-inclusive or statistically representative.

Data gathered in Strands 1 and 2 of the UPMAP project are being used to construct trajectories of participation in mathematics and physics of school students in the 12-17 age group and Strand 3 data are used to construct detailed understandings of reasons for participation (or not) in mathematics and physics from individuals in the 18-20 age group. The central research question for Strand 3 is "What are the 'narratives of choice' that undergraduates tell?". Our 'choice narratives' (i.e., our interpretations of the interview data) will be used to inform Strand 2 interview schedule design (e.g. the time we spend in the Strand 2 interviews exploring the relative influence of school, family and other out-of-school influences on mathematics) and themes extracted from the Strand 3 data analysis will be used to interrogate the longitudinal Strand 1 data (e.g. the importance of the utility of mathematics as opposed to its intrinsic appeal).

Methodology

In Strand 3 the aim is to find out about how or why first year (under 21 years of age) undergraduates got to be on the course that they are enrolled in. Our method involved interviewing them and then analysing the resulting transcribed text. Each of the

undergraduate interviewees was informed about the project: their invitation was via a website or an email which explicitly said "please come and talk to us about your choice of course". In part because the undergraduates knew why they were being interviewed, asking questions like "why did you choose study mathematics?" might produce rehearsed, standard responses like "I enjoy it" or "It's my best subject" and, thus, underlying reasons for decision-making that arise from the individuals' subjectivities might remain hidden. It was these reasons which we aimed to uncover and make sense of. We took as a theoretical underpinning a Kleinian-based psychoanalytical model (e.g. Waddell, *op. cit.*) that theorises that all persons defend against anxieties, thus taking as a principle that there are unconscious and subconscious influences on individuals' decision-making about critical life events. These influences, while not separable from the cultural, gender and socio-economic positioning people experience, are not as well-defined as these sociological categories of 'race, class and gender'. We aim to uncover complexities of decision-making that go deeper than the contextual background of students, recognising that individuals are not consciously aware of all of the reasons for decisions.

Furthermore, there are mathematics-specific anxieties that we anticipated detecting (Nimier, 1993; Black et al., 2009). Hence the approach we have used is based on a narrative approach to interviewing (e.g., Hollway and Jefferson, 2000) which aims to elicit a story from the interviewee in which they reveal not merely the rehearsed and public face of their decision making, but also positionings, whether by themselves or by others, they've experienced (due to gender, class, ethnicity, etc.). A narrative approach also elicits memories of critical events and 'random' occurrences that have brought the person to where they are. The aim is to allow each interviewee to tell a story that includes details they were not fully aware of or, at any rate, did not necessarily consider relevant to their choice of university course.

Our interviews were conducted as follows: we had established mobile or email contact with the undergraduate prior to the interview and welcomed them to the room in their university where the interview was to take place. After a few minutes of helping them to feel relaxed and telling them orally about the project, we generally started the interview by asking each student to talk about their education, encouraging them to start from wherever they wanted. The interview then proceeded from what the undergraduate offered, but with the interviewer's aims of finding out about their early childhood experiences both in and out of primary school, their secondary school years and any out-of-school activities they were involved in. We also wanted to find out about any family, cultural or community influences on their decision making and we were alert to opportunities for asking students if there were any critical events they could remember where decisions about subject choices were made.

In practice, interviewing 'defended subjects' by means of conducting such narrative style interviews requires the interviewer to manifest skills similar to those used in a counselling interview: we aimed to bond with the interviewee, so they'd be relaxed enough to tell their story, and we aimed to remember details within the approximately 50 minute interview so that we could pick up on an interesting thread at a later point in the interview, encouraging them to tell their story of their choices. Nevertheless, there is a fine line between a question that 'follows' a previous point and a direct question. Although asking formal, direct questions tends to elicit rehearsed or 'defensive' replies and is generally to be avoided, in conversational (narrative) interviews, a curious and engaged interviewer, having established a 'working bond', will not always adhere to this protocol (e.g., humour sometimes gets the upper hand and in such a case, typically, the subsequent conversation has an improved emotional tone). The resulting conversation is thus a co-construction

between the undergraduate and the researcher: an audio text that was subsequently typed up to form the written text, 'the transcript'.

A post-interview pro-forma explained about how the undergraduate's interview would be used and also provided the interviewee with the opportunity of making a comment on how the interview process had felt. Several interviewees were explicit in making a comment that indicated that the process of talking about their choices had brought to mind things they had not been aware of, validating the methodological process; for example: "I found it very interesting as it offered self-reflexivity which I had not considered before" (Becky – all names are pseudonyms that indicate gender); "It was a nice relaxed atmosphere – and the questions asked were actually insightful for me in analysing my own education" (Peter).

'Face value' and defended subjects

Analysis of the interview texts is still on-going and we are employing different lenses to interrogate the data (Black, Mendick & Solomon, 2009). We refer to the different lenses we are using as: 'face value', 'defended subject', 'discourses' and 'Strand 1 constructs'. At the time of writing we have analysed interview texts from University A in the first two of these ways, 'face value' – where a grounded approach to the text was used to extract themes – and 'defended subject', and these analyses are the basis of this report.

The brief 'face value' account, which is presented first, gives the reader a glimpse of the sort of things the undergraduates said in interview about their relationship with mathematics and their choices; it is not a quantitative presentation of categories of views but a background to the more detailed analysis that follows. Then, in the discussion of Dan, below, we indicate why a 'face-value' approach is not sufficient to understand these young peoples' decision-making and yet is useful to foil a 'defended subject' interpretation. This is because the defended subject analysis proceeds by first locating indicators of defence – for example, inconsistencies, exaggerations, avoidances and silences are typical of ways that feelings are revealed, whether or not the person is aware of the revelation – and then by interpreting these indicators of defence in terms of the narrative the undergraduate is sharing and developing in the interview.

This interpretative way of reasoning about affects is different to published reasoning in mathematics. In this interpretative work we read into the text, locating indicators of defence 'between the lines', as well as reading face value data from the text; in mathematics, of course, inference is strictly from premises via set rules of inference. A mathematician may well say that there are logical alternatives to the given interpretations and yes there are! The justification for the claim is not based on a closed, formal argument but on a holistic and intuitive appraisal, prepared for by study and experience of psychoanalytic methods and literature, and presented as a narrative itself which has also been adapted in the light of critical interrogation by others.

Analysis and findings

This section gives an overview of undergraduates and their reasons for course of study from University A[1], and then looks in detail at an indicative case. Our central aim is to better understand what has steered these students to the course on which they have embarked. In particular, for this BCME context, we examine the research question focussed on mathematics: "What are the 'narratives of choice' that undergraduates tell concerning mathematics?".

On 'face value' reasons for doing or not doing mathematics

Of the 13 interviewees who were studying STEM subjects, predictably, those doing science or engineering considered mathematics as a tool, whereas those doing single or joint honours mathematics did not give reasons for choosing mathematics that included its usefulness – except at a meta level of it being a degree you could do other things with. Of the four reading mathematics, as joint or single honours, each of them expressed the intimacy of their relationship with mathematics; for example, "I always wanted to do mathematics and only mathematics" (Vera), "I couldn't have lost my maths" (Chloe).

Eight of the nine interviewees not studying a STEM subject had A level mathematics and six of them had grade A in it. These undergraduates' stories of their choices frequently referred to 'enjoying' mathematics at school, but employed opposing discourses like 'lack of relevance' or 'difficulty' to explain why they had not applied for a mathematics-related degree; all of the six A-grade non-STEM interviewees used the discourse of being 'good at' mathematics to explain their reasons for their choice of A level mathematics. Discourses of challenge-and-interest and enjoyment-and-creativity were used to explain non-mathematical choices:

> Maths was the easiest of A-Levels for me by far … I liked maths because I was good at it, and I found it challenging but I didn't find it interesting. But English, … I just found … more interesting and I'd rather do something that I find interesting and challenging than something I find easy and less interesting.

> (Becky, reading English, other A levels English and history; all A grades)

> And maths, which I did to A-Level, I didn't really enjoy as much. … I suppose to say I didn't enjoy it is a bit extreme but I didn't get the same enjoyment out of it as I did with English and History. I sort of liked the creativeness, like writing essays. … I sort of saw [maths] as a job to be done, so I suppose that's the reason that I didn't enjoy it as much.

> (Peter, reading history, other A levels English and history; all A grades)

These undergraduates expressed their reasons for their choices, we have to assume, genuinely. Nevertheless, we can learn more about their decision making when we interrogate their interview text further. For example, early on in Dan's interview he tells of switching from sociology to mathematics A level within the first few weeks of Y12. His reason being that "with maths I knew I knew stuff". At face

---

[1] We have completed our interviewing at University A and have 22 interviews from first year undergraduates who have at least one of mathematics and physics A level. Twenty one of these interviewees had mathematics A level, and 17 of these achieved grade A. Of the 21 with A level mathematics, 11 attended a maintained comprehensive school or post-16 college, 7 attended a maintained selective school and 3 attended fee-paying (i.e. private) schools (one of whom was a scholarship student).

value this is a rational decision by a young man who has the option of continuing with sociology or switching to mathematics. But even his utterance that communicates the reason for his switch is open to other interpretations: from the 'defended subject' perspective, a reader could ask, for example, 'why does he have the need to know stuff at the beginning of a course?'. This 'I know I know' aspect of Dan's relationship with mathematics is then threatened later on in his Y12, producing anxiety:

> I found the, the very last, the very last exam that we did for the further maths AS was FP1 and it was really, really tough

and he did not continue to A2 further mathematics. In the interview he had said that when he was younger:

> I was always quite good at [maths] I guess. And then I don't know I just like found it quite interesting like kind of got it quite easily as well and did it quite well easily so, I just found it interesting I guess.

Yet:

> I didn't really enjoy maths like as I got older, like it was quite, it just got to a level where it got too complex and just stuff I could never see myself ever using in my day-to-day life.

> (Dan, reading English and drama, other A levels English and chemistry, all A grades, further maths AS grade B)

A defended subject reading of Dan's story is that Dan's relationship with mathematics is through performance, so when performance is perceived as weak – B grade for an AS – the relationship is under threat and defences are evoked. When Dan says "I could never see myself ever using in my day to day life" we see an example of Nimier's 'mathematics is remote' defence against mathematics (Nimier, *op. cit.*).

Robin: studying history with economics

The following discussion of Robin shows that defences themselves are interwoven in complex ways. The claim is made that Robin does not choose to read mathematics at university because of certain anxieties which he seems to defend himself against outside his conscious awareness. This report starts (1) by presenting background information on Robin. Then in (2) we present extracts concerning being 'good at maths' and in (3) indicate the importance of history and engineering to Robin while explaining how he was construed as a defended subject *vis à vis* his choice of course. The section concludes (4) by justifying the claim above.

(1) Robin is from a town in the North of England where he went to a middle school, high school and post-16 college. He has one sister six years his junior, his father is an accountant turned manager in the NHS, and his mother was a journalist, then became a full-time mum, and is now a teaching assistant. He was at the beginning of his second term, aged 19, studying history with economics at University A when he was interviewed. His A levels (grade) were: mathematics (A), further mathematics (A), physics (A) and history (A) and he has an AS in economics (B).

(2) Early in the interview Robin says: "I've always been good at maths from basically whenever I could start thinking". He also tells that he got National Curriculum level 8 when in Y7, was in an express GCSE mathematics class on starting high school in Y9 and took the GCSE in Y10; he adds that he "only" got an A for GCSE, had his paper remarked but did not get it upgraded to A*.

Mathematics is the subject chosen by a close peer group with which Robin identifies: At post-16 college, he studied maths entirely within a 'further maths' class;

there were 11 students in the class, three of whom who went to read mathematics at prestigious universities had been 'maths class-mates' with Robin since Y6:

> It's weird because all this time with them doing maths even when I was doing engineering they were suddenly like we came back at Christmas and they all had work for the next semester and stuff and I was kind of interested to know what it was and suddenly this was stuff that I couldn't do because obviously they'd been taught.

After saying this he says that recently "[I]found all my further maths stuff but I didn't want to throw it out because I don't want to get rid of maths".

(3) While at school, Robin had been involved with two engineering enrichments: one in the summer between Y8 and Y9 when he won a prize for making a robot. The other was a longer project, Engineering Education Scheme in Y12 ('EES' – organised by his history teacher), with three other students from his college; all four of these EES students started engineering courses at University H straight from school, but Robin left there after a term, to do history and economics at University A. University H was Robin's second choice for engineering after Cambridge – where he was rejected after interview. He was also rejected (at an engineering careers event) by the RAF to train as a pilot as he had asthma. Robin's history teacher is closely linked to his engineering as well as his degree choice:

> And I had a brilliant history teacher again at both AS and A2 years and she really got me involved with history so … She was really an enthusiastic person and she was as enthusiastic about engineering as she was about history. Her profession was history, she had a history degree from York but she really enjoyed, in a sense she's quite a lot like me, she enjoys history but she enjoys working out how things work and stuff like that. So, yeah, I enjoyed the last year of history at A-level.

Thinking of Robin as a defended subject, we note two rejections related to engineering: no place for him at Cambridge nor in the RAF. So Robin, initially groomed for engineering, to defend himself, has to look elsewhere. Towards the beginning of the interview he spontaneously says:

> Quot. B: I really enjoyed maths. Which is why I didn't want to do a straight history degree, my ideal would have been history with maths but obviously you can't do that. So economics was the closest thing to it basically.

(4) How come he's not doing mathematics? Short answer: because it is too risky; thought of failure elicits anxieties from which Robin has to defend himself. Robin has a mathematical self-identity going back to early awareness of "thinking", yet he was not quite an A* and could not follow his former classmates' mathematics degree work, so failure is imaginable. He experienced rejection by engineering, despite having been positioned to participate in that field and he defended himself against further rejection by avoiding continuing with engineering. His history teacher inspired hope in a different arena; he fantasises identification with her and follows her path.

Discussion

The extracts from the analysis of Robin's narrative exposes complexities around decision making, like choice of university course, that is central to young people's lives. We have shown that certain defences of the self (where defences are theorised as a normal part of personhood), can disrupt a potential career. Adolescent concerns do not sit comfortably with a rational, 'flow chart' model of decision making. Indeed, the very discourse of 'Choice' (Salecl, 2009) that demands individuals' attention to their 'life-making' provokes anxieties. Ultimately, withdrawals and approaches are

emotional acts: Robin is attracted to his history teacher and withdraws from feared failure in his primal intellectual domain (mathematics).

There are some features of mathematics as experienced in education that could be changed and, although we will never know whether this would have been enough to keep Robin in the fold, issues that might not have provoked the defences that resulted in his a-mathematical path are:

- Believing that 'good enough' is, indeed, good enough;
- Having an imaginative projection of doing mathematics in the future;
- Developing a personal, emotionally salient, view of mathematics.

These three items will be developed in other writing. Briefly, the discourse of having to be the 'best' in mathematics to continue is still pervasive and puts many young people off. Continued careers development, while on-going, is needed to seed a personal fantasy that can turn accountants into lion tamers (youtube webref) and recent developments in illustrating opportunities a mathematics degree may offer are very encouraging (mathscareers webref). And, as our Strand 1 preliminary findings already reinforce, students' mathematics-specific 'self concept' and confidence in conceptual tasks, are key psychological factors that have an influence – independent of gender, ethnicity and class – on intention to choose mathematics. Thus, as these mathematics self concepts are experienced differently, honouring the notion of 'student diversity' requires a range of approaches into conceptualising, learning, enjoying and becoming inspired by mathematics.

References

Black, L., H. Mendick, and Y. Solomon, eds. 2009. *Mathematical Relationships: Identities and participation*. London: Routledge.

Black, L., H. Mendick, M. Rodd, and Y. Solomon. 2009 'Pain, pleasure and power: selecting and assessing defended subjects' in Black, L., et al. eds. *Mathematical Relationships: Identities and participation*. London: Routledge.

HM Treasury. 2004. *Science & Innovation Investment Framework 2004 – 2014*. London: HM Treasury.

Hollway, W., and T. Jefferson. 2000. *Doing Qualitative Research Differently: Free association, narrative and the interview method*. London: Sage.

Nimier, J. 1993. Defence mechanisms against mathematics *For the Learning of Mathematics*, 13(1), 30-34.

Reiss, M. J., 2000. *Understanding Science Lessons: Five years of science teaching*. Buckingham: Open University Press.

Royal Society. 2007. *A State of the Nation Report on Science and Mathematics Education*. London: The Royal Society.

— — — 2008. *Science and Mathematics Education 14-19*. London: The Royal Society.

Salecl, R., 2009. 'Society of Choice' *Differences*, 20(1), 157-180.

Waddell, M., 1998. *Inside Lives: Psychoanalysis and growth of personality*. London: Routledge.

Webrefs

esrcsocietytoday.ac.uk/ESRCInfoCentre/PO/releases/2008/may/investment.aspx (accessed 09.09.09)

mathscareers.org.uk (accessed 27.09.09)

youtube.com/watch?v=XMOmB1q8W4Y (accessed 29.09.09)

Using longitudinal, cross-system and between-subject analysis of the TIMSS study series to calibrate the performance of lower-secondary mathematics education in England

Kenneth Ruthven

University of Cambridge

Evidence from the TIMSS study series is used to calibrate trends in lower-secondary mathematics in England between 1999 and 2007, relative to other educational systems, and in comparison with science. Over this period the proportion of students displaying high achievement in mathematics rose, apparently in response to reforms associated with the national *Strategy*. However, the proportion of students displaying positive attitude to mathematics fell substantially. In both respects, performance in mathematics continued to compare unfavourably with that in science.

Performance trends; international comparisons; mathematics education; science education; student achievement; student attitude.

Introduction

The focus of this paper is on using findings from the TIMSS international study series to examine trends in the performance of the English educational system within a framework of cross-system and between-subject comparison. This work has been undertaken in the context of a current research initiative (Economic and Social Research Council [ESRC] 2006) intended to inform ongoing efforts to secure significant enhancements in young people's school achievement in science and mathematics, and significant increases in their participation in further study and employment in these areas. As part of the initiative, the *Effecting Principled Improvement in STEM Education (epiSTEMe)* project (Ruthven et al. 2010) is undertaking research-based pedagogical development, designed to be suited to implementation at scale within the English educational system, and aimed at improving student engagement and learning in early secondary-school physical science and mathematics.

In recent years, schooling in England has been reshaped by two waves of reform aimed at standardising educational provision and improving educational outcomes. From 1989, a national curriculum was introduced, and a system of national assessment at primary and lower-secondary phases, accompanied by a strengthening of mechanisms of professional accountability, notably through teacher evaluation and school inspection. From 1997, these policies were extended by setting targets for improved performance, particularly in student achievement, and by launching a national programme of school improvement, supported by extensive professional development. From 1999, in particular, the teaching of mathematics in primary schools in England was expected to follow the approach to lesson planning, teaching method, and classroom assessment established by the national *Strategy* (Brown et al. 1998; Reynolds and Muijs 1999). Initially developed for use at primary level, this pedagogical model was extended to lower-secondary level both in mathematics (from the student cohort entering in 2001) and in science (from the cohort entering in 2002).

Reporting on implementation of the *Strategy* at lower-secondary level, the main trends noted by school inspectors in mathematics were towards "improvements in the planning of teaching, with a greater focus on learning objectives, the structure of lessons and teachers' use of questioning" (Office for Standards in Education [OfStEd] 2004, 21), and the introduction of "systems… for regular monitoring of pupils' performance, with action taken to help them improve" (OfStEd 2004, 24). The approach developed in science was similar: inspectors found that the recommended lesson structure was near universal, as was emphasis on teaching to explicit objectives (OfStEd 2004, 31), and that teacher assessment was "being increasingly used more successfully to guide pupils on how to improve their work" (OfStEd 2004, 29).

Using the TIMSS study series to analyse changes in system performance

International study series such as the Trends in International Mathematics and Science Study [TIMSS] provide a framework for examining educational trends in terms of cross-system and between-subject comparison. In addition, their estimates of student achievement are less open to inflation as a result of the "teaching to the test" that high-stakes national testing encourages. Furthermore, they provide assessments not just of student achievement but of student attitude, broadening the types of outcome available for consideration. Compared to the PISA study series, the achievement measures of TIMSS are more curriculum based, and so better indicators of the developing knowledge-base necessary for further study (Ruddock et al. 2006); important for the particular concerns of the ESRC initiative with raising student participation in more advanced mathematical and scientific courses.

Equally, the TIMSS series usually allows an age cohort to be tracked from the elementary/primary level in one study to the middle/lower-secondary level in the next (as shown in Table 1). Thus the entire schooling of what has been termed the "earlier" English TIMSS cohort took place following the initial reforms but well prior to the subsequent ones associated with the *Strategy*. TIMSS provides less information about the "intermediate" cohort[1] which was the last to have virtually no experience under the subsequent reforms, while the "later" cohort was entirely schooled under these subsequent reforms. Thus any improvement in system outcomes as a result of the later wave of reforms should be indicated by changes between the earlier and intermediate cohorts on the one hand, and the later cohort on the other.

Table 1: Progress of student cohorts through school phases and the TIMSS study series

Cohort	Entered primary phase	Studied at Grade 4 (Year 5)	Entered secondary phase	Studied at Grade 8 (Year 9)
Earlier cohort	1990	1995	1996	1999
Intermediate cohort	1994	*not studied*	2000	2003
Later cohort	1998	2003	2004	2007

Information and data used in this paper have been extracted from the most recent TIMSS reports (Mullis, Martin and Foy 2008a; 2008b; Sturman et al. 2008). The findings for England will be situated within the distribution of results across

---

[1]   The intermediate cohort was not studied at Grade 4 level in 1999, and did not complete the main attitude measures at Grade 8 level in 2003. There were also sampling weaknesses: England nearly satisfied guidelines for sample participation rates only after replacement schools were included. In addition, 3 of the other systems used as comparators here did not participate in the 2003 TIMSS study.

those 18 educational systems that participated in both the 1999 and 2007 TIMSS studies, and which, like England, taught a general/integrated science curriculum at this level[1].

Establishing benchmarks for student achievement and attitude

TIMSS defines (in terms of test scores) several "benchmarks" for student achievement. Here, the focus will be on what TIMSS terms the "high international" benchmark. In Mathematics at Grade 8 level, this is characterised in terms of students being able to "apply their understanding and knowledge in a variety of relatively complex situations", rather than, at the next benchmark, to simply "apply basic mathematical knowledge in straightforward situations". This provides, then, a suitable marker for the level of capability required for more advanced study in STEM fields. Likewise, in Science, students who achieve the high benchmark are characterised as being able to "demonstrate conceptual understanding of some [science]", compared to the lower "recognise and communicate basic scientific knowledge". Here then, the operational index of "[subject] achievement" in an educational system will be the percentage of students reaching this high achievement benchmark in the subject.

Turning from achievement in a subject to attitude towards it, the most relevant outcome measure available in the TIMSS study series is of "high positive affect towards [subject]". We can treat this as a benchmark of attitude paralleling the benchmark of achievement analysed above. Operationally, to figure in this category students had to respond affirmatively (on average) to three statements (agreeing that they like the subject, that they enjoy learning the subject, and disagreeing that the subject is boring). This benchmark provides, then, a suitable marker for the kinds of attitude conducive to participation in more advanced study in STEM fields. Thus here, the operational index of "[subject] attitude" will be the percentage of students in an educational system reaching this positive attitude benchmark in the subject.

This apparatus will now be used to analyse trends in system performance. Each set of index scores from the 18 systems creates a distribution. For example, in Table 2 below, the proportion of students reaching the high achievement benchmark for science was 39% for the median system in 1999 and 38% in 2007. The median system change in this percentage between cohorts was +1. While the main analysis will focus on these cohorts because fuller information is available on them, the intermediate cohort will later help to triangulate and sharpen findings on achievement.

Examining between-cohort changes in system performance on student achievement

In science, the markers in Table 2 show little overall change in levels of achievement between the earlier and later cohorts. For England specifically, performance did not change significantly in absolute terms, and remained well above the median position and a little below the upper quartile in relative terms. The situation as regards mathematics is rather different. Overall levels of achievement drifted downwards

---

[1] As well as England (EN), these 18 systems comprise British Columbia (BC), Chinese Taipei (TW), Hong Kong (HK), Iran (IR), Israel (IL), Italy (IT), Japan (JP), Jordan (JO), (South) Korea (KR), Malaysia (MY), Massachusetts (MA), Ontario (ON), Quebec (QC), Singapore (SG), Thailand (TH), Tunisia (TN), United States (US). Although some data from Massachusetts is included in relation to the United States, the portion is so small, and state-level education policies sufficiently distinctive, that both entities can reasonably be included as independent 'systems' within the analysis. Likewise, the relative independence of the Canadian provinces in educational matters justifies their treatment as distinct 'systems'.

between the cohorts. However, there was a marked improvement (the second largest such improvement) in the absolute performance of England. In relative terms, too, England moved from well below the median position (closer indeed to the lower quartile) to a little above the median. Comparing the two subjects, then, whereas England's performance in science changed negligibly between the two cohorts, performance in mathematics improved, although not to a level matching science.

Table 2: Distribution of system performance on Grade 8 achievement:
Proportion of students reaching the TIMSS high achievement benchmark[1]

Distribution	Mathematics			Science		
	Earlier cohort scores	Later cohort scores	Between cohort changes	Earlier cohort scores	Later cohort scores	Between cohort changes
Upper quartile	65%	59%	+ 1	47%	53%	+ 3
Median	33%	32%	- 2	39%	38%	+ 1
Lower quartile	20%	17%	- 6	23%	22%	- 2
England	**25%**	**35%**	**+ 10** ↑	**45%**	**48%**	**+ 3** NS

To what degree, though, might this rise in mathematics performance be a legacy of improvement during the primary years (to Grade 4/Year 5) rather than indicative of enhancement over the middle/lower secondary years (to Grade 8/Year 9)? Unfortunately, TIMSS data is available at Grade 4 level for only 8 of the 18 systems under consideration. Of these 8 systems, England had the largest improvement in mathematics achievement at both levels. Nevertheless, the improvement at Grade 8 level (of 10 percentage points) was notably smaller than at Grade 4 level (where the rise was 19 percentage points, from 24% to 43%). However, smaller but still substantial improvements at Grade 4 level in Hong Kong and Ontario did not feed through to Grade 8 level. These patterns suggest, then, that there is no straightforward relationship between changes at the two levels. Nevertheless, given the magnitude of the English improvement at Grade 4 level, it is plausible to conjecture that this contributed to the subsequent improvement at Grade 8 level.

Triangulating between-cohort changes in performance on student achievement

To investigate further, it is necessary to take account of evidence about the intermediate cohort, and helpful to triangulate TIMSS findings against the results of national testing (DCSF 2008). Over the period under review, English schools and teachers were under enormous pressure to improve the performance of students in national tests: for lower-secondary (KS3) assessment at the end of Year 9, level 6 represented the key benchmark of high achievement. The graphs for science and mathematics (Figure 1) show that the relative demands of the national and TIMSS benchmarks differ between subjects: in science, the level 6 benchmark in national testing is more demanding than the TIMSS high achievement benchmark; in

---

[1]   Significance of differences between cohorts are those reported in the TIMSS studies, and indicated as follows: later cohort: ↑ significantly higher; NS not significantly different; ↓ significantly lower

mathematics, this pattern is reversed. In TIMSS terms, then, the expectations of English national testing are higher in science than in mathematics.
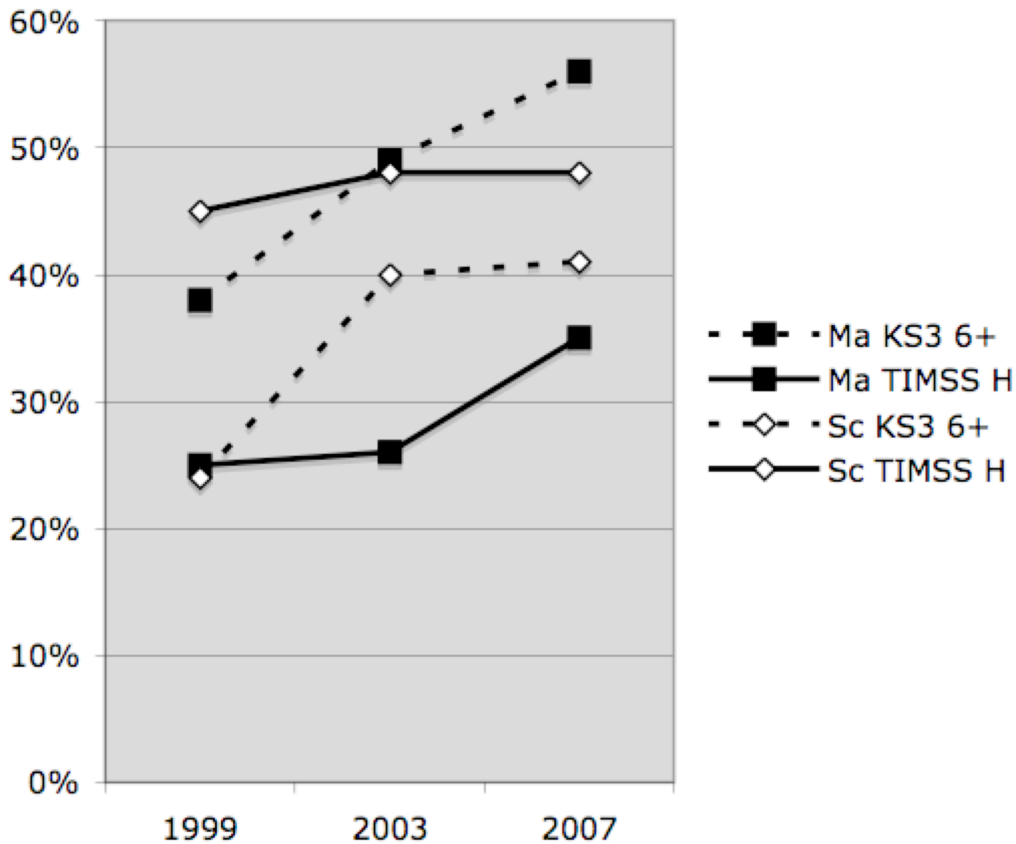

Figure 1: Change in English system performance between cohorts: Proportion of students achieving the TIMSS high benchmark compared to level 6 in KS3 tests.

On trends over time, the graphs show that performance on national tests improved markedly between 1999 and 2003 in both science and mathematics, while performance on TIMSS remained static: this suggests a fairly superficial form of improvement, enhancing student performance specifically on national tests. Between 2003 and 2007, however, trends for the two subjects diverged. Both in national tests and in TIMSS, performance remained static in science, whereas it rose markedly in mathematics. This suggests that the reforms associated with the *Strategy,* which affected only the later cohort, led to a more fundamental form of improvement in achievement in mathematics, but not in science[1].

Examining between-cohort changes in system performance on student attitude

Turning to assessment of student attitude within TIMSS (Table 3), in both subjects, the markers for the overall distributions of system scores, and for the distribution of individual system changes, indicate that the predominant pattern was one of decline in performance. In both subjects England saw a very substantial fall between the two

---

[1]   Some caution needs to be exercised over TIMSS 2003 results because of the sampling problems referred to in Note 2. Expert review of TIMSS items has also found their likely familiarity to English students to be rather greater in mathematics than in science (Ruddock et al. 2006). However, this did not lead to a subject-differentiated pattern of change in performance between 1999 and 2003; over that period rising performance in national tests in mathematics was not paralleled in TIMSS.

cohorts, markedly greater than the general trend. In science, England displays a marked absolute decline (the second largest of any system): in relative terms, it moved from just below the upper quartile to the median position (which was quite close to the lower quartile). In mathematics, too, England displayed a marked absolute decline in performance (the largest of any system): in relative terms, it moved from just below the upper quartile position to just above the lower quartile. Comparing the two subjects, then, England performs more strongly in science than in mathematics, certainly in absolute terms, rather less clearly so in relative terms.

Table 3: Distribution of system performance on Grade 8 attitude: Proportion of students reaching the TIMSS positive attitude benchmark

Distribution	Mathematics			Science		
	Earlier cohort scores	Later cohort scores	Between cohort changes	Earlier cohort scores	Later cohort scores	Between cohort changes
Upper quartile	67%	59%	- 1	77%	68%	+ 1
Median	58%	47%	- 7	63%	55%	- 6
Lower quartile	46%	39%	- 12	59%	52%	- 10
England	**65%**	**40%**	**- 25 ↓**	**76%**	**55%**	**- 21 ↓**

Examining between-cohort changes in system performance on combined outcomes

In some systems, there was very little change in combined performance on achievement and attitude between cohorts, either in science (Figure 2) or in mathematics (Figure 3): in Tunisia (TN) for example (at the top left of both figures). In others, Malaysia (MY) for example (again towards the top left of the two figures), there were marked falls in both achievement and attitude in both subjects.

In both subjects, trend lines relating attitude to achievement across the systems as a whole indicate that higher achievement tends to be associated with lower attitude; this is likely to reflect underlying system-level differences (in factors such as economic development and educational orientation) that mediate both achievement and attitude. The shift between the trend-lines from 1999 (dashed) to 2007 (solid) reflects some form of decline between the two cohorts; and the segments indicating movement of individual systems again show that this is predominantly due to changes (downwards) in attitude rather than (backwards) in achievement.

On achievement, the consistent strong riser is Massachusetts (MA), which improved 13 percentage points in science (from 43% to 56%), and 19 percentage points in mathematics (from 33% to 52%). On attitude Massachusetts declined 5 percentage points in science (from 59% to 54%), and 6 percentage points in mathematics (from 47% to 41%), but this is in line with the median decline across systems in each of these subjects. On achievement, England (EN) was the other strong riser in mathematics, but not in science. On attitude, as already noted, England fell very substantially in both subjects, at or near the extremes of decline.

The Massachusetts example shows that such a fall cannot be attributed to any inevitable within-system trade-off between achievement and attitude. Consequently, in both subjects, England surrendered a considerable lead over Massachusetts on attitude, and fell behind on achievement. Like England, Massachusetts has had a relatively longstanding systemic improvement programme based on establishing

common professional standards and ambitious achievement targets, backed by extensive professional development and strong accountability mechanisms. However, whereas the pedagogical model promoted by the *Strategy* in England was shaped by older research on the effective teaching of basic skills (Reynolds and Muijs 1999), the *Standards* influencing reform in Massachusetts reflected more recent research on developing higher-order thinking (Massachusetts Department of Education 1999).
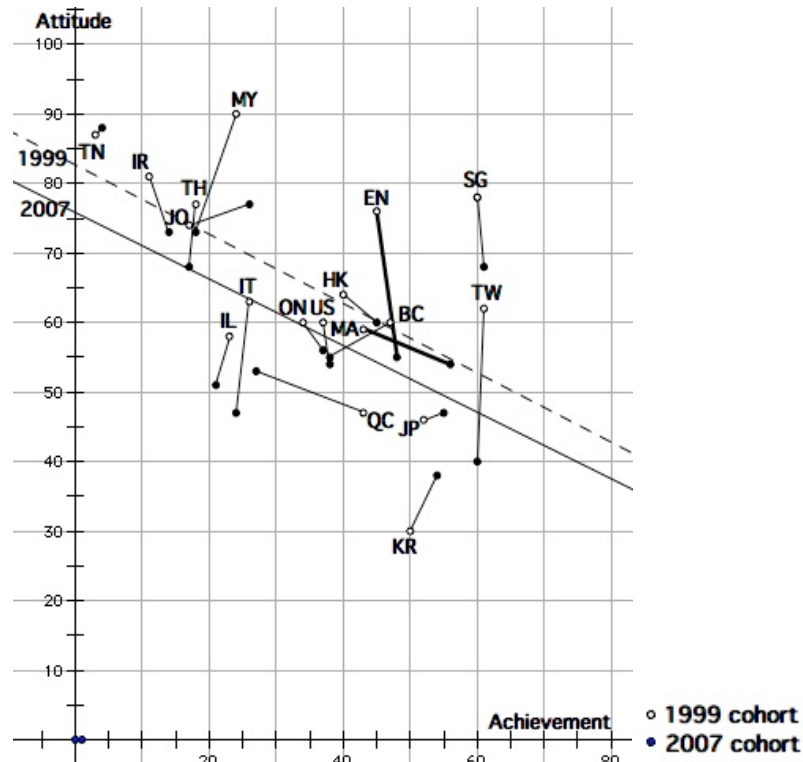

Figure 2: Change in system performance in Grade 8 science between TIMSS cohorts.
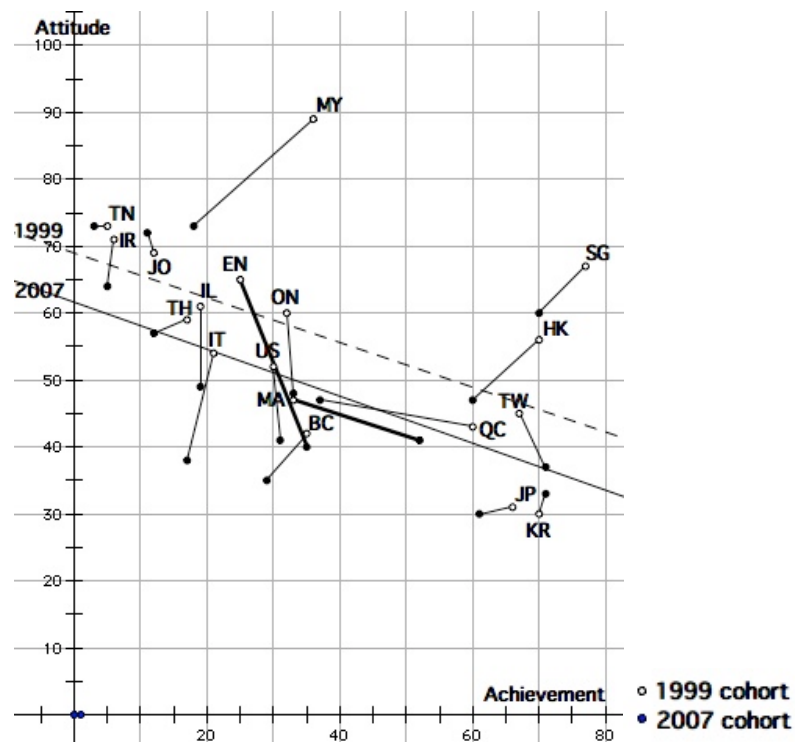

Figure 3: Change in system performance in Grade 8 mathematics between TIMSS cohorts.

Conclusion

This paper has shown how the TIMSS international study series can be used to construct a framework for cross-system and between-subject comparison capable of illuminating trends in system performance on student achievement and attitude. Applied to the case of English lower-secondary schooling, this challenges the optimistic picture from national assessment. Compared to other systems in terms of the TIMSS high achievement benchmark, fundamental gains by English students have taken place only in mathematics, and then only in response to the later reforms associated with the national *Strategy*. Compared to other systems in terms of the TIMSS positive attitude benchmark, declines in both mathematics and science have been exceptionally severe amongst English students. Finally, in these international terms, English performance remains lower in mathematics than in science.

These findings emphasise the importance of developing a better understanding of student affect and identity, a salient feature of many of the projects in the ESRC research initiative. Equally, new insights are clearly required into ways of raising student achievement: in England, there is scope for further improvement in mathematics from a still relatively average performance, and for improvement in science beyond a relatively high, but static, performance. Major recent shifts in English education policy – notably, revision of the national curriculum to reduce prescriptiveness; abolition of compulsory national testing at lower-secondary level; abandonment of centrally-driven school improvement – may now have created the conditions for projects such as *epiSTEMe* to collaborate with schools and teachers in undertaking pedagogical development towards these goals in the light of the most recent research on effective teaching and learning.

References

Brown, M., M. Askew, D. Baker, H. Denvir, and A. Millett. 1998. Is the National Numeracy Strategy research-based? *British Journal of Educational Studies* 46(4): 362-385.

Department for Children, Schools and Families [DCSF]. 2008. National Curriculum Assessments at Key Stage 3 in England, 2006/07 (Revised). *Statistical First Release* 06/2008. London: DCSF.

Economic and Social Research Council [ESRC]. 2006. *Targeted Research Initiative on Science and Mathematics Education*. Swindon: ESRC.

Massachusetts Department of Education 1999. *Partnerships Advancing the Learning of Mathematics and Science*. Malden MA: Mass. Dept. of Ed.

Mullis, I., M. Martin, and P. Foy. 2008a. *TIMSS 2007 International Mathematics Report*. Boston: TIMSS International Study Center.

——— 2008b. *TIMSS 2007 International Science Report*. Boston: TIMSS International Study Center.

Office for Standards in Education [OfStEd]. 2004. *The Key Stage 3 Strategy: evaluation of the third year*. London: OfStEd.

Reynolds, D., and R.D. Muijs. 1999. The effective teaching of mathematics: a review of research. *School Leadership and Management* 19(3): 273-88.

Ruddock, G., T. Clausen-May, C. Purple, and R. Ager. 2006. *Validation Study of the PISA 2000, PISA 2003 and TIMSS 2003 International Studies of Pupil Attainment* (DfES Research Report 772). London: DfES.

Ruthven, K., C. Howe, N. Mercer, K. Taber, S. Luthman, R. Hofmann, and F. Riga. 2010. Effecting Principled Improvement in STEM Education. *Proceedings of the British Society for Research into Learning Mathematics*.

Sturman, L., G. Ruddock, B. Burge, B. Styles, Y. Lin, and H. Vappula. 2008. *England's Achievement in TIMSS 2007: National Report for England*. Slough: NFER.

Effecting Principled Improvement in STEM Education: Research-based pedagogical development for student engagement and learning in early secondary-school physical science and mathematics

Kenneth Ruthven, Christine Howe, Neil Mercer, Keith Taber, Stefanie Luthman, Riikka Hofmann and Fran Riga

University of Cambridge

The *epiSTEMe* project forms part of a national initiative researching means of improving young people's participation and achievement in mathematics and science education. The project involves collaboration between researchers and teachers to devise an intervention, suitable for widespread dissemination, to enhance student engagement and learning in early secondary-school physical science and mathematics. Drawing on the now extensive research base examining US experience of *Standards*-based reform, and parallel research and development efforts in the UK and elsewhere, the project aims to translate promising pedagogical principles into an operational apparatus for viable professional practice.

effective teaching methods; mathematics and science education; redesign research; researcher-practitioner collaboration; secondary schools; student engagement and learning

Introduction

This paper presents the rationale for the *Effecting Principled Improvement in STEM Education* [*epiSTEMe*] project as part of a current research initiative (Economic and Social Research Council [ESRC] 2006) intended to inform ongoing efforts to secure significant enhancements in young people's school achievement in science and mathematics, and significant increases in their participation in further study and employment in these areas. The *epiSTEMe* project is undertaking research-based pedagogical development aimed at improving student engagement and learning in early secondary-school physical science and mathematics, in a form suited to implementation at scale within the English educational system.

From an initial proposal made in January 2007, the project has been funded to run from August 2008 to January 2012. It is organised in three main phases, associated with consecutive school years. During Phase 1 (2008/09) we worked with science and mathematics teachers from partner schools to devise a classroom intervention, including trialling and refining components of teaching modules and research instruments. During Phase 2 (2009/10) we are studying classroom implementation of the full modules by the participating teachers, and the functioning of the research instruments, with a view to finalising both for Phase 3 (2010/11). We are currently recruiting further schools and teachers to be involved in implementation and research during this forthcoming phase. The focus of this research is on evaluating the effectiveness of the intervention – in terms of changes in student attitude and growth in student knowledge – and analysing its operation – in terms of core classroom processes identified by the integrative theory informing its design.

From key questions to a research goal

The call for the ESRC Initiative posed, in general terms, four key questions to the research community (ESRC 2006, 2):

RQ1) What are the key factors that shape patterns of participation, engagement and achievement in science and/or mathematics education by children and young people and what does this tell us about the kinds of intervention that are likely to have greatest impact on participation, engagement and achievement?

RQ2) What can we learn from the effectiveness of past and current interventions, initiatives and practice to inform the design and development of more effective future interventions, initiatives and practice?

RQ3) How can research-informed approaches help to understand and address key challenges in enhancing participation, engagement and achievement in science / mathematics [in particular to address differences linked to socio-economic status, gender, and ethnicity]?

RQ4) What specific new interventions, or changes in policy or practice, offer the greatest potential to improve engagement and learning in science / mathematics and how could their potential effectiveness and feasibility be assessed more fully?

In response to these questions, the *epiSTEMe* proposal was developed by an interdisciplinary team made up of colleagues with specialisms in psychology of education (Howe), language in education (Mercer), mathematics education (Ruthven), and science education (Taber). Had it proved possible within the timespan and personnel available, it would also have been desirable to call at this formative stage on expertise in sociology of education and in educational improvement. The goal was to draw on a spectrum of relevant research fields to fashion a cogent proposal that integrated their insights to exemplify a more strongly interdisciplinary and cross-subject approach to research in the emerging area known as STEM [Science, Technology, Engineering and Mathematics] education. The proposal aimed to achieve this through deriving promising principles from prior research and development, and applying these to design a classroom intervention (and associated teacher training) suitable for wide-scale implementation. The research envisaged not only evaluating this intervention but creating and testing integrative theory to explain its mechanisms.

A key decision was to focus on the early secondary years. It is during this phase of schooling that students meet specialist study of mathematics and science for the first time, and it is known to be particularly important in forming young people's orientation towards further study of these subjects (Osborne, Simon and Collins 2003). In addition, from the point of view of implementation, this phase of schooling is the earliest one in which reform becomes possible through working with relatively small cohorts of specialist secondary teachers rather than a very large cohort of generalist primary teachers. Moreover, because this phase is relatively distant from the pressures of high-stakes external assessment, it offers better prospects of teachers, students and parents being willing to explore new approaches, providing a foundation for change to subsequently work its way upwards through secondary education.

From research base to a pedagogical proposal

Examining the research base at the start of 2007, the amount of relevant British work was limited. While we were aware of a range of interesting development activity, much of this proved to be un- or under-researched. However, the Evidence for Policy and Practice Initiative [EPPI] (Bennett et al. 2005) had conducted some useful systematic reviews. These were highly focused: several of the science teaching reviews, for example, examined small group discussion in relation to particular types

of learning process or outcome (now summarised in Bennett et al. 2010), while a mathematics teaching review had examined strategies to raise pupils' motivational effort at mid-secondary level (Kyriacou and Goulding 2006). These reviews, too, had had to face the lack of relevant British studies. For example, while Kyriacou and Goulding identified 25 such studies related to their topic, they judged "relevance of the focus of the study for the review question" to be "low" in 19 cases and "high" in none; likewise, the "appropriateness of design and analysis for the review question" was judged to be "low" in 19 cases and "high" in only one.

Nevertheless, in scoping *epiSTEMe*, there was relevant British research and development that deserved attention. In particular, previous work had sought to develop well-theorised pedagogical approaches running across science and mathematics teaching. One longstanding programme had developed a pedagogical model for lessons aimed at "cognitive acceleration" (Shayer and Adey 2002). Reflecting on this programme, Shayer and Adhami (2007) reported that the original intention was to use periodic lessons within science or mathematics as a context for more fundamental cognitive acceleration that would then support conventional instruction. In the light of experience, however, they suggested that the intervention had been most successful where it had served not as a complement to conventional instruction but as a constructive critique of it, leading teachers to incorporate elements of the new pedagogical model into their normal teaching. Another longstanding programme had developed a discourse-based approach that teachers had used successfully to promote "thinking together" in science (Mercer et al. 2004) and mathematics (Mercer and Sams 2006). Findings indicated that students could be enabled to use talk more effectively as a tool for reasoning; and that talk-based group activities could help develop individuals' mathematical and scientific reasoning, understanding and problem-solving.

In early 2007, too, recent developmental research had given rise to widely circulated professional materials aimed at improving the quality of teaching and learning in secondary mathematics. Drawing on earlier precedents, Watson and De Geest (2005) had worked with teachers to develop pedagogical strategies to raise the quality of learning of lower-attaining students at early-secondary level. Respecting the professional autonomy and pedagogical preferences of participating teachers, this project emphasised personal innovation rather than collective development of common methods. But, while the project found important differences in teaching strategies, it identified a common commitment to offer students an inclusive and empowering engagement with mathematical thinking. While systematic evaluation proved difficult, available findings were encouraging. Building on previous research into diagnostic teaching, Swan (2006) had carried out design research around the development of resources to support teachers in improving the quality of learning in retake examination courses in further education. The underlying pedagogical model emphasised the use of collaborative discussion to elicit and reshape students' existing knowledge and understanding. A systematic evaluation indicated that learning gains were greatest when the resources were used in such more student-centred ways.

However, by far the largest corpus of directly relevant research had arisen from long-term, programmatic efforts in the United States to formulate *Principles and Standards for School Mathematics* (NCTM 1989; NCTM 2000) and *National Science Education Standards* (NAS 1995). These principles had been operationalised in "*Standards*-based" programs intended to foster coherent understanding of fundamental ideas and their relationships, by helping students to explore and make sense of the material that they are learning, and showing that knowledge is a tool for solving problems (Trafton et al. 2001). Thus, in response to the ESRC RQ2, the

epiSTEMe proposal sought to take account of pedagogical principles common to well researched programs that had been judged "exemplary" (by a Mathematics and Science Expert Panel of the US Department of Education) on the basis of evidence of effectiveness in multiple sites (on a large scale, in rural and urban locations, across US states) for multiple subpopulations (by age, gender, ethnicity, ability).

Nevertheless, while the US exemplary curricula were research-informed in being extensively evaluated (e.g. Reys et al. 2003; Riordan and Noyce 2001), and in appealing to views of learning consonant with theoretical syntheses then available (e.g. Bransford, Brown and Cocking 2000; Kilpatrick, Martin and Schifter 2003), their design had been weakly framed in theoretical terms and their evaluation correspondingly restricted (Confrey 2006; Harwell et al. 2007). Our proposal sought to adapt principles proven in the US, framing them in theoretical terms which dovetail with complementary research undertaken in the UK and elsewhere, and using the resulting principled framework to design an intervention suitable for implementation in England in the first instance (but potentially also other parts of the UK).

The *epiSTEMe* proposal was also designed to throw light on the other three research questions. In relation to RQ4, it sought to illustrate a type of innovation with potential to improve engagement and learning – and hence longer-term participation – and to do this in a way that would exemplify a powerful approach to assessing effectiveness and feasibility of innovations. In relation to RQ3, its background pedagogical principles had already been found to be effective in the US in boosting participation and achievement, and the proposed theory-guided refinements aimed to enhance this further. Finally, in relation to RQ1, the proposed research was based on the hypothesis that those features that make *Standards*-based curricula exemplary are key shapers of engagement and achievement.

From pedagogical principles to operational apparatus

Many of the exemplary US curricula share a pedagogical model organised around carefully-crafted problem situations, posed so as to appeal to students' wider life-experience, to inculcate ideas of acting as mathematicians/scientists, and to develop key disciplinary ideas. Material is developed in lessons that cycle through whole-class introduction by teachers, collaborative problem solving in small groups, whole-class synthesis by teachers, and individual practice and consolidation by students. The *epiSTEMe* project aims to build on this pedagogical model, encouraged by its compatibility with teaching methods and curricular activities that have already been successfully deployed in the earlier British research surveyed above.

In addition to respecting principles that have emerged from research into *Standards*-based curricula, we have sought to refine such principles in the light of insights from broader theorisation and investigation. Central concerns have been how to build on students' interests and experiences (Freudenthal 1983) while also addressing the affective and epistemic complexities of knowledge growth (Pintrich, Marx and Boyle 1993). Activities have been designed to support reflexive, intentional learning (Bereiter and Scardamalia 1989) recognising that this is a process of identity formation as much as cognitive organisation (Sfard and Prusak 2005). Concern with collaborative activity, social interaction and classroom dialogue has been informed by earlier work that has analysed the crucial contribution of these processes in bringing students to engage with differing perspectives so as to support effective learning (e.g. Howe et al. 2007; Mercer et al. 2004; Mortimer and Scott 2003).

These ideas have guided the design of an introductory module intended to help establish the dialogic processes and supporting ground rules fundamental to the

intervention. To operationalise key principles further in terms of classroom teaching and learning, they have been translated into design criteria for topic modules:

> To cover those aspects of the topic prescribed for the early secondary (Key Stage 3) curriculum (specifically Year 7 in mathematics).
> To fill out these prescriptions to build strong conceptual foundations for the topic.
> To show the human interest and social relevance (including, in mathematics, scientific application) of the topic.
> To make connections with widely shared student experiences relevant to the topic.
> To take account of students' informal knowledge and thinking related to the topic.
> To provide means of deconstructing common misconceptions related to the topic.
> To provide for the exploration, codification and consolidation of key ideas.
> To exploit whole-class, group and pair discussion activity on a dialogic model to support these processes.
> To build in individual checks on student understanding with developmental feedback.

In the light of the underlying ideas, these criteria have guided the design of illustrative modules on particular curricular topics: proportionality (linked to fractions in mathematics and forces in science); probability (in mathematics); electricity (in science). While all the modules seek to make connections between mathematics and science, this is a particular feature of the modules on proportionality. Proportional reasoning is known to be challenging, yet *Standards*-based approaches have proved to be effective (Ben-Chaim et al. 1998). Indeed, the principle of 'simplification by integration' (Iran-Nejad, McKeachie and Berliner 1990) suggests that an approach which co-ordinates and integrates mathematical and scientific treatments will provide students with additional capital to appreciate and benefit from the significance of engaging problems. Furthermore, quantitative representation may facilitate conceptual growth in science (Schwartz, Martin and Pfaffman 2005).

This operational apparatus of design criteria and illustrative modules is intended to support pedagogical change by scaffolding the professional development of teachers (Ruthven 2005). First, such apparatus has a *symbolic* function, giving visible substance to change. Second, it has a *pragmatic* function, providing concrete frameworks for classroom activity. Third, this apparatus and its associated discourse have an *epistemic* function, crystallising central ideas –as expressed in the design criteria. Finally, inasmuch as such apparatus and its associated discourse explicitly incorporate a degree of flexibility, or are seen as doing so, they can serve a *heuristic* function, assisting thoughtful interpretation and local reformulation of practice. For, while pedagogical apparatus plays an important part in supporting, structuring and spreading good practice, successful reform depends on deeper understanding of, and flexible thinking about, the practice and its apparatus.

From principled apparatus to viable practice

Indeed, although *Standards*-based programs have been shown to have the potential to support effective pedagogical development (Remillard 2005), particularly if classroom materials have been designed to be "educative" for teachers (Davis and Krajcik 2005), they also run a risk of being assimilated to established pedagogies, often by teachers replacing or revising materials to make lessons more skills-oriented and less open-ended, demonstrated to reduce their effectiveness (Schoen et al. 2003). Fundamentally, successful implementation depends on teachers believing that they and their students have the capacity to engage productively with this type of approach (Arbaugh et al. 2006). In particular, teachers and students cannot simply be expected to be able to participate effectively in the necessary forms of interaction: the

development of their communicative skills and metacognitive awareness needs to become an explicit goal if it is to be successful, as also does the creation of a classroom environment in which there is clear, shared understanding of the value and functions of dialogue for learning (Alexander 2004).

This represents a considerable challenge for any project which aims to design for implementation at scale. It requires what might best be described as "redesign research": namely, an approach to design research that recognises that the successful planning of change has to take account of the existing state of affairs as well as an intended one. In particular, such an approach emphasises the need to establish a viable trajectory from the existing state to the intended one; indeed, it makes such a trajectory a condition for the plausibility of any intended state. Furthermore, the viability of such a trajectory depends on identifying what wider institutional change and professional learning are required. Such a trajectory, then, lies in a zone of proximal professional development, conditioned at its leading edge by ideals for intended practice, but at its trailing edge by the current state of practice and thinking (and notably of teachers' craft knowledge).

Moreover, as emphasised in the New Zealand approach to "best evidence synthesis" (Anthony and Walshaw 1997), reform at scale depends on successful negotiation, across the constituencies forming a professional community, of a new collective understanding of effective practice. Thus, the *epiSTEMe* project is deliberately designed to foster sustained interaction between university-based researchers and school-based practitioners. Huberman (1993, 34) has pointed to the benefits of such interaction "in which researchers defend their findings and some practitioners dismiss them, transform them, or use them selectively and strategically in their own settings". Reframing ideas in order to collaborate successfully with teachers appears to trigger a decentring process amongst researchers. In particular, it creates a need to address the counter-examples, qualifications and challenges which arise as ideas are tested out by teachers. In doing so, researchers are obliged to go outside the study at hand, to marshal a broader range of scholarly thinking and research experience, and to bring them to bear on these claimed anomalies. In the *epiSTEMe* project to date, the design and trialling of the topic modules has been an important site for such dialogue. Equally, Bromme and Tillema (1995, 262) emphasise the part that scholarly knowledge can play in supporting development of practitioner knowledge when they argue that "from a cognitive point of view, professional knowledge is developed as a product of professional action, and it establishes itself through work and performance in the profession, not merely through accumulation of theoretical knowledge, but through the integration, tuning and restructuring of theoretical knowledge to the demands of practical situations and constraints". In the *epiSTEMe* project to date, the translation into practice of ideas about dialogic teaching has provided an important focus for such development.

Finally, during the first year of the *epiSTEMe* project, major shifts took place in national policy: notably, revision of the national curriculum to reduce prescriptiveness; abolition of compulsory national testing at lower-secondary level; abandonment of centrally-driven school improvement. These shifts reflected growing recognition that the gains in student achievement achievable through the improvement approaches and pedagogical models associated with the National Strategies had been largely exhausted, as well as increasing awareness that such gains had been at the expense of a marked decline in student attitudes towards mathematics and science, so emphasising the importance of teaching approaches that are effective in terms of affective as well as cognitive outcomes (see Ruthven submitted). These changes have created more favourable conditions for the dissemination and uptake of innovative

pedagogical approaches aimed at improving student engagement and learning, such as the one being developed and researched in the *epiSTEMe* project.

References

Alexander, R. 2004. *Towards Dialogic Teaching*. York: Dialogos.

Anthony, G., and M. Walshaw. 2007. *Effective Pedagogy in Mathematics/Pàngarau: Best Evidence Synthesis Iteration*. Wellington NZ: Ministry of Education.

Arbaugh, F., J. Lannin, D. Jones, and M. Park-Rogers. 2006. Examining instructional practices in *Core-Plus* lessons: implications for professional development. *Journal of Mathematics Teacher Education* 9(6): 517-550.

Ben-Chaim, D., J. Fey, W. Fitzgerald, C. Benedetto, and J. Miller. 1998. Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics* 36(3): 247–273.

Bennett, J., F. Lubben, S. Hogarth, and B. Campbell. 2005. Systematic reviews of research in science education: rigour or rigidity? *International Journal of Science Education* 27(4): 387-406.

Bennett, J., S. Hogarth, F. Lubben, B. Campbell, and A. Robinson. 2010. Talking science: The research evidence on the use of small group discussions in science teaching. *International Journal of Science Education* 32(1): 69-95.

Bereiter, C., and M. Scardamalia. 1989. Intentional learning as a goal of instruction. In *Knowing, Learning, and Instruction*, ed. L. Resnick, 361-392. Hillsdale NJ: Erlbaum.

Bransford, J., A. Brown, and R. Cocking, eds. 2000. *How People Learn: Brain, Mind, Experience, and School*. Washington DC: National Academies Press.

Bromme, R., and H. Tillema. 1995. Fusing experience and theory: The structure of professional knowledge. *Learning and Instruction* 5: 261-267.

Confrey, J. 2006. Comparing and contrasting the National Research Council report on evaluating curricular effectiveness with the What Works Clearinghouse approach. *Educational Evaluation and Policy Analysis* 28(3): 195–213.

Davis, E. A., and J. S. Krajcik. 2005. Designing educative curriculum materials to promote teacher learning. *Educational Researcher* 34(3): 3–14.

Economic and Social Research Council [ESRC]. 2006. *Targeted Research Initiative on Science and Mathematics Education*. Swindon: ESRC.

Freudenthal, H. 1983. *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Kluwer.

Harwell, M., T. Post, Y. Maeda, J. Davis, A. Cutler, E. Andersen, and J. Kahan. 2007. *Standards*-based mathematics curricula and secondary students' performance on standardized achievement tests. *Journal for Research in Mathematics Education* 38(1): 71-101.

Howe, C., A. Tolmie, A. Thurston, K. Topping, D. Christie, K. Livingston, E. Jessiman, and C. Donaldson. 2007. Group work in elementary science: organizational principles for classroom teaching. *Learning and Instruction* 17(5): 549-563.

Huberman, M. 1993. Changing minds: The dissemination of research and its effects on practice and theory. In *Research on Teacher Thinking*, ed. C. Day, J. Calderhead and P. Denicolo, 34-52. London: Falmer.

Iran-Nejad, A., W. McKeachie, and D. Berliner. 1990. The multisource nature of learning. *Review of Educational Research* 60(4): 509-515.

Kilpatrick, J, G. Martin, and D. Schifter, eds. 2003. *A Research Companion to Principles and Standards for School Mathematics*. Reston VA: NCTM.

Kyriacou, C., and M. Goulding. (2006). *A systematic review of strategies to raise pupils' motivational effort in Key Stage 4 Mathematics*. London: EPPI-Centre.

Mercer, N., L. Dawes, R. Wegerif, and C. Sams. 2004. Reasoning as a scientist: ways of helping children to use language to learn science. *British Educational Research Journal* 30(3): 367-385.

Mercer, N., and C. Sams. 2006. Teaching children how to use language to solve maths problems. *Language & Education* 20(6): 507-527.

Mortimer, E. F. and P. Scott. 2003. *Meaning Making in Secondary Science Classrooms*. Maidenhead: Open University Press.

National Academy of Sciences 1995. *National Science Education Standards*. Washington DC: National Academies Press.

National Council of Teachers of Mathematics 1989. *Standards for School Mathematics*. Reston VA: NCTM.

——— 2000. *Principles and Standards for School Math*. Reston VA: NCTM.

Osborne, J. F., S. Simon, and S. Collins. 2003. Attitudes towards science: A review of the literature and its implications. *International Journal of Science Education*, 25(9): 1049–1079.

Pintrich, P., R. Marx, and R. Boyle. 1993. Beyond cold conceptual change: The role of motivational beliefs and classroom contextual factors in the process of conceptual change. *Review of Educational Research* 63(2): 167-199.

Remillard, J. 2005. Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research* 75(2): 211-246.

Reys, R., B. Reys, R. Lapan, G. Holliday, and D. Wasman. 2003. Assessing the impact of *Standards*-based middle grades mathematics curriculum materials on student achievement. *Journal for Research in Mathematics Education* 34(1): 74-95.

Riordan, J., and P. Noyce. The impact of two *Standards*-based mathematics curricula on student achievement in Massachusetts. *Journal for Research in Mathematics Education* 32(4): 368-398.

Ruthven, K. 2005. Improving the development and warranting of good practice in teaching. *Cambridge Journal of Education* 35(3): 407-426.

——— submitted. Using international study series and meta-analytic research syntheses to scope pedagogical development aimed at improving student attitude and achievement in school mathematics and science. *International Journal of Science and Mathematics Education*.

Schoen, H., K. Cebulla, K. Finn, K., and C. Fi. 2003. Teacher variables that relate to student achievement when using a Standards-based curriculum. *Journal for Research in Mathematics Education* 34(3): 228 – 259.

Schwartz, D. L., T. Martin, and J. Pfaffman. 2005. How mathematics propels the development of physical knowledge. *Journal of Cognition and Development*, 6(1): 65-86.

Sfard, A., and A. Prusak. 2005. Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher* 34 (4): 14-22.

Shayer, M., and P. Adey, eds. 2002. *Learning intelligence: Cognitive acceleration across the curriculum from 5 to 15 years*. Buckingham: Open University Press.

Shayer, M., and M. Adhami. 2007. Fostering cognitive development through the context of mathematics: Results of the CAME project. *Educational Studies in Mathematics* 64(3): 265-291.

Swan, M. 2006. Learning GCSE mathematics through discussion: what are the effects on students? *Journal of Further and Higher Education* 30(3): 229-241.

Trafton, P., B. Reys, and D. Wasman. 2001. Standards-based mathematics curriculum materials: A phrase in search of a definition. *Phi Delta Kappan* 83(3): 259-264.

Watson, A., and E. De Geest. 2005. Principled teaching for deep progress: Improving mathematical learning beyond methods and materials. *Educational Studies in Mathematics* 58(2): 209-24.

Enrichment and engagement in mathematics

Sara Santos and Patrick Barmby

The Royal Institution of Great Britain, London and Durham University

In this paper we examine the notions of engagement and enrichment in mathematics. The Royal Institution of Great Britain (Ri) facilitates the Secondary Mathematics Masterclasses project and has been involved in a QCA project to follow teachers' journeys when developing 'rich tasks' for use in school for whole class teaching, both with the aim of 'Engaging mathematics for all learners'. The Masterclasses were evaluated in 2008 by the CEM centre in Durham, and the Ri conducted case studies for the QCA project on how ideas and methods traditionally reserved for the gifted and talented cohort can be used for a wider range of learners. Drawing on the results of these studies and the research literature, this paper will discuss what is understood by 'enrichment' and 'engagement' in mathematics. In clarifying what we mean by enrichment, we present a structure for enrichment which differentiates between the inputs and the outputs of any enrichment activity, engagement in mathematics being one of the desirable outputs. The findings show that the Ri masterclasses brought about the enrichment outputs we would expect from such activities, with additional outputs for the teachers involved. The Ri's participation in the QCA project has aimed to build on this model in order to maximize such outputs. We explored the impact in teachers' practice when developing rich tasks and the impact on learners. Based on our research, we put forward recommendations for carrying out enrichment activities in mathematics.

Key words: enrichment, engagement

Introduction

The need for mathematical enrichment activities is highlighted by ongoing concerns over students' attitudes towards the subject. Smith (2004) identified a perception by young people that mathematics was boring and irrelevant. He underlined this point by noting the 10% drop in the take-up of A-level mathematics in the 1990s, highlighting possible factors for this decline including the perceived poor quality of the teaching and learning experience, the perceived relative difficulty of the subject, the failure of the curriculum to excite interest and provide appropriate motivation, and the lack of awareness of the importance of mathematical skills for future career options and advancement. Despite a rise in recent years of students taking A-level mathematics, the numbers in 2007/08 were still about 6000 students down on the numbers taking the subject in 1990. In addition, the London Mathematical Society (1995) raised concerns in the past that students entering higher education lacked the necessary ability and skills that might be expected on university mathematics courses. Therefore, it would be hoped that mathematical enrichment programmes tackle these two problems of attitudes towards maths and students' mathematical skills.

Enrichment in mathematics

The question of what is meant by enrichment has been an ongoing question for researchers. Feng (2005) provides a useful summary and quotes Barbe (1960): "an aura of vagueness and confusion seems to surround the term". Feng's own conclusion is that "no overall consensus has yet been reached on the definition and nature of enrichment". The phrases mathematical or analytical thinking and problem solving are also often used when attempting to define the term enrichment. These are discussed by Piggott (2004), and although she creates a model in which these terms are integral to a concept of mathematical enrichment, the difficulty still remains of defining just what is meant by these terms. Despite this difficulty, Feng (2006) put forward four 'paradigmatic positions' on enrichment in order to show the different types of mathematical enrichment activities: (1) Development of mathematical talent, including extending mathematical skills and heightening interest in the subject; (2) Popular contextualisation of the subject, including tackling negative stereotypes and deepening mathematical understanding; (3) Enhancement of mathematical learning processes, including developing learning skills; (4) Outreach to the mathematically underprivileged, including widening student access to mathematics. A point to note here is that these positions on enrichment detail the hoped-for outcomes to enrichment activities only – they do not in themselves define what constitutes enrichment activities. In developing our ideas on what enrichment is in the context of learning mathematics, a definition will be offered at the end of the paper.

Engagement in mathematics

In addition to trying to define 'enrichment', we can also clarify what is meant by engagement in mathematics, which is the overall aim of the enrichment tasks being considered in this paper. Engagement is important primarily because of its relationship with the academic achievement of learners (Peterson and Fennema 1985; Park 2005). Newmann et al. (1992, 12) defined engagement as "students' psychological investment in and effort directed toward learning, understanding, or mastering the knowledge, skills, or crafts that academic work is intended to promote". Past research (Fredericks et al. 2004; Kong et al. 2003) has highlighted three dimensions to engagement: behavioural, emotional/affective and cognitive. Behavioural engagement is about the active participation in learning activities, emotional engagement is the students' attitudes (e.g. perceived value, interest in) towards the activities, and cognitive engagement can be seen as the 'psychological investment' mentioned previously. Meece et al. (1988) defined cognitive engagement in terms of students' use of metacognitive and self-regulation strategies. Kong et al. (2003) found that approaches to learning (e.g. deep, surface) were closely related to cognitive engagement. Fredericks et al. (2004) included motivational goals and self-regulated learning under cognitive engagement. Looking at Feng's paradigmatic positions on enrichment, we can see that the notion of engagement in fact covers most of these positions – whether it is increasing interest or tackling negative stereotypes through attitudinal engagement, increasing participation in mathematics through behavioural engagement, or it is the enhancement of mathematical learning and development of understanding through cognitive engagement. Therefore, a key outcome from enrichment activities seems to be these different aspects of engagement in mathematics.

Evaluation of the Ri Secondary Mathematics Masterclass programme

In 2008, the Royal Institution commissioned the Curriculum, Evaluation and Management (CEM) Centre to carry out the evaluation of their Secondary Mathematics Masterclass programme (Barmby et al. 2008). There were ten specific objectives for the masterclasses, which we categorised under the following areas of impact: (a) Attitudes towards mathematics; (b) Doing mathematics; (c) Participation in masterclasses; (d) CPD opportunities from masterclasses; (e) Facilitation of the masterclasses. The first three objectives again came under the notion of engagement and are hoped for outcomes. An additional outcome of CPD opportunities for teachers was identified, along with the aim of facilitating and developing the masterclasses over time. We describe an example of a masterclass ('The Power of Two') in the Appendix. In order to evaluate the programme therefore, and based on the previous discussions on enrichment and engagement in mathematics, we recognised the importance of differentiating between what can make up enrichment activities (one could say the input) and the desired outcomes from such activities (the output). In order to clarify our view of mathematical enrichment then, especially with regards to the Secondary Masterclasses, we put forward the following structure of the enrichment process:
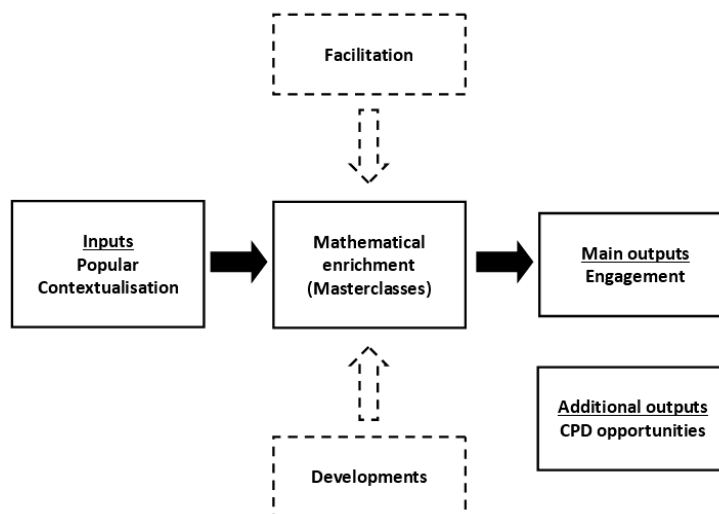

Figure 1: A structure for the mathematical enrichment process

In the model, we specifically split up the inputs and outputs of mathematical enrichment. For example, we have provided an example characteristic of a possible enrichment activity, namely contextualising mathematics within pupils' experience. Looking at the outputs, the evaluation clearly identified particular student outcomes from the Secondary Masterclasses, both from the quantitative results of a student questionnaire, and from qualitative interviews with students carried out during case study visits to masterclasses. The student questionnaire (see Appendix and also Barmby et al. 2008), completed by 917 students, produced the following key findings:

- 64.2% of the students agreed or strongly agreed that their attitude towards mathematics had improved due to the masterclasses;
- 69.6% of the students felt that their ability in mathematics had improved due to the masterclasses.
- 59.3% of the students also agreed that the masterclasses had encouraged them to study maths in the future.

These findings correspond to the three components of engagement identified previously – namely attitude, cognitive (or the result of cognitive engagement) and behavioural. These specific outcomes were also highlighted in the interviews with students:

> "Masterclasses have made me like maths more and now if I've finished, Sir gives me more stuff to do from different books and things."

> "When I was in primary, I didn't really get it at all and after coming here I am really good at it."

> "Yes, I'm more confident in maths now and I seem to enjoy it because I'd like to go on to a maths career when I'm older."

In addition to these outputs for students, some additional benefits to the masterclasses were identified as well, in particular benefits for teachers attending the masterclasses, especially in terms of professional development and providing materials that they could use in the classroom:

> "There are always plenty of ideas! We always come with a book and write things down and come back to school and try and use some of the things that we have picked up."

> "What is really nice is the teachers coming, because they can then take it back to school. What my teachers in [town name] usually do is the two or three that have been, go back and teach the rest of the class. Not all the topic, but bits of the thing."

Having identified the important outputs from the programme, the next step was to identify why the masterclasses were perceived positively, and the evaluation highlighted three factors. The role of the activities was identified by the researchers during their visits to various masterclasses, where practical activities were seen to enthuse students and get them involved in 'doing mathematics'. Activities were also highlighted by students in their interview comments.

> "I just like puzzles, sometimes in maths you just get told stuff or you just have to work out things, whereas in those ones you're actually making things and doing things that are imaginative."

> "You learn yourself through experimenting rather than just being told something."

> "It is different methods and stuff and we put into practice Pythagoras' theory and type of stuff. You cover it at school, but here you actually do it."

In addition to the activities being practical, the evaluation highlighted that the activities were different to what students were doing in school.

> "I think it is definitely more outside the box than addition and subtraction!"

> "I like maths, but I don't enjoy it very much in school at the moment … I think it is just in school, you do not get much chance to learn something new, most of you are just going over the same things."

> "Before this I didn't think paper folding was maths, so it's made me see that maths is more than just numbers and algebra. I have seen the wider side of maths."

Finally, the fact that the masterclasses had shown students the usefulness and importance of mathematics was highlighted.

> "I think they are good because they show you how maths is used in the real world. Used to solve problems."

Case studies from the QCA engaging activities project

The evaluation of the mathematics masterclasses above identified both benefits for students and teachers participating in the classes. In the project 'Engaging mathematics for all' led by the then QCA (now QCDA), one of the authors looked into how masterclass-type activities could be used in the classroom for a wider range of learners. We worked with two institutions; a very high-achieving girls' state school and an all-boys technology college (perceived by the staff as a difficult school). We gathered written and oral statements, some of them on video, from both pupils and teachers. Other information came from observations carried out by one of the authors. We attempted to isolate here the characteristics of the developed tasks (inputs), the impact on learners (outputs) and teachers (additional outputs). The work on the case studies drew our attention to issues that require further research.

Characteristics of the developed tasks (inputs)

Teachers from both schools wished their pupils to think more deeply about what is learnt; make not just connections across the curriculum but between different areas of mathematics; and expand their comfort zone. Both welcomed hands-on activities; in the girls' school the majority of the pupils could feel out of their depth when asked to put their knowledge into practice. The boys' school identified kinaesthetic tasks as a way to engage their pupils and use the sense of achievement that comes from 'making something' as a way of creating a positive experience associated to mathematics (Le Roux and Santos 2009). Pupils worked on the mathematics underlying fashion design and architecture, connecting geometry with algebra. They studied representations of 3D objects in 2D; Picks' Theorem; perimeter and area through the story of Queen Dido of Carthage; stability of shapes; paper folding dodecahedra and buckyballs; links between fractions, algebra and paper folding through folding 'perfect thirds'. Some visited central London's architectural icons and interviewed a team of architects and designers. In some activities older pupils that had previously been taught the activity helped as classroom session leaders and mentors. For details see Le Roux and Santos (2009), Santos (2009), Teachers' TV (2009) and Ri (2009). The accompanying videos 'Cutting the cloth to fit' and 'The Gherkin shapes up' can be found on the QCDA's curriculum website http://curriculum.qcda.gov.uk.

Impact on learners (outputs)

Some impacts were observed by the teachers and by the researcher, whereas others were verbalised by the pupils involved. Although the collected accounts are few, the information gathered motivates a future research study. During the activities, pupils were on task, showed commitment, interest and effort, had a positive attitude in class and behaved well. The pupils engaged in conversations related to the work proposed; in lessons that were led by pupils, their response was positive with commitment and respect from both the mentees and the mentors. It was observed that the pupils actually did mathematics. The teachers summarised it as a 'general buzz around the mathematics department', that for the duration of the case study pupils manifested improved attitudes towards mathematics inside and outside the classroom, such as positive attitude while working, pride in their work, had fun and built confidence in their own abilities to tackle the tasks. Pupils felt challenged, made connections within mathematics and across subjects, claimed to think more deeply about the concepts involved, were able to improve their practical skills within the context of

mathematics. In the corridors several pupils asked the teachers about the mathematical paper folding work on display at the school entrance. Teachers from other subjects got interested in the project as a consequence of the displays. The younger pupils benefitted from being taught by their older peers as they felt more comfortable to ask questions. The older students explaining mathematics to younger ones benefited from having to think deeper about the concepts involved.

Impact on teachers (additional outputs)

The process of development, trialling and tailoring such activities in schools resulted in professional development for the teachers. From their reflections and from observations made by the researcher, the following specific benefits were identified: expansion of one's comfort zone; coping with being challenged; understanding what it is to be a learner; feeling excited about mathematics and the teaching of it; feeling energised and motivated to change practice to incorporate more rich-tasks in everyday teaching; learning and doing mathematics; understanding the value of collaborative work as a teacher; building positive experiences to refer to in future. This work subsequently raises questions about engagement in teachers and not just the pupils. Can teachers provide their pupils with an engaging experience if they are not having regular engaging experiences in the subject? What constitutes engaging experiences for teachers and how do these translate into the classroom? These are issues to examine in future research activities.

Discussion of the findings and implications for enrichment activities

In looking at the literature on enrichment in mathematics, we identified the fact that the term 'enrichment' was actually difficult to define. What we did identify was that possible definitions were based on the hoped-for outcomes from enrichment activities. In fact, when we also examined the notion of 'engagement' in mathematics, what we found was that this largely described the outcomes we were looking for from enrichment. Therefore, based on our discussions in this paper, we propose a rather simplified definition of enrichment as simply activities that bring about engagement in mathematics. In this context then, we need to be clear about what we mean by engagement, and we have used the definition provided by the research (Fredericks et al. 2004; Kong et al. 2003) which includes the three strands of behavioural, emotional/affective and cognitive engagement. When we are considering enrichment activities then, we need to look for these three outcomes. Research has suggested that the first two components – behavioural and emotional/affective – may in fact be quite closely associated, with a person's attitudes possibly predicting behaviour (Crano and Prislin 2006). However, we need to insure that the third component of cognitive engagement is present as well. We want students to 'do mathematics', to take part in mathematical thinking, to develop their knowledge and understanding of the subject, as well as simply taking part in activities and enjoying or being interested in them.

In fact, from the two enrichment programmes that we have examined in this paper, we can see this broad coverage of the components of engagement from the enrichment activities. For example, in the engaging activities projects, although teachers did talk about 'hands-on' activities and kinaesthetic tasks, and also positive experiences, they also spoke about their wish for students to think more deeply about the mathematics. From the secondary mathematics masterclasses, we identified the importance of the activities themselves, rather than say the presenters of the masterclasses, or the facilities available to the students, as being the important factor

in whether students were engaged in mathematics. Although the novelty and the usefulness of the mathematics were also positive issues that were raised by the students, in themselves, these would only perhaps impact on the affective and behavioural aspects of engagement. However, students also reported their cognitive engagement in the masterclasses, how the sessions promoted active thinking, 'making them think' and 'keeping their brain going', as they were encouraged to find solutions to problems and be engaged in harder mental work.

> I don't feel like a master either! If I was a master I would get all the problems straight off! But it does make me think. It doesn't make me think I am amazing, it makes me think how I can solve this.

Therefore, when designing enrichment activities, it is important that all these components of engagement are catered for.

This recommendation for enrichment brings us back to the view of enrichment put forward by Piggott (2004) where problem solving and mathematical thinking was integral to the enrichment process. Indeed, we emphasise once again that clarity is required in ensuring that cognitive engagement is an outcome of the enrichment process, whether that is achieved through problem solving activities or other open-ended activities. We end with the following quote from Leone Burton (1984, 9):

> "Mathematics is used to solve useful problems; it can be played with in a creative way to see what can be discovered; it is the basis on which amusing puzzles can be invented; it has a great power to inform. But they are not the best reason. The greatest value of this approach is in the effect it has in the classroom."

In engaging students through enrichment activities in mathematics, the possible benefit is the impact on the learning and understanding of mathematics, over and above the enjoyment and participation of students.

References

Barmby, P., K. Jones, D. Kokotsaki, F. Ndaji, and J. Searle. 2008. *Evaluation: Royal Institution Secondary Mathematics Masterclasses programme*. Durham: CEM Centre.

Burton, L. 1984. *Thinking Things Through: Problem Solving in Mathematics*. Oxford: Blackwell.

Crano, W. D., R. Prislin. 2006. Attitudes and persuasion. Annu. Rev. Psychol. 57: 345-374.

Feng, W. Y. 2005. Conceptions of Enrichment. *Paper presented at Cambridge Educational Research Association Conference, Cambridge, April 2005*.

Feng, W. Y. 2006. Conceptions of Mathematics Enrichment. *Paper presented at BERA conference, Warwick, September 2006*.

Fredericks, J. A., P. C. Blumenfeld, and A. H. Paris. 2004. School engagement: Potential of the concept, state of the evidence. *Review of Educational Research* 74(1): 59-109.

Kong, Q-P, N-Y Wong, and C-C Lam. 2003. Student engagement in mathematics: Development of instrument and validation of construct. *Mathematics Education Research Journal* 15(1): 4-21.

Le Roux, H., S. Santos. 2009. Unfolding geometry. *Mathematics Teaching* 216: 16-19.

London Mathematical Society / Institute of Mathematics and its Applications / Royal Statistical Society. 1995. *Tackling the mathematics problem*. London: LMS.

Meece, J. L., P. C. Blumenfeld, and R. H. Hoyle. 1988. Students' goal orientations and cognitive engagement in classroom activities. *Journal of Educational Psychology* 80(4): 514-523.

Newmann, F. M., G. G. Wehlage, and S. D. Lamborn. 1992. The significance and sources of student engagement. In *Student engagement and achievement in American secondary school*, ed. F. M. Newmann, 11–39. New York: Teachers College Press.

Park, S-Y. 2005. Student engagement and classroom variables in improving mathematics achievement. *Asia Pacific Education Review* 6(1): 87-97.

Peterson, P. L., and E. Fennema. 1985. Effective teaching, student engagement in classroom activities, and sex-related differences in learning mathematics. *American Educational Research Journal* 22(3): 309-335.

Piggott, J. 2004. Mathematical enrichment: What is it and who is it for? *Paper presented at BERA conference, Manchester, September 2004*.

Ri. 2009. *Engaging Mathematics Activities*, www.rigb.org/engagingmaths.

Santos, S. 2009. Stretching the comfort zone. *Mathematics Teaching* 216: 27-31.

Smith, A. 2004. *Making Mathematics Count: The Report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education*. London: DfES.

Teachers' TV. 2009. *Secondary Maths – Mathematics for all - 3D awareness*. http://www.teachers.tv/video/37908.

Appendix - Questionnaire

The questionnaire on the Masterclasses for pupils contained the following items:

How I feel about the Masterclasses
(five point scale of strongly agree, agree, neither agree nor disagree, disagree, strongly disagree)
Example questions from the 20 items in this section included:
1. I enjoyed the classes
2. The masterclasses showed me the importance of mathematics
3. Following the masterclasses, I now expect to do better in maths courses that I take
4. have enjoyed the social side of the masterclasses
5. learnt a lot of mathematics from the classes
6. The masterclasses have encouraged me to study maths in the future

Quality of the masterclasses
(five point scale of excellent, very good, good, poor, very poor)
1. Quality of the presenters and their presentations
2. Quality of the activities you carried out during classes
3. Quality of the facilities where the classes were held

Possible impact of the masterclasses
(five point scale of strongly agree, agree, neither agree nor disagree, disagree, strongly disagree)
1. My attitude towards mathematics has improved due to the masterclasses
2. My ability in mathematics has improved due to the masterclasses

Appendix - 'The Power of Two'

A mathematics masterclass is a two-and-a-half hour interactive session. The Power of Two is inspired in the Josephus problem: given a circle of *n* people where every alternate living person is killed in succession, where should one stand in order to be the last person to be killed and hence to survive? Through a mixture of pattern-recognition, problem solving and group work, the challenge provides an informal and accessible introduction to number systems (in particular, place value and binary representation) and to the notion and importance of proof.

Saving Further Mathematics?

Dr Jeff Searle

Centre for Evaluation and Monitoring, Durham University

The crisis in science, technology, engineering and mathematics (*the STEM crisis*) has come to the fore in recent years, but problems in mathematics education have persisted for many years. Although there have been many reviews and resulting initiatives none of the attempts to solve the problem has been successful in the long term. This paper reports on research carried out for Mathematics in Education and Industry (MEI) on the impact of the Further Mathematics Network (FMN), which was formed in 2004 to attempt to halt and reverse the large decline in the numbers of students taking further mathematics at A-level. The origin of the FMN in the light of the *STEM* crisis and *Curriculum* 2000 is discussed, as is an analysis of the statistics on the subsequent growth in student numbers. This analysis highlighted two types of schools and colleges; those where numbers had grown substantially in recent years, and those where the institution was able to offer further mathematics despite a small take up. Interviews were conducted with the teacher responsible for Key Stage 5 Mathematics in both these types of institution, and reasons sought for why numbers had grown or how courses could be offered "in house" to a small cohort of students. Factors that emerged included the reputation of the department and institution within its locality, the changes made in 2004 to mathematics specifications by the awarding bodies, flexibility in option block and timetabling arrangements, recognition by students that a qualification in further mathematics is a valuable career asset and support for the students and enthusiasm for mathematics from the teaching staff and the Network. These findings are discussed further in the light of the recent consultation on the future of level 3 Mathematics by QCA.

Introduction and background

There have been concerns for many years about the number of young people coming through the education system who are both suitably qualified and sufficiently motivated to study the subject beyond age 16 and into mathematics related higher education courses and careers. In the government enquiry into the decline in university entrants in science and technology, (Dainton, 1968) the report noted the importance of mathematics to a modern economy. Cockcroft (1982, paragraph 619*)* noted the shortage of good teachers of mathematics but that all pupils should be taught by teachers who are well qualified and enthusiastic. Despite several subsequent initiatives to engage and enthuse young people with mathematics a joint report by the London Mathematical Society, The Royal Statistical Society and the Institute of Mathematics and its Applications (1995) highlighted the persistent problem.

> There is an unprecedented concern amongst mathematicians, scientists and engineers in higher education about the mathematical preparedness of new undergraduates. There is also a long-standing worry about the number of prospective students in these disciplines.

Smith (2004, paragraph 0.3) in his inquiry into post 14 mathematics education, cites the Roberts Report (2002) that during the 1990s there had been a near 10% decline in the number of candidates taking A-level mathematics to about 60,700 in 1999. Smith highlighted the importance of mathematics both as an intellectual discipline in its own right, but also as underpinning much of the scientific and industrial research essential to a modern economy. Smith identified a perception by young people that mathematics is boring and irrelevant, noting this and many other factors were influential in the decline in numbers. These factors included the perceived poor quality of the teaching and learning experience, the perceived relative difficulty of the subject, the failure of the curriculum to excite interest and provide appropriate motivation and the lack of awareness of the importance of mathematical skills for future career options and advancement.

Porkess (2006, 2-5) in his review of recent changes in the structure and provision of A-level mathematics quoted more startling figures, with numbers taking A-level mathematics falling from about 85000 in 1989 to 66000 in 1993 and then to an all time low of about 50000 in 2002. Porkess highlighted a similar drop in numbers taking further mathematics A-level from about 15000 in 1980 to around 5000 in 2005. In further mathematics the perceived difficulty of the subject by 16 year olds was compounded by a shortage of teachers able to teach it and the disinclination of head teachers and college principals to staff it as a timetabled curriculum subject due to relatively small take up by students. In many schools and colleges further mathematics disappeared from the curriculum. The major revision to all A-level subjects in *Curriculum* 2000, accentuated the problem for further mathematics as part of the philosophy behind *Curriculum* 2000 was a broadening element to a sixth form student's curriculum, whereas to take both mathematics and further mathematics at A-level was seen by many to be narrowing.

In this climate of a declining number of students taking further mathemtics, in 2000 the curriculum development organsiation, Mathematics in Education and Industry (MEI) initiated its *Enabling Access to to Further Mathematics Project*. The aim of the Project was to make further mathematics available to all sixth form students who could benefit from studying it irrespective of whether their school or college offered it in its curriculum. The Project also aimed to raise the general awareness of further mathematics as an AS-level and A-level subject amongst sixth form students as many might have been unaware of its existance let alone the benefits it had to offer to any student planning to study a mathematics related course at university. The Project began with four centres in England, where students could meet with each other and a personal tutor to discuss their progress in self-supported study using text books and web based resources provided by MEI. These four centres would evolve to become the 46 centres of the Further Mathematics Network (FMN).

The Smith Report (2004, paragraph 4.38) was instrumental in bringing about the FMN.

> Following the revision of the GCE criteria for Mathematics following the Curriculum 2000 debacle, many respondents are in no doubt that A-level mathematics has been made easier for the very best candidates. In terms of the potentially most able mathematics students, the Inquiry believes that far too few able candidates are entered for AS or A-level Further Mathematics because their school or college do not have sufficient resources to provide these courses …. University departments in all subjects identified as vulnerable in the Roberts report would benefit greatly if more candidates were qualified at this level.

The response by the DfES (2004, 43) initiated the formation of the FMN.

> To encourage the increased take up of Further Mathematics we will also develop proposals to replicate and expand the current MEI Project with a view to establishing a Further Mathematics Centre in each of the 47 local Learning Skills Council areas.

The setting up of the FMN by MEI can be seen as part of the response to what had become to be called the *STEM* crisis, where similar concerns to those in mathematics had been raised in science, technology and engineering as well. The *STEM* crisis and the responses to it were highlighted in the DfES/DTI report (2006), and again by Sainsbury (2007) in his review.

The Centre for Evaluation and Monitoring at Durham University was commissioned by MEI, to monitor the development of the FMN and to evaluate its impact. Part of the evaluation was to analyse the data on the take up and achievement in A-level and AS-level further mathematics during the development of the FMN.

Data analysis 2003-04 to 2007-08

The baseline year for the data analysis was taken as 2004 as that was the year in which the FMN formally started work, with funding provided by the DfES. The data on A and AS-level entries was derived from the National Pupil Database for Key Stage 5, made available by the DCSF. The figures in Tables 1 and 2 below represent the number of candidates that were awarded a grade, subdivided by gender.

Table 1   Entries in A-level further mathematics

A-level 2004			A-level 2006			A-level 2008		
female	male	total	female	male	total	female	male	total
1418	3711	5129	1924	4598	6522	2605	5918	8523
28%	72%		30%	70%		31%	69%	
Percentage increase on 2004			36%	24%	27%	84%	59%	66%

Table 2   Entries in AS-level further mathematics

AS-level 2004			AS-level 2006			AS-level 2008		
female	male	total	female	male	total	female	male	total
897	1689	2586	1542	2533	4075	2021	3463	5485
35%	65%		38%	62%		37%	63%	
Percentage increase on 2004			72%	50%	58%	125%	105%	112%

It can be seen that growth in numbers has been substantial in both the A-level and the AS-level entries, the latter more than doubling over this 5-year period. A substantial number of the total entries achieved were at grade A, this being 58% for the A-level and 39% for the AS-level in 2008, with similar figures for the previous years. It is notable that there are considerably more male entries than female entries; the male entry has constituted about 70% of the total entry for A-level and about 65% of the AS-level entry over the last few years. However, in terms of growth in numbers since 2004, it can be seen that proportionally more girls are being attracted into the subject than boys. Further analysis of the data showed that the growth in numbers was substantially in the state sector compared to the independent sector. Schools and colleges in the state sector accounted for 74% of the growth in A-level numbers between 2004 and 2008 with a corresponding figure of 90% for the AS-level. The growth in numbers represents a substantial influence of the work of the FMN.

The FMN was set up to address the problem of students who could benefit from taking further mathematics but who could not access tuition. The FMN also

offered support to institutions with a small number of students but also encouraged student take up of the subject in general. In the light of this we analysed cohort sizes from 2004 to 2008 to assess the impact of the FMN at institution level. This analysis is summarised in Table 3 and Table 4.

Table 3   Cohort sizes in A-level further mathematics

A-level 2004			A-level 2006			A-level 2008		
Cohort size	Schools Colleges	Students	Cohort size	Schools Colleges	Students	Cohort size	Schools Colleges	Students
	1088	5129		1267	6522		1482	8523
1	21%	5%	1	20%	4%	1	21%	4%
5 or less	74%	42%	5 or less	69%	35%	5 or less	66%	29%
10 or less	92%	69%	10 or less	90%	64%	10 or less	87%	58%
15 or less	97%	83%	15 or less	95%	77%	15 or less	93%	70%
30 or less	100%	100%	30 or less	99%	93%	30 or less	99%	91%

Table 4   Cohort sizes in AS-level further mathematics

AS-level 2004			AS-level 2006			AS-level 2008		
Cohort size	Schools Colleges	Students	Cohort size	Schools Colleges	Students	Cohort size	Schools Colleges	Students
	933	2585		1109	4075		1319	5485
1	45%	16%	1	34%	9%	1	34%	8%
5 or less	89%	63%	5 or less	83%	49%	5 or less	78%	40%
10 or less	98%	87%	10 or less	93%	71%	10 or less	93%	66%
15 or less	99%	93%	15 or less	98%	85%	15 or less	97%	79%
30 or less	100%	99%	30 or less	100%	96%	30 or less	99%	90%

The data on changes in cohort sizes reflects the growth in student numbers over the time period as the cohorts have in general got larger. However many students are still studying in cohorts of 5 students or less but it notable that the number of institutions offering further mathematics has also shown a substantial increase.

The growth in entries as seen in Tables 1 and 2 and the cohort sizes as seen in Tables 3 and 4 raised two research questions. Firstly were there particular institutions where there had been a large growth in numbers in either A-level or AS-level entries, and if so could the factors that influenced this growth be identified. Secondly, where institutions had small entries but were teaching further mathematics "in house", how was this being achieved. Telephone interviews were conducted, with the teacher responsible for Key Stage 5 Mathematics, usually the Head of Department (HoD), to investigate these questions.

Interviews with Heads of Department on growth in student numbers

In May 2008 approaches were made to the HoD at 40 institutions where growth had been identified between 2005 and 2007, asking to interview them. The criterion for selection was that an institution had entered 5 or fewer students in 2005 and 10 or more in 2007 at either A-level or AS-level in further mathematics. This criterion was met by 32 institutions at A-level and by 39 at AS-level. The sample drawn from these institutions included comprehensive schools, state grammar schools, post 16 colleges and independent schools. The criterion chosen was arbitrary in the sense that other criteria would have identified other institutions, but was considered adequate to gather HoD's views on why growth had occurred in their school or college. The response rate was 9/20 for the A-level (45%) and 13 /20 for the AS-level (65%). HoDs were asked to comment on both A-level and AS-level. The five main factors identified as leading to growth were as follows.

1 The reputation of the school and department in the locality.

If a department has built up a reputation for providing good quality teaching that leads to good results at A-level, it attracts students. It was apparent that once further mathematics became established, it attracted students. Some HoDs had started by offering the subject as extra curriculum using a post 4:00 pm slot or a sports afternoon to make time available for lessons. With growing interest and success, senior management had eventually allowed further mathematics onto the timetable. The commitment of the HoD was notable here, particularly those who had been appointment fairly recently, in wanting to offer further mathematics to their students.

2 The changes made in 2004 to the mathematics specifications

Many HoDs noted that further mathematics had become accessible to a much greater range of students due to the changes in the specifications in mathematics modules introduced in 2004. Some departments are now able to offer a 9 module course in mathematics, in which students aim for both A-level mathematics and AS-level further mathematics within one class over a two year period. In some schools and colleges where numbers made it possible, there was considerable flexibility in that students were able to continue to the full further mathematics A-level if they wished to or drop back to just mainstream mathematics. Most HoDs noted that the 2004 changes had done much to dispel the image of further mathematics as difficult and demanding and thus only for the most able. In this context some HoDs raised fears about changes that may be made again to the modular structure in 2012 in that changes could result in similar problems to those that followed *Curriculum* 2000, and numbers in A-level mathematics and further mathematics could plummet again just as they are becoming re-established and growing steadily.

3 Changes to timetable and option block structure

Most post-16 institutions which offer A-levels operate a timetable based on option blocks. In some schools and colleges the popularity of mathematics has grown to where it appeared in several option blocks, and if numbers warranted it, further mathematics would also be an option with an intention that those students who study both would take at least 12 modules. Again the flexibility of the modular structure of mathematics and further mathematics was stressed in meeting particular student's needs. Some HoDs had struggled to get further mathematics onto the timetable, and it was apparent that the attitude of the senior management to further mathematics was essential to this. Often, where numbers had grown and senior management gave full support to the development of further mathematics, a member of the team was a mathematician or scientist. Senior management in general endorsed further mathematics once it was seen to be bringing success to the school or college, as seen in the examination results.

4 Recognition that a qualification in further mathematics is a career asset

Many HoDs spoke of a growing enthusiasm for mathematics in their institution, which is probably related to enrichment type activities. It was reported that many students went on to study mathematics related degree courses in higher education, such as engineering, computer science and economics. Students were coming to see the value of studying further mathematics, at least to AS-level, as being an asset on their application forms, and also of being of benefit to their studies once their degree

course had begun. Some HoDs had introduced further mathematics because they had students who were asking to study it.

5 Support and a general enthusiasm from the teaching staff for mathematics.

Many of the HoDs noted they were fortunate in that their staff were largely well qualified in mathematics. Most staff were involved in supporting students in some way. For A-level study, this often involved a "drop-in clinic", or a specified time when staff would be available for help with any problems. Some HoDs had introduced mentoring systems, in which older students helped younger ones. Nearly all HoDs spoke enthusiastically of the involvement of their staff and students in the UKMT Challenges, at junior, intermediate and senior level and face-to-face team competitions. Some HoDs spoke of their aim to make mathematics both challenging and fun. Thus from Year 7 onwards, they would have extra-curricula activities like puzzle solving, or they would extend the mathematics curriculum through for example, GCSE statistics or Free Standing Mathematics Qualifications (FSMQ) in Year 11. Opinion was split as to whether it was a beneficial idea to introduce the AS-level mathematics core work in Year 11. Some thought this gave students an insight into sixth form study and a helping start, whilst others thought it premature and better to pursue other aspects of mathematics. However, all departments were active in encouraging suitable Year 11 students to continue their study of mathematics after GCSE, often involving talks from external speakers on the importance of mathematics and the career opportunities a qualification in mathematics can bring.

No HoD put the development of further mathematics down specifically to the FMN. However, most commented that they thought the Network was important and they valued it, noting it was doing a good job in raising the profile of mathematics.

Interviews with Heads of Mathematics Departments concerning "in-house" teaching of further mathematics to small cohorts of students

Towards the end of 2008, the FMN identified schools and colleges in the Network where a decision had been taken to teach at least some of the further mathematics modules themselves in 2008/09 despite having relatively small numbers of students, rather than using FMN tutors. In total, 52 such schools and colleges were identified of which a sample was contacted for interview. Interviews were conducted during January 2009 with the teacher responsible for Key Stage 5 Mathematics in 33 of these institutions, covering a wide range of types of school and colleges and most areas of England. The main points arising from the interviews are as follows.

1 Managing provision in further mathematics

Most HoDs had persuaded their senior management that further mathematics should be in the sixth form option blocks and given timetabled time. However, this was rarely a full allocation of teaching time. Typically well established subjects would have 5 hours, whereas further mathematics would be allowed 2 or 3 hours. This was often said to be adequate for a small group of able and committed students; in some institutions the time was supplemented with voluntary extra time. Some senior managers supported further mathematics as it brought kudos to their institution and as such, was a good marketing tool in both attracting and retaining able students.

There is an apparent difficulty for HoDs as to when students actually begin their study of further mathematics. For some it is only offered as an AS-level course

in Year 13, whereas others "parallel run it" with the main mathematics course in both year groups. These are the extremes of many flexible arrangements that allow students to go as far in mathematics as they want to. The main problem encountered is the pre-requisite knowledge required for some of the further mathematics, particularly in the pure mathematics but it was said able students can cope with this. Some believed only high calibre students could successfully begin further mathematics at the start of Year 12. It was noted that for students who wish to take extra modules beyond examination requirements, funding is not available to cover any teaching.

2 Involvement with the Further Mathematics Network

All these institutions were registered with the FMN and were to some extent in contact with their local Centre Manager, even if this meant just receiving information through e mail. Some HoDs had appreciated the help and advice they had received from the local Centre Manager in initiating further mathematics, some saying they had found it invaluable. Many made use of the MEI online resources and encouraged their students to attend revision and enrichment events. Students who had attended such events had largely found them enjoyable and useful; they particularly liked meeting with students from other institutions. However, other commitments often prevented attendance. The timing of these events was raised by many as an issue and some noted small local events might be preferable to large regional events.

3 Mathematics in the institution as a whole

In general the HoDs considered the attitude of pupils and students in their institution to be positive towards mathematics particularly amongst the more able. In all schools there was some sort of enrichment in the mathematics programmes for all year groups. Involvement in the UKMT Maths Challenge and Team competitions was common and also in some more local competitions. Pupils clearly enjoyed taking part of in these. Many schools had "maths days" or similar in which pupils would solve puzzles or be given "real world" problems. External visits, or visitors to the school, were occasionally organised, with *Maths Inspiration* and *Murderous Maths* being mention by several HoDs. Pupils in Year 11 who had shown an aptitude for mathematics were encouraged to consider taking further mathematics. All schools and colleges had at least some members of the department who are well qualified in mathematics and able to teach mathematics at A-level and at least some of the further mathematics modules. Many noted the need for Continuing Professional Development (CPD) in this respect, particularly for younger members of staff, and noted the need for further development of online provision of CPD.

Discussion

It was clear from the interviews that the FMN has provided far more that just support for students. Enrichment events and revision sessions provided by the FMN on most A-level mathematics modules, especially the pure core, are valued. Enrichment events are aimed at inspiring both sixth form and younger students, particularly in Years 10 and 11, from whom the next cohorts of AS and A-level students of mathematics and further mathematics will come. These events often take place in the local university letting young students experience this environment for the first time and also to see a presentation by a professional mathematician. If the numbers taking mathematics and *STEM* related courses at A-level and in higher education are to grow then it is

essential to engender interest and enthusiasm for mathematics prior to age 16. The FMN enrichment events contribute substantially to that aim.

It was also clear from the interviews that one of the main reasons for attracting students to further mathematics was the flexibility of the current arrangement in which some modules can count towards either mathematics or further mathematics. Many institutions have used creative timetabling in which Year 12 and 13 students can study some modules together. If this flexibility were to be removed as was proposed in the QCA consultation on A-level provision in 2009, then there could be dire consequences for the number of students taking further mathematics. It is notable that some professional bodies, such as The Royal Statistical Society and MEI, wrote a position paper in response to the consultation, proposing there be no change. Many of the issues involved have been discussed in the Institute of Mathematics and its Applications publication, *Mathematics Today*. (2009). At the time of writing the outcome of the QCA consultation is not known, but the research reported here shows the post 2004 specifications have been instrumental in creating the growth in student numbers and it would seem sensible to leave the current arrangements alone. Any substantial change could have a detrimental affect on future student numbers.

References

Dainton, F. S. 1968. Enquiry into the flow of candidates in science and technology into higher education. London: HMSO.

Cockcroft, W. H. 1982. Mathematics counts: Report of the Committee of Enquiry. London: HMSO.

London Mathematical Society / Institute of Mathematics and its Applications / Royal Statistical Society 1995. Tackling the mathematics problem. London: LMS.

Roberts, G. 2002. SET for Success: The supply of people with science, technology, engineering and mathematical skills. London: HMSO

Smith, A. 2004. Making Mathematics Count: The Report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education. London: DfES.

DfES 2004. Making Mathematics Count; response to the Smith Inquiry into post 14 Mathematics Education.

DfES / DTI. 2006. The Science, Technology, Engineering and Mathematics Programme Report. London: DfES.

Porkess, R (2006). Unwinding the vicious circle, paper presented to the IMA Conference, 2006.

Sainsbury, D. 2007. The race to the top: A review of the government's science and innovation policies. London: HMSO.

MEI. 2009. QCA consultation on level 3 mathematics: MEI position paper.

The Royal Statistical Society. 2009 Position statement on GCE AS/A level mathematics subjects for the QCA consultation.

The Institute of Mathematics and its Applications, 2009. *Mathematics Today*, vol 45, No. 4.

A Student's Symbolic and Hesitant Understanding of Introductory Calculus

Angela Smart, PhD Candidate

University of Ottawa, Canada

In this paper I discuss a study that looked at one student's understanding of calculus, and used the framework of Tall's theory of Three Worlds of Mathematics to determine the embodied and symbolic nature of that understanding. Initially, the student's understanding of calculus was explored through a task interview using calculus questions designed to elicit embodied and/or symbolic understanding. Results showed that this student predominantly demonstrated a symbolic understanding, with a very limited embodied understanding on the particular tasks. It was also during this interview that the student exhibited the phenomenon of searching for reassurance as to whether he was answering the task and interview questions correctly. This paper discusses this search for reassurance, speculates on potential causes, and argues that there may be a relationship between this search for reassurance and the student's symbolic understanding of calculus.

Keywords: reassurance, calculus, symbolic understanding, didactic contract, institutional norms, three worlds of mathematics

Introduction

The research presented in this paper was part of a study that sought to explore the experience of a first year undergraduate calculus student and how that student's experience in calculus may have contributed to his symbolic (algebraic procedural) understanding and/or his embodied (physical or geometric) understanding, of calculus, and particularly the connections between these understandings. This initial question was examined with regard to a portion of a particular philosophical theory, namely Tall's three worlds of mathematics. The data for this study were collected from a clinical task and reflection interview with one university student who had recently completed a first year calculus course. The purpose of the interview was to examine the understanding the student demonstrated on calculus tasks and how the student spoke about his understanding and experience in introductory calculus. During the clinical interview, an unexpected phenomenon appeared: the student's need for reassurance that he was correct. This was surprising, as the student in this study had completed introductory calculus with over a 90% average and thus, it was assumed that the student was confident with calculus. This paper will begin by briefly describing the study as well as summarise the results. I will then discuss the student's need for reassurance that appeared during the clinical interview. In the conclusions of this paper, I will hypothesise some reasons why the student in this study, as well as other students in general, may feel the need to look for reassurance about how they are doing their mathematics.

Background

I developed the research problem of this study from a combination of my own experience as a mathematics student, tutor and instructor as well as an examination of Tall's philosophical theory of three worlds of mathematics (Tall 2003, 2004a, b). This philosophical theory suggests that all types of mathematics can be divided into one of three domains, or worlds: the Conceptual-Embodied world, the Symbolic-Proceptual world, or the Formal-Axiomatic world. The Conceptual-Embodied world involves a combination of Bruner's Enactive and Iconic modes of representation (Bruner 1966; Tall 2003). According to Bruner, the Enactive mode of representation signifies an action or actions while the Iconic mode of representation is attributed to all visual or image organisation. Thus, Tall's Conceptual-Embodied world of mathematics is a combination of actions on organised visual or image representations of the mathematical world (Tall 2003). The Symbolic-Proceptual world is a combination of processes and concepts acting together, or procepts, which utilise the formal symbolic systems such as arithmetic, algebra, or, in the case of this study, symbolic calculus. The Formal-Axiomatic world uses logical deductions, formal definitions, and axioms to construct axiomatic mathematical systems, such as analysis or field theory. For the purposes of this paper, I will refer to the Conceptual-Embodied world as the embodied and the Symbolic-Proceptual world as the symbolic for the sake of brevity.

Tall's theory has been applied to many areas of mathematics. For example, one study used Tall's theory to investigate university students' embodied and symbolic understanding of a specific concept within linear algebra (Stewart and Thomas 2006). Similar to my own investigation, these researchers focused on individual student's understanding within the explicit worlds. Their findings conclude that developing an understanding of the connections between the embodied and symbolic world can contribute to overall success. Nogueira de Lima and Tall (2006) used the theory of the three worlds to try to explain 15-16 year old students' misunderstandings and difficulties with learning the concept of equations. Similar to the previously mentioned study, Nogueira de Lima and Tall also concluded that developing a connection between the embodied world and the symbolic world can contribute to a better understanding of the concepts.

For my study, Tall's theory, and particularly his descriptions of the different worlds of mathematics, was applied to the participant's answers on introductory calculus tasks. This was intended to analyse the different worlds the student's answers could be categorised in to see if there was a cross over between worlds in the questions. From my experience, a traditional first year undergraduate calculus class focuses on the graphical ideas of rate of change and cumulative growth and the use of the rules of differentiation and integration for symbolic manipulation. This was also the case for the course the student in the study took. Thus, I made the assumption, which Tall's theory supports, that there was the potential for a different understanding of the embodied and symbolic worlds in calculus. I then wanted to explore how the experiences of a student could potentially contribute to the different understandings. That is, if a student experienced mathematics as either mostly from the symbolic world or from the embodied, or a combination of the two, would this correlate to the understanding the student developed? The Formal-Axiomatic world was not included in the study since it was not a significant part of the introductory calculus course that the participant had taken.

The Problem

For this study I wanted to know if there was something about an individual's experience that contributed to his or her developed understanding of the mathematics, as categorised by Tall's worlds of mathematics (Tall 2003, 2004a, b). I chose to position this study within the context of a first year undergraduate calculus class[i1] since calculus is required by numerous undergraduate programs, and is one of the most commonly taught mathematics courses at university. For example, at one university in Canada, undergraduate programs in Mathematics, Physics, Commerce/Business, Economics, Engineering, Health Science, General Science, and Environmental Science all require students to complete a course in calculus. Thus, because of the volume of students who are required to take calculus, studying how students' experience contributes to how they develop their understanding could be informative to educators. Thus, the research question for this study was: 'How does a particular student's experience in a calculus course contribute to his development of a symbolic and/or embodied understanding of the mathematics that was taught?'

I chose to begin my investigation by looking at only one student's experience, as this was a pilot study to test the interview task questions. My participant for this study was not chosen at random as I knew Andre prior to the study. Andre had hired me as a private tutor to help him through some pre-university mathematics classes. Despite this connection, I had no contact with Andre in any capacity while he was completing his university calculus course and felt that our prior connections would not bias the data.

I designed the initial research study after the clinical interview techniques for mathematical thinking outlined by Ginsburg (1981). Ginsburg recommends three clinical interview techniques for mathematics education research. These include a discovery technique, which is aimed at discovering the cognitive processes actually used by the individual in a variety of different contexts, an identification technique, which is aimed at identifying the describing the intellectual phenomena that are discovered, and a competence technique that is aimed to establish competence, not just performance. For this study, my interview with Andre was a combination of Ginsburg's discovery and identification mathematical clinical interview techniques. I designed 10 open-ended mathematical tasks to be completed by Andre during our interview for the discovery portion (partially reproduced in Figure 1). These tasks were designed so that they could be answered both in a symbolic and/or embodied way, in an attempt to try to comprehend the understanding that related to the processes that Andre was using. A professor in Mathematics Education with calculus experience reviewed and agreed with me about the design of the tasks. The interview also consisted of a series of semi-structured questions about Andre's calculus class and his previous experiences in mathematics.

Once Andre had attempted all of the tasks, I engaged him in a reflection about the tasks. I prompted him to speak about how his previous experiences in mathematics, and specifically in calculus class, contributed to how he completed the tasks. At all points when I was with Andre, I made written notes of comments Andre made that I deemed important. The semi-structured reflection interview was audio-recorded. Further to my field notes and the audio recording, I collected the paper that Andre used when solving the mathematics tasks in order to have a written record of his problem-solving steps. After the interview was completed, my written notes and

---

[1] Henceforth, when calculus is mentioned, it refers to first year introductory undergraduate calculus, unless otherwise stated.

the interview transcripts were thoroughly read, analysed, and coded according to themes that emerged. I initially analysed the data for signs of Andre's embodied or symbolic understanding of the calculus concepts in terms of Tall's descriptions of these worlds (2003, 2004a, b). For example, I looked for physical drawings or graphs to demonstrate an embodied understanding and I looked for symbolic manipulation and algorithms for a symbolic understanding, to name a few, as well as any combination of these and other processes Andre may have used. I then examined how this related to how he spoke about his experience.

---

2) For the function f (whose graph shown) arrange the following numbers in increasing order and explain your answer: $0, 1, f'(2), f'(3), f'(5), f''(5)$

4) At what point on the ellipse $x^2 + 2y^2 = 1$ does the tangent line have a slope of 1? Show your work and explain your answer.

5) What is meant by "rate of change" in terms of calculus?

9) If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)dt$ represent? Explain your answer.

10) If $f(x)$ is the slope of a trial at a distance of x miles from the start of a trail, what does $\int_3^5 f(x)dx$ represent? Explain your answer.

Figure 1: Example of Calculus tasks

Results

To summarise, the results showed that Andre demonstrated a very strong symbolic understanding of calculus but not a solid embodied understanding. He had a dominant tendency to want a procedure when solving different types of calculus tasks and avoided the use of graphical representations to help in the solving process.

A good example of Andre's symbolic understanding of calculus occurred when he was trying to complete the second question of the tasks (Figure 1). This question showed a randomly drawn curve. Andre began the task by first wanting to identify the function equation of the curve and decided that the function was actually f(x) = sin(x). When I asked Andre why he first wanted to make the curve a sine curve he reported it was: "Because I am probably used to calculating the derivatives and integrals, so I was thinking that what was the derivative of a sine function or cosine function and I tried to see what it would look like..." This response shows that Andre was inclined to gravitate towards what he described as calculating when it came to calculus questions.

The desire to 'calculate' in the questions continued throughout the entire time Andre completed the tasks. When questioned about why he wanted to find procedures or calculations for the tasks, Andre said he was not taught any theory about the mathematics, but that he was only given the formulas and then used them to do example after example. He also said that the examples were very similar to what would then be on the tests, so that's what he learned. Thus, the way in which Andre described his experience in calculus showed an emphasis in the teaching on what Tall would categorise as a symbolic understanding of mathematics (Tall 2003, 2004a, b).

Another example that suggests Andre's strong symbolic understanding had to do with sketching the curve of functions. When I asked Andre what he found easiest in calculus, he stated curve sketching and said that was because "you are given an easy function to trace, you follow your steps...". Andre then explained to me the procedural steps he would follow to sketch a curve. During the explanation, Andre mentioned using a grid method to record the answers. Interestingly, this grid method has nothing to do with the mathematics, but could be described as a socio-mathematical norm (Yackel and Cobb 1996). Thus, from his initial comment and further explanation, including the mention of the grid, I concluded that to Andre, 'easy' means a step-by-step process to follow, or procedures, which usually take place in the symbolic world of mathematics.

Despite Andre's overwhelming tendency to respond to calculus from a symbolic understanding, he did have a selected understanding of calculus as it related to the embodied world. For example, on question 2, once it was clarified for Andre that he was looking at an arbitrary curve, he was able to solve the question by visually representing the different slopes. When I asked him what he looked for when solving this question he said "slopes" and commented on placing them in order according to how much they increased in steepness. Likewise, with question 5, when I probed Andre to explain what he meant by variation in his definition of rate of change he said he meant that he would look between two x values and see how much y has changed in that period. Although this is not a perfectly clear answer, it shows that Andre is still thinking about rate of change or slope as something he can physically see as well as calculate. Overall, from the interview, I concluded that the understanding Andre demonstrated was related to his experience in how mathematics was taught to him. Andre stated that calculus was taught focusing on the procedures and practicing examples using the procedures. Taking this into consideration and the fact that procedures in introductory calculus tend to involve the symbolic world, it is not surprising that symbolic procedures are mostly what Andre learned from the class.

Need for Reassurance within a Symbolic Understanding

It was during the analysis that another theme appeared. This was Andre's tendency to show a need for reassurance that he was doing the tasks correctly or answering questions correctly, such as hesitating on questions to ask me if he was answering correctly. In this section I will summarise my findings in conjunction with the theoretical framework of Sierpinska's that focuses on a similar theme, namely the student's need for the teacher to tell them if they are correct (2007). Throughout the summary, I will demonstrate how Andre's responses can be situated inside this framework and suggest that this need for reassurance can also be linked to Andre's symbolic understanding.

Sierpinska's findings were similar to my own in that the participants of her study demonstrated and expressed a need for reassurance from their instructor (2007). Sierpinska speculates that this need for reassurance, or for the teacher to tell the

student if they are correct, could be attributed to epistemological, cognitive, affective, didactic, and/or institutional reasons (2007). Reviewing Andre's responses, I would categorise his need for reassurance into specifically institutional and didactic reasons, but there could most likely be affective reasons as well.

Andre demonstrated a need for reassurance, which could be attributed to institutional rules and norms, while solving task question #4 (Figure 1). While attempting this question Andre stopped midway through, drew a smiling face, and went on to the next question. This was surprising since Andre was actually solving the problem in a correct manner. When questioned as to why he stopped, Andre said, "Because I wasn't really sure…since we didn't have the word implicit *(referring to implicit differentiation)* I didn't know if I was supposed to do that." This speaks to the convention Andre was taught in class – to only solve a particular way if primed by a particular word. Since that word was not present, Andre was not sure about his action and thus stopped trying to solve the task question altogether. This example speaks to his need of reassurance not just from the instructor, but from his prior experience with calculus questions. In other words, Andre had experienced reassurance when taking calculus in that when he saw the words 'implicit differentiation', then he felt reassured that he was supposed to solve the question in a particular way. Andre's experience of needing a priming word in the question is an example of an institutional norm that somehow became viewed by Andre as necessary for mathematical validity. This example also gives support to the results discussed earlier that Andre has a strong symbolic understanding of calculus. With a symbolic understanding, Andre is comfortable with questions that require step-by-step procedures for solving. Andre had most like developed his understanding in the way Siepinska described, as "justified on the basis of their acceptability by the school authorities, not on their grounding in an explicit mathematical theory" (2007 16). Andre demonstrated here that one of the steps he required for solving question like #4, which was based on his experience with acceptable school authorities, was to see the word 'implicit' written within the question. Sierpinska suggests the reassurance phenomenon could exist because of the institutional rules and norms that students are taught, which equate to mathematical validity. This reassurance can also be attributed to some of the procedures used in solving calculus problems. In class, a particular step-by-step procedure might be taught as the only right way to solve a problem, even if it is only conventions set by the instructor.

Andre also demonstrated a need for reassurance that could be linked to the didactic perspective. When I asked Andre why he had solved task question #9 a certain way and particularly why he sketched the area under the curve, which happened to be a correct way for solving the question, Andre got a little upset and stated "Umm, because that's what I think it was...Okay, I'm a little bit confused." This example shows that when I asked Andre to justify why he solved something in a particular way, he claimed to become confused. I speculate that he was not necessarily confused but started to question his answer and lose his confidence when I asked him to explain what he had done. On further probing it became clear that he thought I was only asking him because he had not solved the question correctly, which was not the case. From the didactic perspective, Sierpinska suggests that in mathematics, the teacher gives the task and it is the student's job to produce an answer. According to this model, Sierpinska claims, "the teacher is assumed to know the correct answer" (2007 15). There is also the suggestion by Sierpinska that there are many tasks in school mathematics where it may be impossible or difficult for the student to verify the answer. This example demonstrates Andre's understanding of the didactic perspective between instructor and student, that is, that the instructor holds

the answers. This example also makes me wonder about the understanding Andre has of the role of questioning in mathematics. Andre lacked the confidence to explain and argue for his method of solving this particular task. Instead, he needed to be reassured that questioning how and why he solved a task in a certain way was not an indication that he had done anything incorrectly. This lack of confidence again may be linked to Andre's symbolic understanding. Since Andre had learned the mathematics as step-by-step symbolic procedures, he lacked the knowledge of why he was doing particular steps and only knew that it was part of the process. I questioned Andre as to where he learned to draw a curve and shade the area underneath to represent integration. He said it was something the teacher always did before he solved the question. Thus, Andre was mimicking a process he experience in class without understanding why it was significant.

There were also numerous times throughout the interview when we were discussing what he had done when Andre would pause during an explanation and ask "Right?", "Correct?", or "Do you know what I mean?" The frequency of these pauses for reassurance was such that it was very noticeable and made me conclude that Andre was either not confident in what he was doing or expected me to reassure him throughout the interview. As was mentioned earlier, Andre had received the highest grade possible in his calculus course and thus, I assumed he would have developed some level of confidence when attempting calculus questions. These verbal pauses for reassurance make me wonder if Andre completed his calculus course with a lack of confidence. Conversely, it may be the case that Andre's didactic expectations were for me to confirm this processes while he was solving.

According to Sierpinska, one of the affective reasons for a student's need for reassurance may come from the certain words that are used in mathematical discourse. In particular, according to Sierpinska, mathematical discourse uses the terms 'right' and 'wrong' instead of 'true' and 'false', the latter of which are more appropriate in mathematics. Sierpinska suggests that the words 'right' and 'wrong' are "emotionally laden, especially when uttered in relation with a student's work" (2007 13). Thus, students desire to hear that their work is 'right', as that has positive emotional connotations as well as connects to the notion that there is one right way to do things, which is a symbolic/procedural view.

Although I did not ask Andre directly how he emotionally felt about the questions, I did notice and recorded a lot of apprehension when he was not sure if he was correct. Likewise, one of the reasons Andre may have stopped solving task question #4 may have been that he did not want to lose morale in the case he was proceeding incorrectly, as well as the lack of the priming words 'implicit differentiation'. This may also have been why Andre continued to ask me for reassurance that he was answering the questions correctly. Similar to the other potential reasons already mentioned, Andre's emotional need for reassurance may also be supported by his symbolic understanding. Without a strong embodied understanding of the mathematics, which provides the knowledge that allows students to develop an understanding of the justification of the procedures they take, students may look for positive reinforcement that they are proceeding correctly as they have no other indicators available to them.

Conclusion

Overall, it was unforeseen that Andre would exhibit throughout the interview a need for reassurance that he was answering the mathematics task questions and/or the interview questions correctly. Now, after examining the need for reassurance in

relation to Andre's strong symbolic understanding, there seems to be strong links between the two. More specifically, it seems that the symbolic understanding, based on step-by-step procedures for solving mathematics problems, may also lead to students' feeling hesitant about mathematics.

I believe it would be beneficial to further explore this phenomenon of a need for reassurance in mathematics and, in particular, if this phenomenon occurs more often in individuals who demonstrate strong symbolic understandings of mathematics. At the same time I had at one point been Andre's tutor and part of me wonders if Andre looking for reassurance from me comes from the fact that he still sees me as a teacher when we are discussing mathematics. This might play a role, but I believe that it has more to do with a combination of the reasons discussed in Sierpinska's (2007) research as well as his developed symbolic understanding of calculus. Thus, my concluding hypothesis is that there is a link between a student's symbolic understanding and demonstrating a need for reassurance. In future research, I would like to investigate on a larger scale whether it is mostly students who demonstrate a symbolic understanding that search for reassurance. This study could potentially produce significant results that could speak to classroom practices.

References

Bruner, J., S. 1966. *Towards a Theory of Instruction*. Cambridge, Harvard University Press.

Ginsberg, H. 1981. The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For the Learning of Mathematics* 1(3): 4-11.

Nogueira de Lima, R., and D. Tall. 2006. The concept of equations: What have students met before? *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Prague* 4: 233-240.

Sierpinska, A. 2007. I need the teacher to tell me if I am right or wrong *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, Seoul, South Korea* 1: 45-64.

Stewart, S. and M. Thomas. 2006. Process-object difficulties in linear algebra: Eigenvalues and eigenvectors. *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Prague* 5: 185-192.

Tall, D. O. 2003. Using technology to support an embodied approach to learning concepts in mathematics. *Historia e Tecnologia no Ensino de Matematica* 1: 1-28.

------------------2004a. Building theories: The three worlds of mathematics. *For the Learning of Mathematics* 24(1): 29-32.

------------------2004b. Thinking through three worlds of mathematics. *Proceedings of the 28th Conference of the international Group for the Psychology of Mathematics Education, Bergen, Norway* 4: 281-288.

Yackel, E. and P. Cobb. 1996. Sociomathematical norms, argumentation, and autonomy. *Journal for Research in Mathematics Education* 27(4): 458-477.

Conceptualising the mediation of mathematics in classrooms as textured narratives

Geoff Wake and Birgit Pepin

University of Manchester, UK & Høgskolen i Sør-Trøndelag, Norway

This paper builds on a framework that conceptualises mathematics teachers as narrators developing narratives that interweave two important strands that we identify as being focused on the mathematical and social. It is these narratives that we consider to mediate the mathematics for students in classrooms that in turn we consider as activity systems and analyse using Cultural Historical Activity Theory. We draw on case study data collected in the ethnographic tradition in colleges as part of a project funded as part of the ESRC Teaching Learning Research Programme (TLRP) to consider how implicit in such narratives are socially emergent and shared understandings of what constitutes mathematics itself and what it might mean to be a mathematician in different settings. Giving a texture to teachers' narratives, therefore, we identify factors relating to shared epistemologies and didactical contracts (Brousseau, 1997) that we find crucially important in defining what it means to study mathematics as a discipline.

Keywords: Pedagogy, Narrative; Cultural Historical Activity Theory

Introduction

The UK's Economic and Social Research Council funded research project, 'Keeping open the door to mathematically demanding courses in Further and Higher Education' involved both case study research investigating classroom cultures and pedagogic practices and individual students' narratives of identity together with quantitative analysis of measures of value added to learning outcomes in an attempt to investigate the effectiveness of two different programmes of AS mathematics for post-16 students. Here we focus on the classroom experiences of students drawing on data that was collected in the ethnographic tradition with video and audio recordings, photographs and researcher notes together with follow-up interviews with students both in small groups and individually and pre- and post- lesson interviews with the teachers involved.

In doing so we aim to problematise the often held assumption that there is only 'one story' to be told in mathematics teaching, and in mathematics classrooms. Instead, we propose 'teaching as narrating' as a way of participating in an activity system (e.g. Engström, 1999) such as a classroom. In this paper, we address a specific aspect of this system - the means by which teachers engage students with, and mediate mathematics through, their 'mathematical story'. We develop a framework to explore how teachers have their own particular 'stories' to tell, their particular ways of facilitating student participation, and consequently a particular process of 'negotiating' (or directing) what it means to 'do mathematics'. However, we also suggest that these individual narrations reflect aspects relating to epistemologies and didactical contracts that have historically evolved and are culturally bound and situated.

Perspectives/theoretical framework

In our analysis of classroom pedagogic practices in an attempt to understand the social construction of mathematics that we observed, we turned to the construct of "narrative", in the sense of Ricouer (1984), and as developed in educational settings by Bruner (1996) and others. We, therefore, conceptualise the teacher as "narrator" revealing a mathematical plot whilst drawing on a range of pedagogic practices in an attempt to engage his or her audience in different ways. We suggest that teachers in operationalising their actions therefore develop different narrative constructions that can be considered as an interweaving of two different strands:

- a mathematical strand, which is often in effect the development of a mathematical argument, and
- a social strand which comprises of social activities (which arise from the teacher's choice of pedagogic practices) and social discourse. (see fig.1).
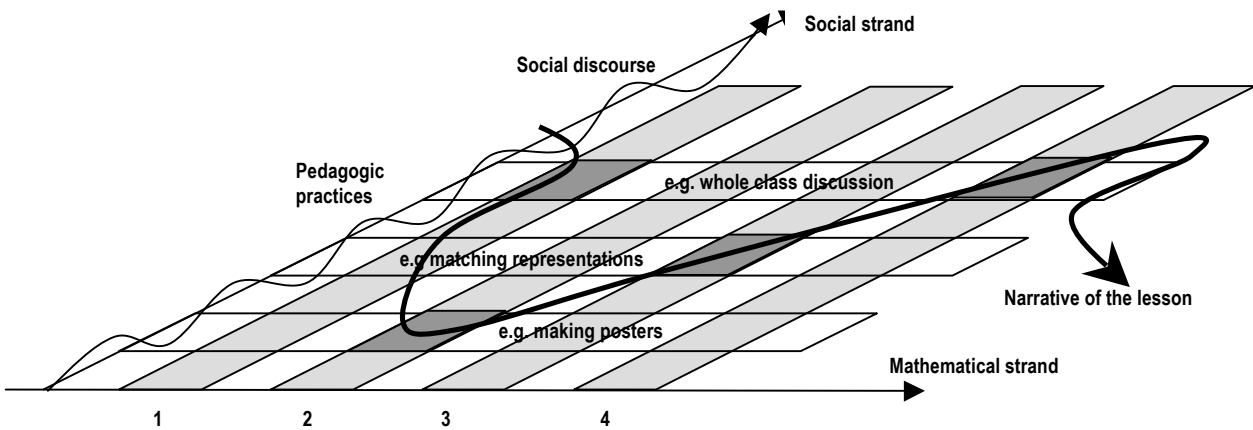

Figure 1: Schema illustrating two dimensional framework used to analyse the narrative of mathematics lessons

We build upon this two dimensional framework drawing on Cultural Historical Activity Theory to identify key factors that we suggest mediate teachers' developing narratives giving additional texture in ways that implicitly define what it means to study mathematics in a particular setting, and indeed to define what mathematics "is" in this setting.

We conceptualise, therefore, the classroom interactions as nested within an evolving systems network, in which teacher and students are mutually constituted through the course of their interactions. The notion of close relationship between social processes and developing knowledge draws on the work of post-Vygotskian Cultural Historical Activity Theory (CHAT) (see e.g. Cole (1995), Engeström (1999) etc.), and it is fundamental to Lave and Wenger's (1991) social practice theory which emphasises the notions of "community of practice" and collective knowledge that may emerge within the spaces people share and within which they participate.

Figure 2 provides a useful conceptual schema for identifying the factors that mediate the actions of a community in relation to the object of its activity (in the case of the AS mathematics classroom taken to be the learning of mathematics). The top triangle draws attention to the "instruments" such as texts, notes, examination questions, teacher exposition, and so on that mediate the individual student's learning.

The lower part of the diagram considers the classroom community more widely. Here, how teacher and student roles are constituted giving rise to a division of labour of the learning community and "rules' both implicit and explicit are taken into account. We find, for example, that rules pertaining to examinations and students' performance in these can drive teachers' actions in classrooms and result in offering particular cultural models of what it means to "do mathematics". For example, we note teachers aligning themselves with students in a joint battle with the examinations as teachers talk of "they" in relation to those who set the assessment papers.

We will return to attempt to understand how these particular factors are powerful in helping shape a teacher's narratives after describing the basis of our narrative framework in a little more detail with reference to an example lesson from one of our case studies. The particular lesson to which we refer here should not be considered as typical, as it certainly reflects the individual style of the particular teacher / narrator who is central. However, as you will see it does illustrate aspects of what we might consider a normative script (Wierzbicka, 1999) of lessons at A Level.
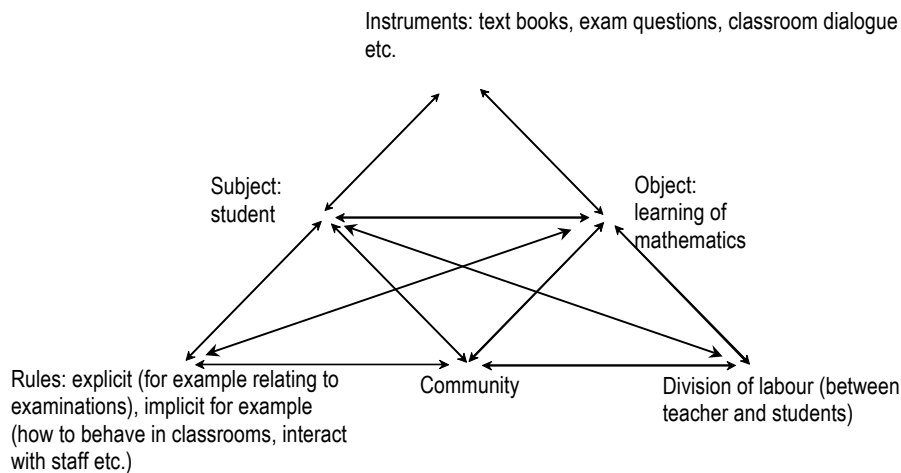

Figure 2: Cultural-historical activity theory schema

The narratives of a lesson

This lesson is in the first term of the GCE AS "Pure" course in a college in the North of England: it focuses on applications of differentiation. In the introductory phase of the lesson the teacher initially drew attention to differentiation as the abstract idea of rate of change of y with respect to x referring to the notation $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. To illustrate that this might be applied to a "real" situation and that different variables other than x and y might be involved the teacher suggests that at issue might be the rate of change of velocity with respect to time, and went on to ask if anyone knows the "special name" for this particular rate of change as the only question in the first ten minutes of the lesson.

Perhaps the transmission style the teacher employed is encapsulated in his statement at this point of the lesson that, "we just need a couple of definitions before we can move on to what I wanted to look at in detail today."

He drew a non-specific/general curved line and emphasised that the gradient at a specific point is given by the differential of the function introducing appropriate notation $f(x)$ and $f'(x)$. At this point he introduced the "new stuff" – the average gradient, or gradient of the chord, between two points (A and B) on the curve, although he did suggest that this idea had been met by the group when differentiation

was first introduced. Here he re-emphasised that the gradient at a point is found using differentiation, $\dfrac{dy}{dx}$, and average gradient between two points is found using the gradient of a chord, and he introduced the notation $\dfrac{dy}{dx}$.

Following this the lesson moved to a second phase in which the teacher modelled how to answer a problem of a form that the students would practise in the final stage of the lesson. However, at this point of the lesson the teacher introduced a 'social' strand of narrative that from this point interweaves, more or less closely, with the mathematical narrative. This revolved around an imagined world in which the teacher developed a problem situation based on the worms in his garden: as you will see this is not a 'real' context but perhaps is 'realisable' as in the RME approach (see for example, Van den Heuvel-Panhuizen (2001)). As this extract of the transcript of the lesson demonstrates this strand of the teacher's narrative with which he expects the students to engage is not insubstantial.

> "So I went into my garden, true story this, and I started digging up some worms. Alright? So I took my fork and I dug up lots of worms and they were all of different sizes so that's quite interesting in the first place. So I thought, well, I wonder if there's any relation between the age of these worms and their length so I collected as many worms as I had time for. For the visual learners amongst you, here's one of them. This is, in fact, it's Japanese. Could be German. Who knows? Ok, this is one of the worms I collected. So I collected till I'd got enough, a decent sample size, right? And I measured these worms, how long they were. I then asked them how old they were. They were quite co-operative. And I plotted how long the worms were at particular ages and, to my surprise, and remember you're not making notes, this is background, to my surprise, when I plotted the age of the worm to its length, all the points roughly lay on what looks to me like a quadratic so, of course, as you yourselves, I got quite excited at that and I thought, well, if I could find the equation of that quadratic, I'm quids in, yeah? I could predict the length of worms at different ages that I didn't have so I got very excited. I also noticed that when the worm wasn't born its age was zero so that was spot on, that fits nicely, so I do know one point that lies on this potential quadratic, quadratic with a negative coefficient of the squared term."

With brief reference to techniques that students had met previously of fitting a quadratic curve to model data such as this the teacher went on to introduce the equation $l = 8t - \dfrac{1}{2}t^2$ that he claimed to have found for a curve that fits his imaginary data of worm length, l millimeters, at time, t years. The first part of the problem that he set was to find the rate of growth of these worms in the first year of their lives which he immediately translated this applied problem for the students into the more abstract mathematical form of having to calculate "delta l" by "delta t". He proceeded to demonstrate how to find the average gradient by firstly finding l when $t = 1$ and then proceeding to calculate the increase in l $\left(7\dfrac{1}{2} - 0\right)$ divided by the increase in t (ie 1).

After brief comment by the teacher that a rate of growth of $7\dfrac{1}{2}$ millimetres per year was, "Quite a lot really" the teacher repeated the procedure carrying out all of the calculations at each stage to find the average rate of growth during the fourth year. In conclusion of this phase of the lesson the teacher "discussed", by asking questions that he answered, the validity of the answers he had found so far:

> "Has the result surprised you or not? 7.5 millimeters per year in the first year, 4.5 millimeters per year in year 3 to year 4. Does that make sense that a worm grows

really quickly at first and then starts slowing down its growth rate? That seems sensible to me, I think we do the same. Obviously, I'm still growing but…ok."

In what may, due to the shift in the mathematics involved, be considered a third phase of the lesson the teacher posed the question,

"What is the rate of growth after 3 years? Not, "what is the average rate of growth?" Exactly, on the worm's third birthday - at that instant, what is its rate of growth?"

Again in this phase the teacher modelled how to find an answer by differentiating the function $l = 8t - \frac{1}{2}(t^2)$ and substituting $t = 3$ to give a rate of change of 5 millimetres per year. Again the teacher asked the class to consider the likely validity of this answer by comparing it with the average rates of change he had found for the first and third years of the worm's life.

In a fourth phase of the lesson the teacher posed the question, "How many years before the worm is fully grown?" After suggesting that the students should think about this in terms of the rate of change of the length of the worm a student made the second intervention of the lesson suggesting that this is at a stationary point. Re-interpreting this, the teacher pointed out,

"In other words, the gradient is zero. When the worm is now fully grown, it's no longer growing so the rate of change of the length with respect to time is zero."

He proceeded to demonstrate that in this case $t = 8$, and once again considered this in the light of the context of the situation, re-introducing a social element of narrative:

"Now, I hope there aren't any biologists here who are going to tell me that worms don't live to 8 years old. They do in my garden, they wouldn't lie to me. I asked them and they said they were honest about their age. So 8 years is when they stop growing. They're not dead. They just stop growing."

Due to space restrictions we leave the lesson here at a point just before the lesson entered a penultimate phase in which students practised answering questions with a similar form to the example considered by the teacher (without the worms!).

Analysis and further theoretical development

In this particular lesson the teacher demonstrates the use of the two distinct strands of mathematical and "social" narrative and interweaves these: at times using the 'social' narrative to motivate, and at other times ensuring it intersects relatively closely with the mathematics in such a way that engagement with the social requires engagement with the mathematical. For example, consider how the teacher's social and mathematical narratives are closely aligned as he discusses how to find when a worm is fully grown with students thinking about growth mathematically (considering the maximum point of the quadratic function) and socially (the teacher emphasises that in this context the function would suggest that the worms would be shrinking and that "we can't have that for worms".) We suggest that the stories the teacher spins about his garden worms require more attention than just focusing on the 'social': the mathematical is at times intricately interwoven with this. Equally sharing the worm story is also a central element of this teacher's dual goal (engagement with, as well as learning, mathematics) directed action: he wants his students to connect to the worm story in order to learn about the mathematics as well as providing a memorable 'event'.

As may be apparent from this extract here this particular lesson had relatively little variation in pedagogic practice, with the teacher in the main choosing to use a predominant transmission style for long stretches with the only interruption being a substantial period in which students practised the techniques that the teacher had modelled. This resulted in a long period of passive activity for the students followed by a period in which they were more actively engaged but on the whole working individually: the result was little or no sociability throughout the lesson. To explore the students' experiences further a CHAT analysis can be captured by the schema of fig.2. This draws attention to key mediational issues regarding informal rules such as those that have developed to determine how students are expected to be passively engaged (!) and the resulting division of labour.

In our CHAT analyses of different lessons, therefore, we note that different teachers operationalise their actions by delivering their own unique interweaving of mathematical and social strands in their narratives. On the one hand the mathematical strand is driven by the mathematical argument that the teacher wants to present and reflects the way in which the teacher understands how mathematical ideas and processes familiar to his or her students may be (re-) introduced and interconnected to develop new (to the students) mathematics. Whilst on the other hand, the social strand contains references to 'why' as the teacher draws on a range of experiences, practices and discourse with which he or she attempts to motivate and engage his or her students in learning.

For any individual teacher their uniquely constructed narratives are dependent to a large extent on their knowledge (content knowledge, pedagogic content knowledge, and knowledge of students), beliefs and the environment (school, department, programme) in which they operate. We suggest that the social system in which teachers operate, therefore, gives an overall feel or *texture* that whilst all-pervasive is perhaps consequently less visible than the narrative strands we have discussed thus far. So far we have not referred explicitly to this but merely hinted at how aspects of teachers' narratives give a flavour of the world in which they are situated. CHAT draws our attention particularly to the implicit and explicit rules of the college / classroom activity system that help develop this texture and we suggest that these combine and are pervasive across all aspects of the teacher's narrative. Consequently this texture, we suggest, encapsulates the teacher's operationalisation of cultural expectations which we consider as normative and particular to different educational phases in relation to studying mathematics. Particularly important in this regard we draw attention to the key elements of cultural expectations in relation to phase specific epistemologies and didactical contracts. To unwrap this a little we suggest that in the A Level mathematics classrooms that we explored there is a normative epistemology that persists across individual teachers and which reflects curriculum specifications and the way these are mediated by texts and assessment. This in turn pervades teacher's narratives and builds lesson by lesson, week by week and term by term to enculturate students into what it means to be a mathematics practitioner in this particular setting. For example, we note the highly procedural nature of student's working that stems from text books that "bite sizes" mathematics into manageable rules and procedures that we suggest inhibits deep understanding. In our observations of student working the procedural approach was stark but perhaps not surprising if students' long term contact with mathematical narratives reflects the type of transmissionist and disconnected teaching that our project found.

We therefore suggest that teacher / student classroom interactions can be analysed using a frame that considers mathematical and social strands of narrative that is textured by normative expectations regarding teaching, learning and values. Our

own use of such narrative analysis of classroom pedagogy in combination with CHAT suggests the mathematics education community and policy makers should widen the debate to include further deliberation about how a range of mediating factors interact to determine the mathematical experiences of learners.

As our project demonstrates teacher's narratives are immensely powerful in shaping learners' identities in relation to mathematics and consequently their likelihood of continued engagement with the subject.

References

Brousseau, G. 1997. *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Cole, M. 1995. 'Socio-cultural-historical psychology, some general remarks and a proposal for a new kind of cultural-genetic methodology', In *Sociocultural Studies of Mind*, ed. J.,V. Wertsch, P. del Rio and A. Alvarez, 187–214. Cambridge, Massachusetts: Cambridge University Press.

Bruner, J. 1996. *The Culture of Education*. Cambridge Massachussets: Harvard University Press.

Engström, Y. 1999. Innovative learning in work teams: analyzing cycles of knowledge creation in practice. In Y. Engström, R.Miettinen & R.Punamäki (eds), *Perspectives on activity theory* (pp.377-404). Cambridge, UK: Cambridge University Press.

Lave, J. & Wenger, E. 1991. *Situated learning: legitimate peripheral participation*. New York: Cambridge University Press.

Leont'ev, A.N. 1981. 'Primaeval Collective Hunt', in *Problems of the 'Development of*

Mind, 210–213. Moscow: Progress Publishers.

Ricoeur, P. 1984. *Time and Narrative, vol.1* Chicago: University of Chicago Press.

Van den Heuvel-Panhuizen, M. 2001. Realistic Mathematics Education as Work in Progress, in Common Sense in Mathematics Education. In *Proceedings of The Netherlands and Taiwan Conference on Mathematics Education*, ed. Lin F.L. Taipei, Taiwan.

Wierzbicka, A. 1999. German "cultural scripts": Public signs as a key to social attitudes and cultural values. *Discourse and Society* 9(2): 241-282.

Secondary mathematics departments making autonomous change

Anne Watson and Els De Geest

University of Oxford, Open University

In the *Changes in Mathematics Teaching Project*[1] three mathematics departments made autonomous changes to their practice in KS3. We chronicled aspects of their practice, tracked a focus sample of students, and obtained KS3 results for the cohort. This paper reports characteristics of the department activity and identifies common features and difficulties which may be associated with the effects of the changes.

Keywords: collaborative practice; teacher change; low attaining students; subject departments

Introduction

This paper reports on the common characteristics of three secondary mathematics departments making deliberate changes to the ways they taught, with the intention of improving the learning of previously low attaining students (PLAS) – that is those who scored at or below a low level 4 in KS2 SATs – during KS3. These changes led in two cases to significant increases in test results for the whole cohort at KS3, and in the third case to sustaining previous levels while other core subjects declined. Several aspects of their work were analysed to identify features that may have contributed to this success. This study is a contribution to knowledge of how mathematics departments can act as self-developing subject communities. They also had at times to resist some attempts by senior management teams to impose other practices on the teachers. It is also worth noting that they were not involved in specific development projects such as lesson study.

What is special about mathematics departments?

School subject departments operate in similar ways for many purposes as social and learning communities (Hodkinson & Hodkinson, 2005), whatever the subject, but we might expect them also to be distinguishable through characteristic epistemic cultures (Knorr-Cetina, 1999). They would also be distinctive in ways which relate to current issues in school mathematics teaching in England: shortage of teachers; high turnover of teachers; pressure for results because schools are compared using mathematics test results; a high political focus; inherent cognitive and emotional difficulties in learning the subject; and a larger body of research about learning maths than for many subjects. The departments on which this paper is based were also distinctive in being subjects of research. One head of department told us: 'we kept going because we knew you were coming in and would ask what we had been doing', and on another occasion: 'it's good to talk to someone outside about what we are doing.'

The study

All three schools contacted us separately to tell us what they were doing because they had been influenced by an earlier project (Watson and De Geest, 2005). Our decision to research their changes was thus opportunistic. The approach taken was non-interventional, and ethnographic in that we watched and listened and asked people about what was going on in their practice. The research team collected data from four main sources:

- interviews, observations and video of teachers and heads of department
- observation of some department and other meetings and inservice events
- interviews with a sample of previously low attaining students (PLAS)
- data, materials, tests and marks, pertaining to the whole cohort.

The changes that were made were initiated by the departments or at least by the heads of departments. They all focused on the cohort that started year 7 in 2005, and they all introduced new schemes of work, new ways of teaching and all-attainment groupings for that cohort. The grouping decision was taken for three different reasons: for timetabling expediency, for an equitable 'fresh start' for everyone, and for the opportunity for teachers to work as a team with parallel groups. One school (SP) continued to teach all-attainment groups in year 8 while others introduced loose setting with highest and lowest attaining students in separate sets. All schools used setting for year 9. Research comparing mathematics teaching using different groupings is always confounded by related changes in teaching, but in this study the approaches to teaching which were developed for all-attainment groups were mostly, but not totally, sustained throughout all three years in various forms.

From the last two data sources we learnt that PLAS developed and maintained a positive approach to mathematics lessons and their own achievement on the whole throughout the three years. Also test scores for PLAS were significantly higher than previous years in two schools (SP and LS), but not in the other. Overall for the whole cohort, test scores at KS3 improved significantly in two schools and were maintained in the third against a background of falling scores across the other core subjects (FH)[1]. The combination of the test scores, the significant improvement for PLAS in two schools, and the reported positive attitudes make it worth knowing more about the departments' work. This paper relates to findings from the first two data sources.

Analysis

We interviewed consenting teachers up to three times during the study. Those who taught the cohort every year were interviewed three times, but due to staff changes and deployment decisions several teachers were only interviewed once or twice. The qualitative data collected from semi-structured interviews about their practice, their teaching and how the departments worked were analysed using a frame devised from activity theory in which each teacher's responses were sorted according to whether they were about how the department activity, or about classroom activity. The activity theory perspective focuses primarily on the role of tools in shaping and being shaped by the object of the collaborative work, the improved learning of PLAS, and is also concerned with the roles and authorities within the system. It derives from a view of human collective activity in which systems inevitably contain contradictions which give rise to adaptations in the system. The work of the system is divided among

---

[1] More details can be found on www.cmtp.cp.uk

individuals and coordinated in the community to achieve shared goals. Individual variations are either subsumed or they contribute to contradictions[1].

For both department and classroom activity, utterances were further sorted according to whether they referred to the purposes of the activity, the people, the tools and other mediating mechanisms that brought about that object, how the community operated in terms of its members, rules, expectations, and how tasks were carried out. Thus for each teacher we had one, two or three sets of data about departments and classrooms that could be collated with other teachers. Where we had two or more years' of interviews we could also look for changes in what individual teachers talked about. Other qualitative data from observations of meetings were analysed under similar headings and added to the overall picture of departments emerging from the sorted data, and also to our own observational knowledge of what changed during the three years. Space is too limited to expand on this process here[2]. Here we further synthesise the data to identify overarching characteristics and difficulties. Our use of the analytical triangle derived from Engestrom (1998) follows its application to school mathematics departments by Venkatakrishnan (2005). She described the activity of a department acting under imposed change using the triangle to show how the department and its local authority systems, while apparently using the same tools, appropriated them in different ways.

Describing the departments' work

In our interviews teachers often talked about connections between the features of their work which are usually represented by the vertices and midpoints in Figure 1. Some of these connections were about individual actions, such as interpretation of tool use, or individual planning, while others expressed how external pressures can constrain the nature of these connections, such as the assessment regime constraining the object – students' improved learning – in ways that were nothing to do with the department system. There were similarities in teachers' talk about individual actions and influences and we saw this as a kind of shadow overlaid onto the triangle giving meaning to the lines. Thus we could include in the diagram the fact that individuals are driven by their interpretations, yet these both shape and are shaped by the system within which they work. The labels on the links in Figure 1 indicate that most teachers talked about their relation with tools as communication and creation; that individual decisions affect the way they enact the object; that their position in the department affects the work they do; the importance of personal professionalism; their responsibility to the school and parents; and relations between various regimes and autonomy.

Changes in object, tools, expectations and labour

In this section we describe common changes during the study which we synthesised by comparing departments over time. The *shared object* of the work of the departments and the classrooms was to improve mathematics learning of PLAS, but the nature of this learning was described differently at different times during the project. At the start, the object was usually described as 'develop mathematical thinking, make mathematics fun, help them believe they can learn maths' but towards the end of the first year the majority of teachers added comments such as 'help them

---

[1] There is not space here to give a more detailed account of activity theory, see Jaworski & Potari (2009) for a thorough account applied to the work of mathematics teaching.
[2] We have written elsewhere about this analytical process (Beswick, Watson & De Geest 2007).

be confident about the basics' to their descriptions. Later still, some teachers from all schools were also talking about 'working on key ideas'. These shifts of focus in the object were reported to us, sometimes as if they were dissenting views, but we learnt from our the discussions at department meetings that all schools did, to some extent, pay more specific attention to arithmetical understanding about half way through the project than they had at the start. These changes of expression of the object then generated changes of *expectations* and the nature and use of *tools*.
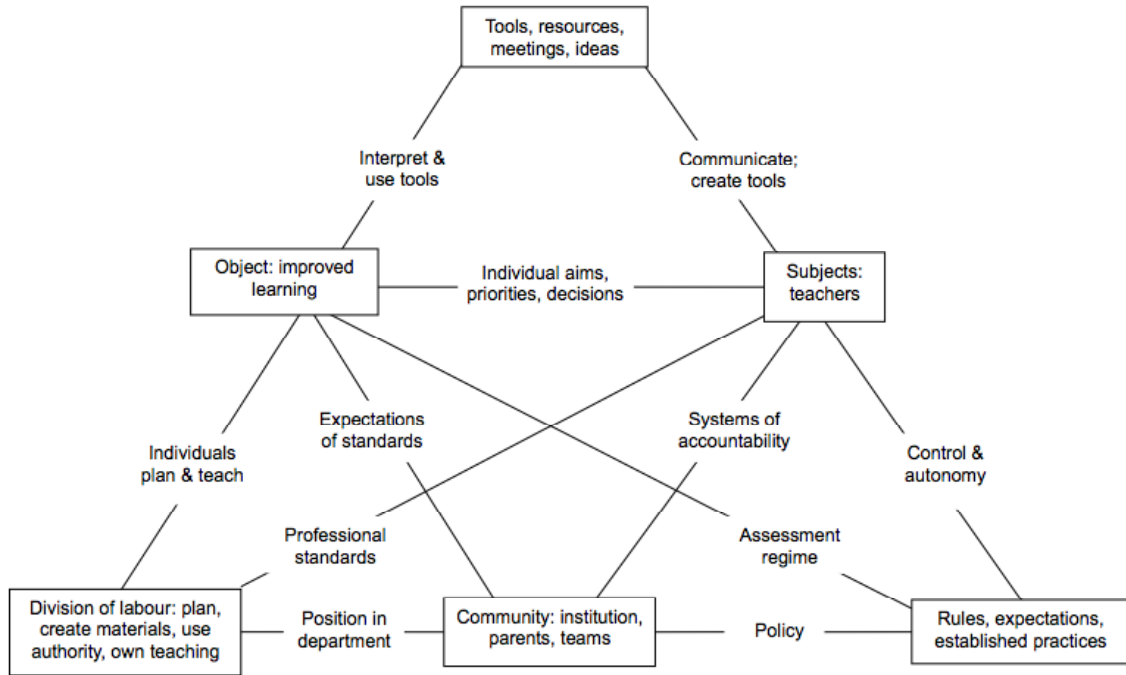

Figure 1: The activity of mathematics departments making autonomous changes to practice

In school LS a day-long end of year review of year 7 work focused on the need to give students more experience of questions that required multi-staged reasoning, and also to shift teachers' discussions from task selection and design to thinking about what students were expected to learn. They believed that they had by then established ways of working and thinking as classroom norms and now, maintaining those norms, needed to move away from the 'how' and think harder about the 'what'. A similar but less explicit shift took place in the other two schools and we noticed that this was combined with a shift away from using open-ended exploratory tasks from published materials towards internal development of resources with specific curriculum foci, and attention to language, examples and methods that might be taught and learnt through teacher-generated tasks. At the start of the project, therefore, the shared tools were mainly published materials promulgating open-ended approaches, but teachers individually and collectively shifted the object to include learning specific mathematical ideas. This shift was supported by, and also influenced, changes in what teachers regarded as their resources, so that the knowledge of the community was treated as the main resource, and the action of communicating and developing that knowledge became the main activity of the department, rather than being caught up rules and expectations about who should produce what and 'put it in the resources file'. When we analysed the *division of*

labour in the department, we were initially ourselves caught up in stories about what was supposed to happen rather than what did happen, so we perceived these stories as 'rules' or, rather, 'expectations' within which teachers worked, while in practice they did other kinds of sharing. The category 'expectations' arose during analysis because these were internally-generated pragmatic understandings about roles, but recognition of pressure on teachers' time in school meant that all HoDs were good humoured about whether these were achieved or not.

Department meetings had a multiple function, attendance being part of the overarching rule-structure for teachers, but they acted as a keystone for the community, holding other aspects of activity in balance with each other. They also functioned as a kind of resource, being the forum for discussion and dissemination that was available for all teachers, even those who were not always around for informal interaction. Meetings with others, and other people's knowledge, came to be used as major *tools for change*, whereas typical mathematics department resource banks were the ordinary *tools for maintaining teaching*.

All the innovations were seen, to various extents at various times, to be in conflict with senior management teams, the inspectorate, beliefs about 'what we are supposed to do', students' previous experiences of mathematics teaching, parents, and also with the views of a small minority of teachers. These 'outsiders' could exert influence through rules, through their position embracing the department in a wider *community*, and through the tools they provided for teachers to use.

Common practices

By about the middle of the second year these changes had become embedded for most teachers. For all schools the approach to planning was internally prescriptive, but different in nature to the scheme of work set out in the National Strategy. For example, SP developed a module of work on logical reasoning, while LS agreed a school-wide approach to teaching equations, starting with the balance metaphor, which would permeate all their teaching. Mostly, teams were united together in seeing their practice as in conflict with 'the strategy' or 'the framework' because in year 7 at least, and to some extent in other years, attention was more on developing mathematical behaviour by spending extended time on core mathematical ideas using a variety of short and long tasks than on coverage of a number of specific topics.

There were other common features which, from our experience, might distinguish these departments from some others:

- The focus on the study cohort was made explicit at certain times which were ring-fenced in various ways; for example, extra department meetings, inservice days devoted to planning for the cohort, major meeting agenda items
- All departments undertook overtly critical use of official documents; in two schools new teachers were enculturated into critical professionalism; in the third a core group of experienced teachers were already critical professionals
- All departments focused on key ideas in mathematics and/or mathematical thinking (albeit defined in different ways by different teachers)
- All departments had a culture of frequent informal discussions about teaching mathematics – whether they had a dedicated subject staffroom or not.
- Small teams created, collated and disseminated resources.
- All departments provided protected timetabled time for co-planning where possible, and all used email prolifically.
- Departments shifted from talking about tasks and activities as if they would somehow 'deliver' learning to developing focused teaching approaches

- The presence of new teachers was seen as important to stimulate talk
- Planning cohered around discussion of parallel groups and also vertical planning, so that students' experience would be coherent year-on-year
- Nearly all teachers were willing to abandon their own past approaches in order to have coherence throughout the school
- All departments included core members who overtly learnt together
- Several members of each team were well-informed, read professional literature and sought research-based approaches to teaching

It is clear from this list that most of the teachers had a strong commitment to the shared aim, and also to some extent saw themselves as 'against' some outside authority. There was also a levelisation of teachers' roles in relation to the study cohort, in that the emphasis was on shared development of teaching and contribution was seen in relation to the time one was willing to spend on the work rather than on qualifications and experience.

Formal meetings were carefully planned by HoDs. Departments discussed particular mathematical topics in depth at these meetings, and often this would include sharing ideas about what the important features were and how they related to other topics. All departments did mathematics together, and this promoted pedagogic discussion best when the shared focus was on classroom tasks rather than on mathematics for personal professional development. Such discussions often revealed different perceptions of the subject matter. Teachers would often review recent teaching and revise approaches for the future, and these discussions also revealed different perceptions of the subject matter. Meetings were not used as a conduit for management information. Organisational and information matters were dealt with by email or informally, not generally in meetings. The teamwork, internal networking, collaboration and use of knowledge from outside networking which took place in the departments appeared to have arisen because of the nature of the department meetings, which were collegial and professional rather than managerial and coercive.

Critical difficulties and differences

School SP had the most stable team, the most stable student population, maintained all-attainment grouping for longer, and had the least disagreement among teachers. This was the school which had the strongest sharing of ideological commitment to the methods adopted, and also in which the head of department took the strongest critical stance towards external imposition. It also had the highest increase in test scores and was the only school to make a statistically significant difference for the weakest of the PLAS. While this confirms the central importance of belief in raising achievement for PLAS (see Watson and De Geest, 2005), this school also had by far the lowest previous test score level, so had more room for test scores to rise.

School FH had the least stable team, including a change of head of department after the first year. In practice, the department changed its focus from the study cohort to the subsequent cohort, so that the maintenance of test scores when compared to falls in other core subjects, while being a great achievement, was not due to a strong department focus after year 7. From teacher reports we think that input in year 7, followed by a spread of new practices to all years could have contributed. Another feature is that of the three schools this one had the highest scores before the study, and hence the least space for improvement.

In all schools there had been dissent among some teachers at the start of the project, but in SP dissent evaporated, with teachers self-reporting their changes of mind. In LS we could not learn much about dissent because the dissenting teachers

had not agreed to participate in the study. In FH we learnt more about dissent because it was overtly expressed to us as covert disagreements with HoDs. There were two main issues that emerged from our observations: *leadership style* and *marginalisation* of teachers.

The heads of department took various roles during the study in relation to their departments: overt change agent, inspirer-leader, presenter, participant, listener, tea and cake provider, and learner. The first kind of leadership appeared to generate most dissent and obstruction, while the other kinds appeared to be accepted and understood. Of course, all HoDs were change-agents and managers, but their role in formal and informal discussions tended to be more as team players. In one case, the HoD genuinely asked a less experienced colleague to help her sort out her management of one class. We noticed that the team participated more in discussion of mathematics when leaders took the role of listener and learner and others presented the mathematical tasks to be undertaken. HoDs would often ask colleagues to prepare tasks or brief talks and presentations for meetings. There were also times in all schools when an HoD or other teacher would present an inspiring insight or observation from research or reading.

When teams work as closely and coherently as these did, the difference between core members and marginalised members becomes very marked. We were able to identify four kinds of marginalisation, which may overlap:

- Institutional: teachers who have other roles in school, only teach mathematics for part of the time, and hence are not around for informal discussions and may not attend all formal meetings
- Ideological: teachers who have reasoned differences with the department policy and voice this either in meetings or outside meetings; they may or may not act out department policy and are unlikely to use available tools for change
- Epistemological: teachers who have a different view of mathematics and may not understand some of the public discussions about mathematics learning
- Self-imposed: teachers who choose not to take part in informal interactions or other opportunities for team membership; they may or may not attempt to act out department policy and are unlikely to use available tools for change

Non-specialist teachers are not necessarily marginalised nor are marginalised teachers necessarily non-specialist. Differences between specialist and non-specialist teachers are indicated by our lesson observation data outside the scope of this paper.

Differences among teachers were not discussed explicitly in departments. For example, teachers were reluctant to challenge each other about mathematical knowledge. This means that some teachers continue with limited or sometimes incorrect mathematical ideas even when working among strong mathematical colleagues. Heads of department who display their own need to learn openly seem most likely to generate helpful talk about mathematics. Teachers were also reluctant, but less so, to challenge assumptions about limitations of students' capabilities in relation to hard mathematical ideas. We witnessed one meeting in which a teacher talked of her class being able to tackle complex tasks but other teachers, who taught parallel groups, claiming that theirs could not be helped to do so. There seemed to be no mechanism in the discussion to examine these differences. At other times, teachers appeared to be conversing about similar issues but when we listened to audio recordings we noticed that some teachers talked about how students *think* while others talked of what they will *know*, and some teachers anticipated what students might *learn* while others talked only of what they will *do* – yet conversation continued as if there was shared meaning.

In FH, after the cohort was reorganised into sets in year 8, one group with the lowest prior attainment was taught by a teacher who exhibited characteristics of three kinds of marginalisation. The teaching methods were not in accord with department ideas and agreements. The decision to staff lowest attaining groups with marginalised teachers is often forced on HoDs by school constraints, and is likely to continue, however a core teacher took the group over for year 9. The relevant sample students exhibited a slight dip in attitudes in year 8, and recovered in year 9. This school was one that had insignificant changes in test result at KS3, and no improvement for PLAS. In terms of our study, therefore, FH could be said to have failed in their initial aim, but as that aim had been abandoned until the following cohort, and a key teacher did not share the aim, we prefer to think instead about how those who taught these students in year 9 rescued some attitudes.

Reflections

When departments decide for themselves to undertake change, the processes are necessarily complex. The successes of these departments are due to collaboration, but we have shown by analysing their activity that this was of a special kind. It was not an imposed structural mechanism for change, nor was it generic; instead it was grounded in mathematics teaching and participants worked together equally and informally around planning through discussion of mathematical tasks. It took time for them to settle on new objects of activity, and to identify and use tools for change in their interactions. Due to various kinds of marginalisation full collaboration was not achieved, but mainly these teachers acted together and were willing to give up their own ideas for the collective work. Teachers participated professionally in the development and use of agreed teaching ideas and resources, and resisted outside pressures to various extents. Our analytical frame allowed us to describe aspects of their complex work and to observe what changed over the three years.

References

Beswick, K., Watson, A. & De Geest, E. 2007. Describing mathematics departments: the strengths and limitations of complexity theory and activity theory. *Proceedings of annual conference of Mathematics Education Research Group of Australasia*. University of Tasmania, Hobart pp. 113-122

Engestrom, Y. 1998. Reorganising the motivational sphere of classroom culture: An activity theoretical analysis of panning in a teacher team pp.76-103. In F. Seeger, J. Voigt, U. Waschescio (eds.) *The culture of the mathematics classroom*. Cambridge: Cambridge University Press.

Hodkinson, H. & Hodkinson, P. 2005. Improving schoolteachers' workplace learning. *Research Papers in Education, Special Issue*, 20(2), 109-131.

Jaworski, B. & Potari, D. 2009. Bridging the macro- and micro-divide: Using an activity theory model to capture socio-cultural complexity in mathematics teaching and its development. *Educational Studies in Mathematics*. 72, 219-236

Knorr-Cetina, K. 1999. *Epistemic cultures: how the sciences make knowledge*. Cambridge, MA: Harvard University Press.

Venkatakrishnan, H. 2005. *The implementation of the mathematics strand of the Key Stage 3 Strategy: A comparative case study*. Unpublished PhD thesis, King's College London.

Watson, A. & De Geest, E. 2005. Principled Teaching for Deep Progress: Improving Mathematical Learning Beyond Methods and Materials. *Educational Studies in Mathematics* 58, 209-234.

Initial teacher education providers and the anxiety spiral: results from the first two years of a longitudinal study

Marcus Witt and Jill Mansergh

School of Education, Bath Spa University

There is considerable evidence that many primary teacher trainees come to their PGCE year with significant levels of anxiety about mathematics. Unless these anxieties are addressed, trainees may fail to remedy gaps in their subject knowledge, may fail to learn the required pedagogical skills and may pass their anxieties on to the children they teach. This suggests that these people are a significant link in the chain that perpetuates mathematical anxiety. The fact that trainees' attitudes to mathematics change considerably during their PGCE year represents an opportunity for training providers to reduce anxiety levels. At Bath Spa we have been running a longitudinal study seeking to track the changing attitudes towards mathematics of our cohorts of trainee teachers and to explore the reasons for any changes through interviews with groups of them. The findings presented here provide a glimpse into the mathematical world of trainee teachers and reveals some interesting (and often surprising) factors which may help other providers of initial teacher education to reduce trainees' anxiety about mathematics in the future.

Literature

There is considerable research evidence to suggest that people entering the teaching profession come with strong feelings about mathematics (Hogden and Askew 2007) often as a result of their experiences as learners of mathematics (Unglaub 1997). For primary teacher trainees, anxiety about mathematics is especially important as there is evidence to suggest that teachers pass their anxiety on to the children they teach (Burnett and Wichmann 1997, Fiore 1999). There is a weight of evidence to suggest that anxiety about mathematics disrupts cognition (e.g. Ashcraft 2002, Miller and Bichsel 2003), thus hampering trainees' ability to learn the associated mathematical pedagogical skills. Trainees with less secure subject knowledge, a possible consequence of high anxiety levels, have been shown to plan and teach mathematics lessons less effectively (Goulding et al. 2002, Rowland 2000). Recently, Rowland et al. (2009) have explored in great depth the link between trainees' subject knowledge and their ability to make appropriate pedagogic choices.

However, a number of studies have shown that trainee teachers' anxieties about mathematics can and do change during the period of training. Gresham (2007) suggests that trainees, like the majority of the population have a poorer understanding of concepts than they do of procedures and that this reliance on learned procedures without a clear conceptual understanding exacerbates anxiety. She designed a mathematics course that provided trainees with concrete experiences specifically intended to improve their conceptual knowledge and found marked reductions in trainees' levels of anxiety. In follow-up questionnaires the majority of the trainees attributed their reduction in anxiety to the use of concrete manipulatives to reinforce concepts. However, it is not clear from the study whether these comments were made spontaneously, or were simply one of a list of possible responses. It is also unclear

from the study whether simply doing some mathematics as an adult and in an environment where motivation levels are high was the cause of the reductions in anxiety. There doesn't appear to have been a control group in that study. Hawera (2004) also found marked reductions in trainee teachers' anxiety about mathematics after an intensive 12 week course (48 hours) of mathematics. This course focused on trainees working together to in what she terms an 'emotionally safe environment'. Again, it is unclear precisely what led to these trainees' reduced anxiety levels. Several of the trainees talked about mathematics becoming more real. It could be that preparing to become a teacher and needing to understand the mathematics in order to be able to teach it, makes it more real and immediate, leading to greater engagement with the subject.

Brown et al. (1999) also report significant changes in trainee teachers' conceptions of and attitudes towards mathematics over the four years of their teacher training, without the issue being explicitly addressed in the training course. The study suggested that the change was gradual and related to the stages of their training. Trainee teachers began with highly affective accounts of their mathematical experiences at school. These were replaced as the trainees engaged with mathematics in a pedagogical context. A further shaping of the trainees' attitudes occurred as they entered school and began were more influenced by the organisational concerns of their placement schools. This again suggests an engagement with the subject on a practical level in a way that reflects the trainees' immediate concerns and interests.

All the above accounts concern trainees on extended courses. Much less is known about the attitudes of trainee teachers on one-year PGCE courses, such as ours. These trainees spend time in schools from the beginning of the course and are concurrently receiving mathematical training in the university. The evidence reviewed above suggests that the PGCE year represents an opportunity for providers of initial teacher training to alleviate anxiety and address trainees' negative attitudes about mathematics. However, time for these trainees is very limited and little is known at present about the factors during initial teacher training that might help alleviate anxiety levels in trainees and therefore the ways in which providers can make their provision more effective for such trainees.

This study aimed to explore the anxiety of primary teacher trainees as they followed a one year PGCE course. The mathematics provision consisted of 16 lectures of 45 minutes, 16 'pedagogy' seminars of 90 minutes with a focus on how to teach primary mathematics and 7 largely optional 90 minute 'subject knowledge' seminars with a focus more on developing the trainees' own understanding of mathematics. The distinction between 'pedagogy' and 'subject knowledge' was largely done for timetabling reasons and, as will be reported later, there were considerable overlaps between the two. This study was not an intervention; we were simply looking to explore whether our trainees' anxiety about mathematics was affected by our provision and, if so, specifically which aspects of the PGCE year had an effect.

Data collection

Here we report a study that sought to answer a number of important questions:

- Does participation in our one year PGCE course lead to changes in primary teacher trainees' anxiety about mathematics?
- Which elements of the PGCE mathematics provision lead to changes in trainees' anxiety?

The study used questionnaires administered at three time points (September, December and April) to explore the changing attitudes towards mathematics of primary teacher trainees throughout their teacher training year. The questionnaires examined the trainees' anxiety levels through a shortened, 30 point version of the Maths Anxiety Rating Scale (MARS, Suinn and Richardson 1972). The questionnaires also asked the trainees to rate themselves as mathematicians and as teachers of mathematics on a scale of 0-100. Data about the trainees' highest mathematical qualification and their prior experiences of working in primary classrooms were also collected in the initial questionnaire.

The factors influencing attitudinal changes were explored through interviews with small groups of trainees. The interviews were semi-structured in nature. There was a loose 'agenda' and the interviewers came with pre-prepared questions. However, if the group began to discuss an interesting issue, the interviewers let the discussion proceed without intervening.

During the two years of the data collection, a total of 254 trainees had full sets of data. A larger number than this had partially completed data sets and these data are used where appropriate. For example, all the data collected at the first entry point is used in the analysis concerning trainees' attitudes on entry to the course. During the first year of the study, a total of 18 trainees were interviewed in three groups. In the second year of the study, thanks to funding from ESCalate, we were able to extend the interviewing with a total of forty trainees were interviewed in eight groups of five.

A conscious decision was made to collect both quantitative and qualitative data. Quantitative data were needed in order to build up a comprehensive picture of the attitudes of whole cohorts of trainees and to indicate whether any change in attitude was actually taking place during the PGCE year. The use of quantitative data enabled us to explore the magnitude of any changes and to see how anxiety about mathematics interacted with other attitudes about the subject.

The use of the quantitative data was not enough in itself to offer us any real indication of why changes were occurring and which aspects of the PGCE mathematics provision were responsible. For this reason, the decision was made to conduct the group interviews. These interviews took place in April. An initial analysis of the quantitative data informed the framing of some of the interview questions and enabled us to target questions at areas highlighted in the quantitative data.

Data Analysis

The quantitative data was analysed using SPSS. Correlational analysis was used to explore the relationships between mathematics anxiety and the trainees' ratings of themselves as mathematicians and as teachers of mathematics. The three data collection points allowed us to look at ways that these relationships changed over time.

T-tests allowed us to compare different groups of trainees within the whole cohort. Specifically, we looked at the outcomes of those trainees who had reported very high levels of anxiety on entry to the course and compared them to the data for the rest of the cohort. Importantly, t-tests allowed us to look at the 2007/8 cohort and compare their scores with the 2008/9 cohort. We were also able to look at other sub-groups within the main cohort. The PGCE course offers two 'routes'. One is a 'general primary' route that covers children between the ages of 5 and 11 and prepares trainees to teach in either Key Stage 1 or Key Stage 2. The other is an 'early years' route that covers children between the ages of 3 and 7 and involves placements

in Key Stage 1 and the Foundation Stage. The use of independent samples t-tests allowed us to explore differences between those trainees on the general primary route and those on the early years route.

The analysis of the qualitative interview data involved an initial coding of the conversations to identify common themes. This initial coding was done by the two named researchers independently. We were interested to see whether there was any consistency in the trainees' responses to questions about their initial levels of anxiety, the origins of this anxiety and the factors that had led to changes in their attitudes towards mathematics, or whether their responses were highly disparate. Although the interviews were semi-structured, we did not begin with specific codes. The initial codes were largely descriptive as the trainees' responses were collected and categorised. A second phase of analysis took place once the initial codes had been discussed and developed; more abstract, inferential codes were identified from the initial coding and used to refer back to the literature. From these inferential codes initial findings and recommendations were independently formulated and then compared. The findings reported below are the result of this coding process. The words of the trainees are illustrative of the themes and feelings that the coding process revealed.

We were keenly aware of the potential problem that the interviewers who were asking questions about the PGCE course were the same people who were responsible for the design and delivery of the course. We tried to be as open as possible, to carry out the interviews in an atmosphere of genuine trust and information gathering and to be open to negative or critical comments. We always asked specifically for suggestions and comments about how we could improve the course and encouraged those trainees who had been less happy with the provision to come to the interviews so that they were able to voice their feelings. Despite this, we were aware that there may have been feelings and critical comments that the trainees wished to voice, but felt that they couldn't.

Findings

The questionnaire data indicated that trainees come to their teacher training year with widely differing levels of anxiety about mathematics, with about 16% (59 out of 358 trainees for whom initial data was available) showing anxiety levels greater that 1 standard deviation above the mean. However, during the interviews, the majority of trainees indicated some level of anxiety about mathematics, almost always as a result of their own experiences as learners of mathematics. This was interesting, as all the trainees had a minimum of a GCSE Grade C, which would make them all relatively 'successful' learners of mathematics. However, a large number of them were aware that they had learned mathematics in a highly instrumental way, knowing 'tricks and recipes' to get to the right answer, without any real understanding.

> 'It feels like I was taught to answer exam questions.'

Correlational analysis showed that trainees' anxiety levels on entry were moderately related to their views of themselves as a mathematician ($r=0.525$, $p<0.001$) and moderately related to their assessment of themselves as a teacher of mathematics ($r=0.536$, $p<0.001$). Initially, their views of themselves as a mathematics teacher also showed a weak correlation with their level of classroom experience ($r=0.184$, $p<0.01$). However, as expected, this relationship became non-significant by the end of their PGCE year ($r=0.042$, $p>0.05$).

During the year there were significant reductions in trainees' anxiety about mathematics (t=19.53, p<0.001), and increases in their assessments of themselves as mathematicians (t=17.86, p<0.001) and their assessments of themselves as teachers of mathematics (t=20.93, p<0.001). There were statistically significant reductions in anxiety levels between September and December and between December and April.

The group of trainees who had levels of anxiety on entry more than one standard deviation greater than the mean were compared with the rest of the cohort. The analysis showed that their levels of anxiety fell significantly more than those of the rest of the cohort (t=2.15, p<0.05). However, this group of trainees were still significantly more anxious than their peers by the end of the course (t=5.83, p<0.001). This particularly anxious group also rated themselves more poorly as mathematicians and as teachers of mathematics than their less anxious peers at the beginning of the course. By the end of the course, there was still a statistically significant difference in terms of their rating of themselves as a mathematician (t=4.68, p<0.001), but no significant difference in terms of their ratings of themselves as teachers of mathematics (t=1.59, p>0.05). This suggests that these two concepts were very closely related at the beginning of the course, but became increasingly dissociated as the course progressed.

The group interviews provided some interesting illumination of the statistical data. Trainees come to their PGCE year with ideas about mathematics and mathematics teaching. While most trainees who had a negative experience as learners of mathematics were determined not to repeat the mistakes of their own mathematics teachers, some found it hard to envisage teaching mathematics in ways that are not very similar to those they 'suffered' themselves.

> "Although you might have had a bad experience, it's just making kids realise how accessible maths is; it shouldn't have to be something you can't look forward to."

> "I also wanted to change their views, because I'd had such a bad experience at primary school. I wanted to make it more fun, so I knew ways not to do it."

Less anxious trainees were content to be 'taught' some subject knowledge and simply to refresh mathematics that they had not used for some time. The more anxious trainees saw the opportunity to do some mathematics as extremely valuable. They needed more than simply to be reminded how to do something. This suggests that providers of initial teacher training need to do more than offer a 'one-size-fits-all' subject knowledge provision. The element of choice as to the level of subject knowledge provision offered seemed to be particularly important to the most anxious trainees. Their being grouped with other trainees who were less confident with mathematics enabled them to engage more with the teaching, to ask questions and to 'have a go' without the feeling that their more confident peers were judging them.

> "I have this problem of, if someone says it to me I understand it, but then when I go away, like on a test and I have to do it for myself, or apply it to different numbers I'm like whooooaaagggggghhhhh."

> "We were shown how to do that and I got it, but having the chance to actually do it, that really helped."

There was a complex relationship between the trainees' level of anxiety, their subject knowledge and their understanding of pedagogy. Pedagogy sessions, although not designed specifically to address subject knowledge, were places where trainees made significant advances in their mathematical understanding. For many trainees, being shown ways of explaining a particular mathematical concept to primary school children acted as a trigger for making leaps in their own conceptual understanding, which in turn led to much greater confidence with their own mathematics.

> "The way you understand something is definitely going to inform the way you feel you can deliver it to the children."

> 'If you don't understand something very well, you won't necessarily be able to think of ways of teaching it that will be interesting to the child.'

> 'Me and J have talked about when we've been to subject knowledge sessions, it's like, I understand it now and you could teach it *that* way.'

These comments reinforce the importance of 'deep subject knowledge for teaching' (DCSF 2008). These trainees, without having been made explicitly aware of this term, were seeing clear links between their own understanding and their ability to find interesting and clear representations of mathematical ideas for their children.

For a number of trainees, their initial anxiety was not expressed in terms of deficits in their own subject knowledge, but rather to the expectation of having to teach mathematics. They were aware that there were areas of mathematics that they could 'do', but were not able to teach.

> "The whole time, what was going through my head was, how on earth am I going to teach this because I can do it, but I don't really know how or why. I became anxious about how I was going to teach it."

It could be that the 'anxious' trainees may have had different sources of anxiety. Some may have genuine anxiety at the thought of doing some mathematics and many seem to be aware of gaps in their own subject knowledge. These gaps may have been covered for the purposes of examinations by learning tricks and recipes, but were thrown into sharp relief as the prospect of teaching mathematics, albeit to primary school children, approached.

Tutors were seen as important in reducing trainees' anxiety. On a simple level, their approachability and patience allowed trainees to have a go at problems and ask questions without the feeling that tutors would be dismissive or impatient.

> "They all work to your level. If you don't get something right, it doesn't matter. They are willing to work through it with you until you do understand it."

More subtly, there was an emerging sense that the tutors themselves were acting as ambassadors for mathematics. The fact that tutors had a sense of humour and were approachable meant that the trainees' perception of mathematics itself was changed. For example, the final lecture before Christmas, which was done as a pantomime and used to reinforce the key messages from the first part of the taught course through songs and sketches, was unexpectedly cited by a number of trainees as a moment of realisation that mathematics itself didn't have to conform to the 'dull and dry' stereotype. A session on problem-solving, which involved acting out the problems to try and make the mathematics clearer, was also mentioned by several trainees as an important point in their relationship to mathematics.

> "Using physical examples of people standing on chairs and tables has proved that you can enjoy maths, it doesn't have to be dull."

During the second year of the project, the subject knowledge sessions were changed from a more 'didactic' teaching focus to an opportunity for all the trainees, regardless of their prior mathematical achievements, to engage in sustained mathematical investigations. This was not an explicit 'intervention' for the purposes of the research project, rather a natural development of our course in response to the data collected from the first year of the project, in which a number of trainees commented on the importance and usefulness of being able to do some mathematics.

There was an indication from the interview data that engaging in sustained mathematical investigations enabled trainees to uncover latent mathematical

understanding. Several of the trainees spoke of a 're-engagement with mathematics' and of being able to draw a line under bad school experiences and start again.

> "For me it showed what I did know rather than highlighting what I didn't know. I do know a reasonable amount of maths and I can work things out without being told. At that stage it was important for me to know that I did know some maths."

Several appreciated the pedagogical benefits of this and saw themselves as 're-learning' mathematics alongside the children they were teaching, with all its attendant benefits. The investigative sessions also served to highlight the importance of collaboration in mathematics, something that served to reduce the anxiety of some of the trainees.

> "It's almost that we're at the same point where I'm learning to enjoy maths as well and recognise that there aren't always right and wrong answers and there are different ways of doing things."

Not all the trainees lost their anxiety about mathematics during the year. Those who remained anxious rationalised their anxiety by realising that it put them in a better position to understand those children who do not find mathematics an easy subject. The trialling of the investigative sessions also produced a dichotomy. The more anxious trainees from the first year of the study had been positive about the fact that they were in a group with other like-minded or similarly anxious trainees. The investigative sessions were run with all the trainees together, something that was appreciated by some of the trainees and a source of concern to others.

Implications

The findings that are described briefly above led to the following implications for our PGCE mathematics provision:

Trainees on primary PGCE courses are not neutral about mathematics. They have clear views about and highly affective accounts of their experiences of learning mathematics. However, these experiences are not homogenous and providers of ITE might try to tailor their provision as closely as possible to the needs of their trainees.

A minority of primary teacher trainees come to the PGCE with significant anxiety about mathematics. The source of this anxiety may well be a lack of subject knowledge as a result of poor experiences as learners of mathematics (Brown et al. 1999). Some trainees may not have thought of themselves as anxious about mathematics in the past. Their lack of conceptual understanding may have been brought to the fore by the prospect of having to teach some mathematics. This suggests that a timetable-driven disconnection between 'pedagogy' and 'subject knowledge' is potentially not helpful.

Tutors play an important role in the alleviation of trainees' mathematics anxiety. Simple things like not putting undue pressure on trainees and being patient with difficulties in subject knowledge are important. More subtly and possibly more importantly, trainees seem to imbue the subject of mathematics with tutors' own personalities. Tutors making sessions humourous and fun might dispel the dry and dusty image with which mathematics seems to be saddled.

Trainees all seem to benefit from actually doing some mathematics rather than simply being told about it. For the trainees who came with high levels of anxiety, or who had had bad experiences as learners of mathematics, doing some mathematics and being successful at it enabled them to draw a line under their prior experiences and to re-engage with mathematics. This is in line with some of the studies cited

above (Hawera 2004, Gresham 2007), but was done within the tight time constraints of a one-year PGCE.

The findings provide a rich insight into the thinking of primary teacher trainees as they grapple with mathematical subject knowledge, pedagogy and their own feelings during their training year. We hope that they represent a source of ideas for providers of initial teacher education to develop their courses so as to make better provision for anxious trainees and help to break the chain of mathematics anxiety.

References

Ashcraft, M.H. 2002. Math Anxiety: Personal, Educational and Cognitive Consequences. *Current Directions in Psychological Science* 11(5): 181-185.

Brown, T., O. McNamara, U. Hanley and L. Jones. 1999. Primary Student Teachers' Understanding of Mathematics and its Teaching. *British Educational Research Journal* 25(3): 299-322.

Burnett, S.J., and A.M. Wichman. 1997. *Mathematics and Literature: an approach to success*. Dissertation submitted to the School of Education, Saint Xavier University, Chicago.

DCSF. 2008. *Sir Peter Williams' Independent Review of Mathematics in Early Years Settings and Primary Schools*. London: DCSF.

Fiore, G. 1999. Math-abused students: Are we prepared to teach them? *Mathematics Teacher* 2(5): 403-406.

Goulding, M., T. Rowland and P. Barber. 2002. Does it matter? Primary teacher trainees' subject knowledge in mathematics. *British Educational Research Journal* 28(5): 681-704.

Gresham, G. 2007. A study of mathematics anxiety in pre-service teachers. *Early Childhood Education Journal 35*(2), 181-188.

Hawera, N. 2004. *Addressing the Needs of Anxious Preservice Teachers* in R.Faragher, M.McLean, and I. Putt (Eds), Mathematics Education for the Third Millenium: Towards 2010 (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, 1: 287-294. Townsville: MERGA.)

Hogden, J., and M. Askew. 2007. Emotion, identity and teacher training: becoming a primary mathematics teacher. *Oxford Review of Education* 33(4): 467-489.

Miller, H., and J. Bichsel. 2004. Anxiety, working memory, gender and math performance. *Personality and Individual Differences, 37*(3), 591-606.

Rowland, T., S. Martyn, P.Barber and C. Heal. 2000. Primary Teacher Trainees' Mathematics Subject Knowledge and Classroom Performance. *Research in Mathematics Education* 2(1): 3-18.

Rowland, T., F. Turner, A. Thwaites and P. Huckstep. 2009. *Developing Primary Mathematics Teaching*. London, Sage.

Richardson, F.C., and R.M. Suinn. 1972. The mathematics anxiety rating scale: psychometric data. *Journal of Counselling Psychology* 19(6): 551-554.

Unglaub, K.W. 1997. Mathematics Anxiety in Preservice Elementary School Teachers. *Journal of Early Childhood Teacher Education* 18(1): 68-74.

Prospective elementary teachers' beliefs about problem solving: A comparison of Cypriot and English undergraduates at the commencement of their courses

Constantinos Xenofontos and Paul Andrews

University of Cambridge

The research reported in this paper draws on semi-structured interviews conducted with first-year undergraduate teacher education students, in the first weeks of their course at one university in Cyprus and one in England. The interviews, focused on students' conceptions of mathematical problems and problem solving yielded substantial, culturally-located variation in students' responses highlighting continuing inconsistencies in the operationalisation of this key concept around the world. Some implications for teacher education and further research in the problem solving field are discussed.

Introduction

Teachers' beliefs have been the subject of extensive research, based on the assumption that what teachers believe is a significant determinant of what gets taught, how it gets taught and what gets learned in the classroom (Middleton 1999; Wilson and Cooney 2002). According to Aguirre and Speer (2000, 327), "being able to identify and describe the mechanisms underlying the influence of beliefs on instructional interactions would deepen and enrich our understanding of the teaching process". Older and recent studies highlight the importance of examining, analysing and changing teachers' beliefs in order to implement successfully mathematics curricula reforms (Ernest 1989; Handal and Herrington 2003), as without a challenge to teachers' underlying beliefs, teachers may exploit new resources or modify practice inappropriately (Handal and Herrington 2003). Reform classrooms are characterised by an emphasis on problem solving (Peterson et al. 1989; Cady et al. 2007) and yet many international comparisons have fallen into the trap of assuming that things with the same name must have the same function in every culture (Grant, 2000). Our aim here is to address some cultural similarities and differences of prospective elementary teachers' beliefs about mathematical problem solving in Cyprus and England.

Teachers' beliefs

Beliefs have been defined as "conceptions, personal ideologies, world views and values that shape practice and orient knowledge" (Aguirre and Speer 2000, 328). Moreover, recent beliefs-related research, continuing to draw on Ernest's (1989) triadic model, has focused primarily on how teachers think about the nature of mathematics, its teaching and its learning. In addition, drawing upon Bandura's early work in the field (Bandura 1977), research has highlighted the role of teacher self efficacy in general (Wolters and Daugherty 2007) and mathematics teaching self-efficacy in particular (Charalambous et al 2008). Thus, we argue, Ernest's original model is appropriately augmented by such additional dimensions.

Research has shown that the relation between beliefs and instructional practices is complex and cannot be described simply in terms of cause-and-effect. For example, while a number of studies have highlighted substantial disparities between

espoused and enacted beliefs (Thompson 1984; Beswick 2005; Raymond 1997), others have indicated that both beliefs and actions are contingent on the changing nature of the classroom context (Schoenfeld 2000; Skott 2001). Thus, acknowledging this problem, we have examined the mathematics-related beliefs of beginning undergraduate primary teacher education students to determine the extent to which they reflect similar, culturally embedded, perspectives to their peers. In this respect, results from comparative studies of serving teachers' mathematics related beliefs (Andrews and Hatch, 2000; Santagata 2004; Correa et al. 2008) and practices (Givvin et al. 2005; Andrews 2007a) indicate that culture plays a key determinant role in both their formation and manifestation.

Curricula reforms, teachers' beliefs and problem solving

A number of papers have focused on the mathematics instruction-related problem-solving beliefs of prospective teachers, including those of Verschaffel et al. (1997) in Flanders and Timmerman (2004) in the USA. In many such studies, prospective teachers' cultural location remained, as a significant influencing variable, essentially unacknowledged. Moreover, a collective definition of *problem-solving-oriented instruction* has been assumed and, it is our contention, although this paper is not the place for a lengthy elaboration, that this assumption has little basis. For instance, as a consequence of the role of the National Council of Teachers of Mathematics (NCTM) in the framing of reform curricula in the US, much problem solving research has been undertaken in that country. The results of these studies have influenced curriculum development in many countries. Nevertheless, variation in definition and implementation can be seen, for example, in a 2007 special edition of Zentralblatt für Didaktik der Mathematik (ZDM) on problem solving around the world that includes articles highlighting the role of problem solving in the curricula of Israel, France, Italy, the UK, the Netherlands, Portugal, Germany, Hungary, China, Australia, Singapore, Japan, Brazil, Mexico, and the US respectively. In other words, despite US influence in the field, both problem solving as an activity and problem solving research continue to mean different things in different countries (Torner et al. 2007), to the extent that problem solving, according to where and by whom the term is used, can mean a goal, a process, a basic skill, a mode of inquiry, a form of mathematical thinking, and a teaching approach (Chapman 1997).

While some single-national studies of teachers' problem solving beliefs and practices have been undertaken in, for example, Australia (Anderson et al. 2004), few cross-national studies have been undertaken in this area. From the perspective of serving teachers' beliefs about mathematics, Andrews (2007b) concludes that English teachers tended to view mathematics as applicable number and the means by which learners are prepared for a world beyond school, while Hungarian teachers perceived mathematics as *problem solving* and independent of a world beyond school. Such findings seem to confirm that the *teachers' perspective* is a neglected dimension in comparative studies of problem solving.

Mathematical problems

Despite the notion of mathematical problem having been used differently by different scholars, not all research in the field shows a lack of consensus. There is much agreement as to the nature of a problem. One key characteristic is that a problem lies with the person seeking the solution and not the problem itself. As Schoenfeld (1985, 74) notes, "being a 'problem' is not a property inherent in a mathematical task.

Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person". That is, a problem for one person may not be for another (Borasi, 1986; Nesher et al. 2003). Such insights, and our interpretation of the work of these scholars, are helpful in framing our study, not least because they allude to three key criteria for defining the relationship between problem and problem solver. Firstly, problem solvers must have encountered a block and see no immediate and obvious way forward. Secondly, they must actively explore a variety of plausible approaches to the problem. Thirdly, they must accept that the search for a solution necessitates an engagement with the problem. With regards to the context of problems, Blum and Niss (1991) distinguish between the problems embedded "in some mathematical universe" (p. 38) and those related to real world. In other words, problems are seen to be either purely mathematical or applied. These perspectives frame the study we report below.

Method

As stated above, in this paper we report on the problem and problem solving beliefs of prospective elementary teachers from Cyprus and England. Participants were in the first weeks of an undergraduate teacher preparation programme at a one reputable, as measured by systemic measures of teacher education accountability, university in each country. Data were collected by means of semi-structured interviews at the beginning of the academic year 2008-2009. The interview questions regarded students' definitions of mathematical problems and problem solving, their beliefs about pedagogy (teaching/learning), and their self-efficacy beliefs both as problem solvers and as potential teachers of problem solving. The comparative dimension is important in helping to identify and understand the ways in which culturally located experiences inform the construction of students' beliefs.

The Cypriot cohort comprised thirteen students (twelve female, one male), while the English comprised fourteen (thirteen female, one male). At the time of the interviews participants had received no problem solving-related university instruction. Therefore, they were seen as products of the school rather than university systems of their countries. Analyses were focused on students' *meaning* (see Kvale and Brinkmann 2009) and drew on both theory-driven and data-driven approaches (Boyatzis 1998; Kvale and Brinkmann 2009). In this paper, due to space limitations, we report on three themes identified by the data-driven analyses. These are students' perspectives on the nature of mathematical problems, the nature of mathematical problem solving, and the characteristics of effective problem solvers. The results are presented alphabetically by nationality, Cyprus then England.

Cypriot students' perspectives on mathematical problem

Eleven students indicated that mathematical problems, usually embedded in text, should be clearly presented with adequate information and data so that solvers can easily attempt a solution. Panayiota's comments were typical of others. She said that a mathematical problem comprises "mathematics related sentences, which include information, data and a desired outcome. We have to think about the data, to process them and get the answer". Eight students commented on the significance of difficulty in defining a problem. For some as reflected in Sofia's comment, a problem by definition "has difficulty and unknown factors within it". Others indicated that notions of difficulty lay, essentially, with the problem solver and not of itself, the problem. Such a perspective could be seen in Demetra's comment that "problems and their

difficulty are connected to certain age groups", while Haroula added that "the criterion is school, if it is primary, gymnasium or lyceum. In primary school, problems are very easy, in gymnasium they are more complicated, and in lyceum you can find the hardest". All thirteen students implied that mathematical problems are contextualised within a real-world framework. Christina's comment was not atypical. She said,

> at the first grade, problems were like "I have two apples, my grandmother gave me two more, how many do I have now?" Later on, at the sixth grade, problems were more complex, let's say something about how many square metres of a wall surface could someone paint with so many litres of paint. In gymnasium, they might be something like "how much it cost to paint a surface", which had to do with area and volume. In lyceum, they were more or less the same.

Cypriot students' perspectives on mathematical problem solving (MPS)

Eleven students indicated, either directly or indirectly, that MPS is a process. Of the four who used the word process explicitly, Pantelis' comments were typical. He said that "mathematical problem solving is a process, the process towards what we are asked to find. It is the process during which you use the given data in order to find the answer to a problem". Of the others, Demetra's comment was typical. She said that "mathematical problem solving is the use of the data in order to find what you are asked to". A recurrent theme in these students' responses was the need to read the problem repeatedly. Panayiota's comment was typical. She said,

> You have to read the problem two-three times, underline some key points, because you know, sometimes problems have unnecessary things in them, you have to find what is important, then start processing all these in your mind, read two-three more times, write down your data and what you want to find, do a shape if it's needed and then do the algorithms.

Cypriot students' perspectives on what makes a good problem solver

Eight students suggested, along the lines of Martha's comments, that good problem solvers "have the skills for organising the given information quickly. They tidy up the data, the questions. Those who are not good don't structure their work". Sofia presented a typical response in respect of distinguishing the expert from the novice. She said that

> there is a big difference. Someone who is a good solver, as soon as he sees the problem, he has a clear picture in his mind about what has to be done, directly. Someone who is less good will have difficulties in finding which way to follow for solving the problem.

Several students added that problem solving requires concentration, as seen in Demetra's comments that "solvers who concentrate when they encounter a mathematical problem perceive what has to be done quickly and manage to resolve it".

Eight students also suggested that, with practice, problem solving competence can be acquired. For example, Angeliki commented that a novice "could spend more time on practice... to develop his mathematical thinking, learn about different types of problems, and develop a clearer idea around mathematical problems". However, the remaining five students indicated that being a good problem solver was natural. As Panayiota noted, it all "depends on the individual, biologically, I think some people are born with it; it's their talent, either you have it or not".

English students' perspectives on mathematical problems

In respect of their conception of mathematical problems, the English students presented a range of perspectives. Common to ten was an explicit invocation of number operations, as seen in Victoria's comment, that a problem was "anything, from adding, dividing, subtracting, timesing, or arrange them and then put together". However, the context in which they described their perspectives varied. For four students problems were essentially mathematical in nature, as seen in Daniel's slightly recollection of his school experiences. He said

> Pythagoras' theorem, if you have the length of the hypotenuse and you need to work out,.. I can't remember how it was. Like we've got sin, cosine and tangent and you need to work out the other two. Like you've got one of the angles and you need to work out another angle or length. That's one which sticks in my head.

Two students indicated that mathematical problems were related to the real world and every-day life, as in Laura's comment that they had to do "with money, we did how, if apples cost 10p each, how much money do I need to buy six apples? Which is 60. Six times 10p". However, the majority of the group, seven students, implied that problems could be construed as either mathematical or real world. Melanie's comment reflected those of others. She said that a "mathematical problem could mean lots of things. It can be a standard two plus two on a piece of paper or how much money I need to go for shopping. (…) Something that uses numbers to come up with 'a' answer, or a series of answers".

English students' perspectives on mathematical problem solving (MPS)

Eleven students indicated that MPS is a structured process during which solvers apply prior knowledge in a structured step-by-step approach. Julia's response, typical of others. Suggested that MPS was "just basing what skills you know on trying to solve a problem in maths, so just applying the knowledge to structure it and work step by step to work out a problem", while for Laura it meant having to "take it step by step and apply things you already know". For some this step-by-step approach meant breaking down a problem into smaller pieces-tasks, working on each piece separately, and finally putting all the pieces together. Rachel, reflecting comments of others, said that "I break it down..., and then do a bit a time and then at the end I put them all together. I do that with most mathematical problems". All students saw MPS as drawing on prior knowledge. Interestingly, Melanie, was the only student who used the word process explicitly. She commented on the

> the processes that you use to solve a problem. So, the way you think, the way you work out a problem, whether you need a resource to do it or whether you do it in your head. What steps you take to come to your conclusion. You need to understand the problem, what you've been asked to find out, cause lots of problems are in a context where you have to pull out the information you need, you need to understand the processes, you need to follow out the procedure and then you need to understand your answer.

English students' perspectives on good problem solvers

Thirteen students suggested that expert problem solvers can see through problems and apply the necessary mathematical knowledge and strategies quickly and efficiently. For example, Laura indicated that "an expert knows how to answer straight away, whereas someone who is not so good does think a long time about it and mainly have one option, whereas an expert might have lots of different ways to think about it".

Expert solvers, she added, "already have the knowledge to work out what you need to do to solve the problems, whereas, otherwise they have to think what steps you need to take to get there". Daniel summarised the nature of expertise thus, "I would say an expert is kind, they already have the formulas in their head so they can just work it out mentally".

The same thirteen students indicated that a prerequisite of expertise was practice. Lucy's comments were typical. She said

> I think is just practice than anything else... if you're learning a language, you have to practice it, don't you? To learn it. So, like if they have different, if they have theories, like an example of how to solve a problem, and if they practice and do it over and over again, like just memorise it and you know how to use that theory then I think that would get better.

Discussion

Space limits the extent of our discussion, although a number of important outcomes have emerged that merit comment. Despite some within-country differences, the major variation lay between countries. Firstly, all Cypriot students saw mathematical problems as tasks related to real world, while their English colleagues fell into three camps; those of the same opinion, those who saw mathematical problems as purely mathematical and a majority who acknowledged that problems could be purely mathematical or related to the real world, along the lines of Blum and Niss (1991). Nevertheless, the Cypriot students tended to talk in general, almost abstract, terms while the English in particularities. For example, in defining a problem many Cypriot students focused on the generic characteristics of a problem while the English tended to offer examples of problem types from which properties could be inferred by the reader.

Similar issues emerged with respect to the nature of problem solving. Both groups of students attended, in some way, to process. The interesting difference lay in the sense that Cypriot students tended to view the process holistically – read, understand, collect data, analyse data and so on – while the English saw the process as one of simplification or reduction of the task to a series of small steps. Such differences are unlikely to be coincidental. Inevitably they will reflect the teaching these students received prior to going to university. Indeed, the English perspectives on the process of simplification find resonance with an earlier study of English and French curricular traditions (Jennings and Dunne 1996). Importantly, from the perspective of future teacher education programmes, these students are not mathematics majors but prospective generalist primary teachers. They are people who, one day, will teach young children mathematics. If problem solving is a key element of that country's curriculum, then English universities clearly need to understand the beliefs their undergraduates bring to their studies. The problems would appear less severe, at least as far as beliefs are concerned, for the Cypriot authorities.

In terms of their beliefs about the characteristics of effective problem solvers there was evidence of genuine similarity across the two groups, with substantial proportions comparing the approaches of effective problem solvers – the ability to see straight through the problem to a solution strategy with no apparent difficulty - with those of novice or inefficient problem solvers. Moreover, there was also a consensus that such skills could be acquired through appropriate practice. Lastly, such findings highlight the plea made earlier that those involved in research on problem solving, at all levels, need to acknowledge not only the lack of definitional consensus but also the key role played by culture in the construction of participants' construal of

mathematical problem, problem solving and the characteristics of effective problem solvers.

References

Aguirre, J., and N. M. Speer. 2000. Examining the Relationship Between Beliefs and Goals in Teacher Practice. *Journal of Mathematical Behavior* 18 (3): 327-356.

Anderson, J., P. White, and P. Sullivan. 2004. Using a Schematic Model to Represent Influences on, and Relationships between, Teachers' Problem-Solving Beliefs and Practices. *Mathematics Education Research Journal* 17(2): 9-38.

Andrews, P., and G. Hatch. 2000. A Comparison of Hungarian and English Teachers' Conceptions of Mathematics and its Teaching. *Educational Studies in Mathematics* 43 (1): 31-64.

Andrews, P. 2007a. Mathematics Teacher Typologies or Nationally Located Patterns of Behaviour? *International Journal of Educational Research* 46(5): 306–18.

Andrews, P. 2007b) The Curricular Importance of Mathematics: A Comparison of English and Hungarian Teachers' Espoused Beliefs. *Journal of Curriculum Studies* 39(3): 317-338.

Bandura, A. 1977. Self-efficacy: Toward a Unifying Theory of Behavioral Change. *Psychological Review* 84(2): 191-215.

Beswick, K. 2005. The Beliefs/Practices Connection in Broadly Defined Contexts. *Mathematics Education Research Journal* 17(2): 39-68.

Blum, W., and M. Niss. 1991. Applied Mathematical Problem Solving, Modelling, Applications, and Links to Other Subjects: State, Trends and Issues in Mathematics Instruction. *Educational Studies in Mathematics* 22(1): 37-68.

Borasi, R. 1986. On the Nature of Problems. *Educational Studies in Mathematics* 17(2): 125-141.

Boyatzis, R. 1998. *Transforming Qualitative Information: Thematic Analysis and Code Development*. Thousand Oaks, CA, SAGE.

Cady, J., S. Meier, and C. Lubinski. 2007. The Mathematical Tale of Two Teachers: A Longitudinal Study Relating Mathematics Instructional Practices to Level of Intellectual Development. *Mathematics Education Research Journal* 18(1): 3-26.

Chapman, O. 1997. Metaphors in the Teaching of Mathematical Problem Solving. *Educational Studies in Mathematics* 32(3): 201-228.

Charalambous, C., G. Philippou, and L. Kyriakides. 2008. Tracing the Development of Preservice Teachers' Efficacy Beliefs in Teaching Mathematics During Fieldwork. *Educational Studies in Mathematics* 67(2): 125–142.

Correa, C., M. Perry, L. Sims, K. Miller, and G. Fang. 2008. Connected and Culturally Embedded Beliefs: Chinese and US Teachers Talk About How Their Students Best Learn Mathematics. *Teaching and Teacher Education,* 24(2): 140–153.

Ernest, P. 1989. The Impact of Beliefs on the Teaching of Mathematics. In *Mathematics Teaching*: *The State of the Art, ed*. P. Ernest, 249-254. London: Falmer Press.

Givvin, K., J. Hiebert, J. Jacobs, H. Hollingsworth, and R. Gallimore. 2005. Are There National Patterns of Teaching? Evidence from TIMSS 1999 Video Study. *Comparative Education Review* 49(3): 311-342.

Grant, N. 2000. Tasks for Comparative Education in the New Millennium. *Comparative Education,* 36(3): 309-317.

Handal, B., and A. Herrington. 2003. Mathematics Teachers' Beliefs and Curriculum Reform". *Mathematics Education Research Journal* 15(1): 59-69.

Jennings, S., and R. Dunne. 1996. A Critical Appraisal of the National Curriculum by Comparison with the French Experience. *Teaching Mathematics and its Applications* 15(2): 49-55.

Kvale, S., and S. Brinkmann, S. 2009. *Interviews*. *Learning the craft of qualitative research interviewing*. Los Angeles, SAGE.

Middleton, J. A. 1999. Curricular Influences on the Motivational Beliefs and Practice of Two Middle School Mathematics Teachers: A Follow-up Study. *Journal for Research in Mathematics Education* 30(3): 349-358.

Nesher, P., S. Hershkovitz, and J. Novotna. 2003. Situation Model, Text Base and What Else? Factors Affecting Problem Solving. *Educational Studies in Mathematics* 52(2): 151-176.

Peterson, P., E. Fennema, T. Carpenter, and M. Loef. (1989) "Teachers' Pedagogical Content Beliefs in Mathematics." *Cognition and Instruction* 6 (1): 1-40.

Raymond, A. 1997. Inconsistency between a Beginning Elementary School Teacher's Mathematics Beliefs and Teaching Practice. *Journal for Research in Mathematics Education* 28(5): 550-576.

Santagata, R. 2004. "Are You Joking Or Are You Sleeping?" Cultural Beliefs and Practices in Italian and U.S. Teachers' Mistake-Handling Strategies. *Linguistics and Education* 15(2): 141–164.

Saxe, G., M. Gearhart, M. Franke, S. Howard, and M. Crockett. 1999. Teachers' Shifting Assessment Practices in the Context of Educational Reform in Mathematics. *Teaching and Teacher Education* 15(1): 85-105.

Schoenfeld, A. 1985. *Mathematical Problem Solving*. New York, Academic Press.

Schoenfeld, A. 2000. Models of the Teaching Process. *Journal of Mathematical Behavior* 18(3): 243-61.

Skott, J. 2001. The Emerging Practices of a Novice Teacher: The Roles of His School Mathematics Images. *Journal of Mathematics Teacher Education* 4(3): 3-28.

Thompson, A. G. 1984. The Relationship of Teachers' Conceptions of Mathematics and Mathematics Teaching to Instructional Practice. *Educational Studies in Mathematics* 15: 105-127.

Timmerman, M. 2004. The Influences of Three Interventions on Prospective Elementary Teacher's Beliefs About the Knowledge Base Needed for Teaching Mathematics. *School Science and Mathematics* 104(8): 369-382.

Torner, G., A. H. Schoenfeld, and K. M. Reiss. 2007. Problem solving around the world: summing up the state of the art. *ZDM Mathematics Education* 39: 353.

Verschaffel, L., E. De Corte, and I. Borghart. 1997. Pre-Service Teachers' Conceptions and Beliefs About the Role of Real-World Knowledge in Mathematical Modelling of School Word Problems. *Learning and Instruction* 7(4): 339-359.

Wilson, M., and T. Cooney. 2002. Mathematics Teacher Change and Development. The Pole of Beliefs. In *Beliefs: A Hidden Variable in Mathematics Education*, ed. G. Leder, E. Pehkonen, and G. Torner, 127-148. Kluwer.

Wolters, C., and S. Daugherty. 2007. Goal Structures and Teachers' Sense of Efficacy: Their Relation and Association to Teaching Experience and Academic Level. *Journal of Educational Psychology* 99(1): 181-193.