**Observations about some UK primary teaching that has been influenced by the mastery agenda**

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**Introduction**

From 2014 the UK government has been financing exchange visits between English and Shanghai teachers of mathematics to young pupils. The political thinking behind the decision was that, despite major differences in training, workload and cultural background, primary school teachers could change their teaching to the methods employed by teachers in Shanghai and use a 'mastery' approach. The political intention may have been to alter pedagogy, that is the environment, expectations and practices of mathematics teaching. However, a major difference observed by the English teachers is the grain size of the focus of lessons, the Shanghai teachers focusing on critical aspects of a mathematical idea where the English teachers have been used to working with varied pedagogy on broader conceptual areas. In this paper we use variation theory as an tool to analyse part of a lesson which was developed after the teacher observed lessons in Shanghai and placed, through the NCETM website, in the public domain.

**Background**

The exchanges described in the previous paragraph have been managed by the National Centre of Excellence in Teaching Mathematics (NCETM). The word 'mastery' has multiple meanings in educational practice and for this paper we are going to use it to encapsulate the idea that all pupils should learn core mathematical concepts together, with appropriate extra support where necessary. The teacher on whom we focus has been successful within past performance parameters, has been involved in the exchange programme, has masters-level experience in the psychology of early mathematics, and holds a teaching advisory position. She knows about variation not only from seeing Shanghai teacher in action but also from literature and workshops held in the UK, based on Chinese and Swedish approaches. Towards mastery the class with whom she works is an all-attainment group, with support sessions organised outside main lessons rather than differentiated work within lessons. Chanting in unison has become a feature of her lessons.

**Theoretical background**

'Variation' is a field of study in education arising mainly from the work of Ference Marton and his colleagues, who understand learning as the result of pupils discerning the variation of some aspects of an idea against a background of invariance (e.g. Marton & Pang, 2006). This idea has strong resonance within mathematics, because mathematical concepts are usually made available to pupils through examples which have some features in common but vary across representations, orientations, numerical content and so on. In order to use variation effectively teachers have to have a clear idea of the particular conceptual focus for their lessons, and the use of variation can be particularly powerful when the lesson focus is a 'critical aspect' of an overall concept. The phrase 'critical aspect' refers to aspects of a concept without which the concept cannot fully be understood, and which has caused some difficulty for pupils and teachers (ref). Aspects of an object of learning are said to be either defining, such as the size of the angle if the object of learning is a right angle, or non defining as for example the length of the two lines of the angle or the colour of the angle. Both defining and non-defining aspect can be critical. Quite a few so-called misconceptions appear when pupils assume non-defining aspects to be defining.

The aim of designing the teaching approach is to make it more likely that what the pupils' experience during the lesson, their lived object of learning (LOL), matches the teacher's intended object of learning (IOL). Between the intended and lived objects of learning is the enacted object of learning (EOL) that consists of the examples, actions, and verbal exchanges that take place in the classroom and make the object of learning available for pupils. It is the EOL that is directly observable for researchers. For example, the teacher may intend that pupils will learn about the infinitude of numbers on a given interval, but if the lesson only presents discrete points on a sub interval the idea of density on the interval has not been enacted, and some pupils may not extend the idea for themselves. However, the enactment of the object of learning does not guarantee that all pupils will experience it; they may create or experience variation against a background of previous knowledge or by shutting out disturbing variation, so the lived object of learning may differ from the enacted object of learning.

Gu at al. (ref) write about two types of variation, procedural and conceptual. 'Procedure' refers to the process of building an awareness of a new concept by presenting a sequence of tasks that lead pupils towards recognising a class of problems that can all be solved using the same method.  'Conceptual variation' is approaching a static concept from a variety of directions, visual variation and various instances and non-instances, 'same/different' distinctions.  Unfortunately the distinction between concept and procedure has developed a different meaning in English language mathematics education literature, deriving from Hiebert's distinction between knowledge and understanding of concepts and knowledge and performance of manipulative procedures (1980). To avoid confusing these different uses of the word 'procedural' we will not use these distinctions and instead focus on the object of learning that can be discerned by the relationship between variation and invariance available to pupils. In Chinese textbooks concepts are developed systematically: at first there is one problem that can be solved in several different ways, using different materials and symbolic representations. This is followed by the same mathematical structure being presented with different combinations of 'givens' and different transformations, possibly also with different, but connected, numerical content. This can be followed with a variety of problems that have the same underlying structure, and therefore similar solution methods. There is interplay between varying what *does* matter in understanding a concept so that it will be identified against an invariant background, i.e. varying the defining aspects, and varying what *does not* matter so that it will be recognised among other features.

Watson and Mason (ref) identify different layers of variation around the concept. One layer is to identify and use the dimensions of possible variation (DoV) of a concept, thus moving towards an inclusive definition of it and a list of its properties; the other is to identify and use the range of permissible change (RoC) in those dimensions, to move towards a well populated collection of examples of the concept. But the importance of underlying structure can get lost in the discourse of variation. Ultimately, a mathematical object of learning is likely to be an invariant relationship that can be experienced through working with a sequence of examples that embody the relationship (Watson ref). A dimension of variation is said to be open if the RoC includes at least two values that are present. Thus presentations of DoV and RoC in a lesson are the enactment of the object of learning. The question then arises about what features should be varied so that pupils come to appreciate the underlying structure, the IOL.

Recognising that the deliberate use of variation is unfamiliar for teachers of mathematics in England leads us to our overall research aim: *to identify the enacted objects of learning when primary teachers in England make use of variation to achieve mastery. We ask what is varied and how is it varied?*

In this paper we demonstrate our method.

**The study**

Although teachers have given permission for the videos to be available on the NCETM website, permission has also been granted separately by the teachers (who are recognisable) and their head teachers for research purposes and publication of our findings. Our purpose is not to make judgements about the quality of teaching, particularly on the basis of one video lesson, but to identify issues related to the use of variation in their teaching.

The videos were watched by the researchers, together where possible but otherwise separately. Initially each researcher produced a chronological report on the lesson describing what examples were presented and how, what questions were posed and what answers were given, with commentaries about the dimensions of variation and the associated invariants that were opened up in the lesson. These reports and commentaries were compared and collated to generate a shared view of the EOL in segments of the lesson, and how this enactment took place by the use of DOV and ROC.

The four researchers have different experiences and knowledge to bring to the initial analysis. Al-Murani (ref) had introduced the idea of variation to secondary teachers in his doctoral work, and compared pupils' learning in related topics to their use of variation in teaching. Designed as an intervention study, he did not find significant differences in the learning of pupils in the classes taught by the teachers who had training in variation and those who had not, possibly because using variation wisely was already a feature of the teaching of some of his non-intervention teachers. However, he did find that interaction between teachers and pupils around variation, whether the teacher had been 'trained' or not, provided bridges between the IOL, EOL and LOL that influenced learning. Watson has used variation as a design tool to construct exercises, example sets and sequenced tasks in which the interplay between variation and invariance draws pupils' attention to underlying concepts and relationships that are constant within varied examples (ref). Kilhamn has supervised a number of learning studies, a kind of lesson study where a group of teachers plan, enact, and revise a lesson using variation theory to enhance learning (Marton & Pang, 2003, 2013). For example finding critical aspect and patterns of variation when comparing and ordering fractions (Drageryd, Erdtman & Kilhamn, 2013) Morgan analyses in terms of the knowledge she has of the teacher's intentions, i.e. the intended object of learning, and also what pedagogic change has taken place and why.

Analysis therefore focuses on the dimensions that are varied and what is kept invariant, how variation and invariance are exploited by the teacher in the flow of the lesson, what dimensions of variation are further opened up and how these are enacted in the dynamic of the lesson. By doing this, we identify what aspects of mathematical concepts are potentially being made available for pupils to learn, described as the EOL. From this analysis we do not make assumptions about the individual capabilities of the teachers, since we only have limited access to their teaching. We also draw on publicly- available interviews where these give additional incomplete information about the teacher's intentions, the IOL, and we use some pupil utterances to give incomplete information about their experience, the LOL.

First we comment on the DOV, ROC and invariance (INV) and LOL evidence, and then provide a commentary about the EOL and its possible relation to the IOL and LOL.

Although some teachers in the exchange and in our overall study are using translations of Chinese lessonplans, the teacher in this paper designed the lesson herself using the idea of making very small steps in conceptualisation. The lesson is the second of a sequence focusing on 'difference' as subtraction with year 1. Her stated IOL for the whole lesson is to move pupils from understanding subtraction as 'take away' to understanding it as 'difference'. Here we report on one excerpt to illustrate our method:

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|  | **What happens** | **Dimensions of variation,DoV/Range of change, RoC/Invariance, INV** | **LOL** |
| 1 | The pupils are asked to represent the word problem: "5 cars in a car park, three drive away" using counters  "Who can tell me about the representation? What do the counters represent?"  "What does one counter represent?" | From the pupils:  DoV1: Geometric arrangements of 5, 3, 2; RoC: "some of you have represented five like this:  ●●●  ●●  and others have represented it like this:  ● ●  ●  ● ● "  DoV2: various actions by children to show subtraction as take away; RoC: moving counters to a different place or by removing completely  INV: cars in a car park, counters to represent cars | Pupils reply that the counters represent cars |
| 2. | T says they are going to represent cars by a line of counters | DoV1: The teacher chooses one representation layout, so this becomes an INV | Pupils represent cars in a line |
| 3. | T says: “I am going to ask you all to draw a part-part-whole. What is the whole and what are the two parts?"   |  |  | | --- | --- | | 5 | | | 3 | 2 | | DoV 3: ways of representing part-part whole; RoC: counters, boxes, numbers, spoken words  INV: cars in a car park, counters to represent cars  INV: same situation, numbers 5,3,2  INV: linear layout of counters  INV: part-part whole diagram use for additive relationship | Pupils use part-part-whole diagrams they have met before |
| 4. | T says: "Write a number sentence ... using part-part-whole" | From the pupils:  DoV 4: transformations of this particular part-part-whole relationship  INV: cars in a car park, counters to represent cars  INV: same situation, numbers 5,3,2  INV: linear layout of counters | Various pupils have written some of these:  5 - 3 = 2  5 - 2 = 3  2 + 3 = 5  3 + 2 = 5  5 = 2 + 3  5 = 3 + 2  These transformations are not new to the students. |
| 5. | "They are all correct, but one tells our story best. Which one?”  5 - 3 = 2 is selected  All chant 'subtraction can tell us about take away' | INV: cars in a car park, counters to represent cars  INV: same situation, numbers 5,3,2  INV: linear layout of counters  INV: symbolic representation of situation |  |
| 6. | "Now represent: there are five red cars and there are three blue cars."  Pupils told to use a row of 5 and a row of 3:  ●●●●●  ●●●  Colours of the counters seen as irrelevant (non defining). | From the teacher:  DoV2: different image; RoC: 'remove' or 'compare' numbers  DoV3: different coloured counters were used and the irrelevance of this was discussed.  INV: cars in a car park, counters to represent cars  INV: numbers 5, 3 and 2  INV: linear layout of counters  INV: symbolic representation of situation |  |
| 7. | "Looking at this picture what is the difference between the number of red cars and the number of blue cars. Tell the person next to you."  T draws a part-part-whole diagram to represent the situation with the cars, and overlays that onto a picture of counters:   |  |  | | --- | --- | |  | | |  |  |  |  |  | | --- | --- | | 5 | | | 3 |  |   All chant "we can use part-part-whole to tell us about difference" | From the pupils:  DoV 5: the ways the difference is worked out and expressed; RoC counting on, using number facts, comparing numbers, one to one matching  From the class discussion:  DoV 6: the way the cars are mentioned; RoC: the difference between the *red* cars and the *blue* cars, later it is asked as difference between the *number of blue counters* and the *number of red counters*, the *number of blue cars* and the *number of red cars*, 'how many more red cars ...?'  From the teacher:  DoV 7: the ways in which the part-part-whole diagram can be drawn; RoC: either schematically or drawing around the physical representation.  DoV 8: what a part-part-whole diagram can be used for; R0C: 'take away' and 'comparison'  INV: cars in a car park, counters to represent cars  INV: numbers 5, 3 and 2  INV: linear layout of counters | Pupils show different ways to express the 'difference':  “3 is 2 less than 5 and 5 is 2 more than 3”  The difference “between 5 and 3 is 2”  “3 is a smaller number than 5” |
| 8. | T gives a new task: “There are 7 children and there are 4 dinner tokens. Represent that.” Students work in pairs using counters and drawing part-part-whole diagrams.  T says: “What [one pupil] said is that 3 and 4 equals 7. And we could use that addition fact to help us find the missing part. To help us find the difference” | DoV 9: numbers change  DoV 5: ways to work out difference  DoV 10: meaning of 'difference'; RoC: compare quantities of two similar objects; do a one-to-one matching of different objects  INV: linear layout of counters  INV: part-part whole diagram use for additive relationship  INV: use of known model for subtraction to find the difference | Some pupils seem to find it hard to identify the whole in the relevant part-part-whole diagram. |

**Commentary**

Subsegments 1 and 2: the variation provided by the pupils in the DoV is expected and acknowledged by the teacher through open choice of representation. She uses the metaphor of “letting the kite out” to describe how she lets the pupils work creatively for a while, showing their LOL to generate an ROC, before “pulling the kite in again” when drawing them all together to continue using the same representation, thus introducing an invariant (INV) notation. The EOL is to know how subtraction as take away can be represented using counters, and in particular to recognise a linear layout. This is achieved through allowing variation and then being explicit about using one layout.

Subsegment 3. IOL is transition from counters in a linear layout to a familiar part-part whole diagram, enacted as a visual transition. EOL is to connect the two representations.

Subsegment 4 and 5: as with subsegment 1 and 2, teacher C allows the DoV to play out as a LOL and then explicitly selects the particular variant required.

EOL is to see how the part-part-whole diagram can describe the story situation of 'take away' and that one symbolic format most appropriately describes the situation.

Subsegment 6: EOL is that a 'compare' situation can be represented using counters, in a linear layout

Subsegment 7: DoV introduced by pupils in the ways they find difference, but the EOL is to compare two lines of counters. The IOL was to focus on part-part-whole so they 'understand it in a way that they can apply it to a different structure’ (quoting the teacher).  
EOL is to represent a 'compare' situation with a part-part-whole diagram and to name the missing part as the 'difference'..

Subsegment 8: EOL is to represent and work out the difference in a new situation using a part-part-whole diagram. A new DoV has been introduced which is the meaning of 'difference': comparing two quantities of similar objects (cars) or matching two different sets of objects (pupils and dinner tickets). LOL is not the same for all students as some of them cannot decide which group is the 'whole'.

In this segment the part-part-whole diagram and the linear layout of counters are invariant models for understanding, and the words of the story provide context which changes from being about 'take away' to being 'compare'. Because so much is invariant in subsegments 1 to 7 the transition from 'take away' to 'compare' seems relatively successful. This shift of meaning is the main variation. However, in subsegment 8 the numbers vary and so does the meaning of 'difference'. Even as analysts it took us several cycles to recognise this variation, which could explain why some pupils hard difficulty in applying the thinking they had just done to this new situation. The word 'whole' in the phrase 'part-part-whole' no longer had its previous meaning. The enacted object of learning can be discerned from looking at differences against a background of sameness; the lived object of learning is how pupils discern difference against a background of sameness.

**Discussion**

The first thing to acknowledge is the mathematical coherence underlying the IOL: that a whole lesson is devoted to developing a new meaning for subtraction. The teacher intends this to be done through using images and language (part-part-whole) that the pupils already know, and the diagrams and the variation of the symbolic number sentences they produced shows that underlying additive structure has been a consistent experience for the pupils. They also talk about knowing number facts. This means that for the majority, and possibly all, pupils the lesson is not about counting but about using familiar representations to compare numbers and calculate difference - a new idea in relation to the subtraction they have done previously. As we analyse the lesson we see one after another various DoV being pinned down to become invariant so that eventually the new idea of 'difference' becomes the only change around, and is then emphasised through a chanted phrase.

The excerpt we analyse lasts for about 25 minutes and is mainly about 5, 3, and 2. The emphasis is on structure and meaning rather than calculating with specific numbers. We are struck by the contrast between this and lessons where the emphasis is on doing the same thing again and again with different numbers. It is also noticeable that the pupils are treated as if they know the relevant number facts that they will need to use, so the subtraction they are doing are not laborious calculations, but adaptations of known number facts. However, because of this teacher's wide previous experience and knowledge we are reluctant to attribute this totally to the Shanghai exchange. Indeed, this level of care for detail and for what children might perceive is a feature of teaching in many countries and contexts. The authors of this paper were aware of these ideas before knowing about so-called 'Shanghai' methods.

We are interested in

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