variation within, and covariation between, representations

Angelika Kullberg Anne Watson John Mason

Göteborg University Oxford University Oxford & Open

Universities

This is a theoretical paper in which we conjecture that some difficulties in learning mathematics may be due to the need to coordinate covariation between different representations, not merely to ‘match’ individual states. We use research within variation theory on learners’ understanding of the infinity of numbers in an interval to draw attention to the need to coordinate variation of decimal representation with position on a numberline. We problematise this relationship and use a small empirical study to hypothesise how learners might relate these implicitly and explicitly. This covariation lens augments the usefulness of variation theory in thinking about mathematics pedagogy and learning.

Introduction

In a series of learning studies, Runesson and Kullberg (Kullberg, 2007; Runesson & Kullberg, in press) developed ideas concerning teaching students about the density of rational and decimal numbers. Over a series of cycles of co-planning, teaching, observing and reviewing with a group of three teachers they constructed a list of core ideas which, when presented to students, appeared to bring about a change in their understanding of the infinitude of numbers in any interval. These ideas, called *critical features*, were then offered to two teachers who were not in the original study, as the basis for their lesson planning. In this paper we look not at this process, but at the critical features from the perspective of a learner who is expected to develop understanding of the mathematical idea by *discerning variation* in what is made available to experience by the teacher and other students in the *space of learning* (Marton & Booth, 1997; Marton, Runesson, & Tsui, 2004).

Critical features and variation

The theoretical framework, variation theory, describes how people learn, perceive and experience the world around us. Variation theory originates from phenomenography (Marton & Booth, 1997) and is influenced by the idea of learning as differentiation (Gibson & Gibson, 1955), though its roots can be traced back to Aristotle. Gibson and Gibson argue that for a child to identify an object, he must be able to identify differences between it and other objects. From this it follows that the way we experience something is a function of features we notice or discern at the same or nearly the same time. A variation (change) in a feature makes it more noticeable than if it remains invariant, and so more open to being discerned and experienced as a feature. For instance, variation in representation of numbers between fractions and decimals makes it possible to discern a finite decimal number as a part/whole relationship, because the ‘part’ and ‘whole’ are shown in the decimal digits. Every concept or phenomena has particular features that are critical for learning. Marton *et al* (2004) argue that the critical features must, at least in part, be found empirically, for instance through interviews with learners and through analysis of learning the specific content in classrooms. The critical features developed in the earlier study (in grade 7) for the density and continuity of number are:

* Decimal numbers, seen as numbers (on a number line)
* Rational decimal numbers seen as a form of expressing rational numbers (where fractions and percentages are other forms)
* Decimal numbers seen as part/parts of a whole

- e.g. 0.97 as 97 centimetres of a one metre ruler

- the interval can be divided in smaller and smaller parts (e.g. hundredths, thousandths)

The theory suggests that for learning to take place, students need to understand and appreciate these features. In variation theory, this means that *dimensions of variation* have to be opened up in lessons so as to allow or prompt learners to see what these features mean. Watson and Mason have developed this idea further to make it more applicable in mathematics (2005, p.5). While one possible dimension of variation of number is the size of the written digit, this dimension is unconventional and it does not afford the possibility to explore continuity, so we restrict the concept to *dimensions-of-possible-variation* which do not change the focus of learning. Watson and Mason also introduced the notion of *ranges of change* to allow the dimension to be explored in different ways by learners, and *range-of-permissible-change* to show that these are constrained by the need for mathematical meaning.

how teachers enacted the object of learning

The two ‘new’ teachers in Kullberg’s study enacted the critical features in different ways despite coplanning, and this will be reported elsewhere, but both initiated activity by asking learners the key question ‘how many numbers are there between 0.17 and 0.18?’ and both offered decimal notation, fraction notation, and a numberline as tools for discussion of this question. Looking at these representations from the perspective of available dimensions of variation we realised that they are very different, and the learners’ task is to see the static representations as equivalent, that is to relate 0.175, 175/1000, and a point mid-way between points marked 0.17 and 0.18 on a line. Acceptance of three disparate signs for the same thing can either be done without meaning as a practice to be internalised, or meaningfully through dynamic relationships between the variation in each representative form. Even if one has doubts about whether variation theory tells the whole story of learning, this insight does lead to something quite useful, because the ranges of change in each of the three representations are so very different. In decimal notation, one dimension of variation is a single digit in the decimal, and the range of permissible change is the digits 0 to 9; on the line the range of permissible change is physical position; in vulgar fraction notation there are two ranges that have to be coordinated, the numbers in the numerator and in the denominator. Considering the first two of these, the learner has to understand and appreciate how variation in the digits relates to change in position on the line, how cycling through digits, and extending the string, matches the simple action of moving a point.

Semantic and syntactic actions with notation

It is widely accepted in several traditions that it is spurious to try to separate concept from representation. However, relating representations as if they have the same meaning seems to require some kind of isomorphism that allows manipulation of one to ‘match’ manipulation of the other (Vergnaud, 1998). Wertheimer (1945) had a sense of this when he showed that some algebraic representations of area express spatial arrangements while others do not. Similarly, Dorfler (2005) sees the internal structures of diagrams as direct expressions of mathematical relationships. He also claims that algebraic notation is diagrammatic in the same sense.

In Confrey and Smith’s (1994) development of the idea of covariation they reject the need for obvious isomorphism and instead talk about how change in one variable shows up as change in another variable. Note that the distinction between dependent and independent variable is situated when you think in terms of covariation, since you can sometimes vary several different variables and detect related variation in others. In their work, all the variables are quantities so a table of ordered data pairs enables comparison to be made on the basis of numerical change. If we regard each of two representations as providing a variable, we can see that what might be required to understand how representations are related is to look at how small changes in one representation are mirrored by small changes in the other.

But in the situation we describe a small change in position might correspond to large change in the number and nature of digits after the decimal point, or the denominator of a vulgar fraction. Thompson’s view of covariation (e.g. 2002, p.205) points to the importance of coordinating actions on each variable with an ‘operative image’ so that one can act out how the two are related. He also shows how even the necessary idea that a point on a line represents a value is itself a potential obstacle, so that 5th and 8th grade students have to work quite hard with his image of bunny hops to make sense of the duality of point and line. This observation, combined with the difficulty Confrey and Smith’s college students had in coordinating data, suggests a further difficulty – that changes in decimal and fraction representations of very close numbers can be visually dramatic, while the shift on the line is tiny. Thus number might be perceived as ‘more discrete’ than continuous movement on a line.

Covariation and a ‘covariation approach’ (Confrey & Smith, 1994) are examples of when several *dimensions of variation* are opened up at the same time for students to experience. In a ‘covariation approach’ to a task involving two variables, variation in two dimensions is used to produce a graph. For instance, Confrey and Smith show that in regard to functions *y*=*f*(*x*), a covariation approach could entail ‘being able to move operationally from to *ym* to *ym+1* coordinating with the movement from *xm* to *xm+1* ’ ( p. 137). In our example, about the density and continuity of number, it would entail coordinating different representations of a decimal number, as well as both the range of change within the same representation and between representations.

covariation in students’ representation of numbers

Dufour-Janvier *et al* (1987) identify difficulties with early use of representation of the numberline on students’ later learning concerning the density and continuity of number. The early use of a discrete numbertrack as ‘stepping stones’ with gaps between counting numbers is a long way from the concept of the density of the real numbers as illustrated by a continuous numberline. ‘It is hardly surprising that at the secondary school so many students say that between two whole numbers there are no numbers, or at most one. Nor should there be much surprise that they also have great difficulty placing a number if they cannot associate it with the gradation already given on the line’ (Dufour-Janvier, Bednarz, & Belanger, 1987, p. 117). Furthermore, when using multiple representations of the same concept it is expected that the learner will be able to grasp the common properties and will succeed in constructing the concept. The learner should also be able to ‘reinvest the knowledge acquired, in contexts embedding different aspects of the same concept’ (Dufour-Janvier *et al*., 1987 p. 112). From a variation theory perspective this means that to be able to grasp common properties students must discern similarities and differences between the representations. This implies covariation.

To explore these ideas of covariation further, we provided some representational tasks for year 7 and 9 students in one school. To persuade teachers to participate the questions were such as could be offered in normal class time by the usual teachers, and the questions had also to be of use to the teachers’s planning. We received completed scripts from 49 year 7 students (11/12 year-olds) and 51 year 9 students (13/14 year olds). The questions which will be reported on in this paper are those that gave us some insight into how students coordinated covariations between numberline and decimal representations of number.

Relating decimal numbers and a scaled interval of a numberline

Three kinds of task each offered a scaled interval from 1.7 to 1.8, with ten subintervals between, and asked students to:

1. identify numbers represented as points on the numberline segment which coincided with scale marks
2. represent specified numbers on the numberline segment, which would coincide with scale marks
3. generate some numbers themselves and place them on the numberline segment.

(i) Write down the numbers marked on this numberline in decimal notation



(ii) Represent these numbers on a numberline: 1.75, 1.76, 1.77, 1.78



(iii) Write down three decimal numbers between 1.7 and 1.8. Show where they would be on this numberline:



Almost all year 9 students and ¾ of the year 7 students did task (i) successfully. Almost all year 9 students did task (ii) successfully, but only half of year 7 did so. This difference in response suggests that ‘reading’ the scale is easier than representing given numbers on a scale, and the relationship between the two (the Cartesian duality) is not fully coordinated. For task (iii), however, when asked to write their own numbers on the same segment, almost all year 9 students were successful and 5/8 of year 7 were also successful. Since some year 7 students were better at labelling points correctly than placing particular numbers they may have used this approach for the hybrid task (iii).

A few year 9 students used three decimal places and numbers ‘off’ the scale marks for task (iii) to demonstrate full understanding.

Some of the failed attempts to relate decimal numbers and scaled numberline representations were illuminating. Several year 7 students labelled the line: 1.17, 1.27, 1.37 … instead of 1.71, 1.72 …. We take this to mean that they appreciate variation along the line, and knew that this related to variation in digits, but not the meaning of those digits. There were other, similar, failed attempts which suggested partial coordination of variation, and these students were all unsuccessful in task (ii).

Relating decimal numbers and ‘unscaled’ segments of numberlines

Two tasks asked students to work with unscaled lines. We had also asked students to say which questions they found hardest, and all students who selected particular questions as hardest chose one of these, as predicted by Dufour-Janvier *et al*. Two students explained their choice as ‘there were no lines to help’. It is also the case that these two tasks could not be done with only two decimal places, so they were inherently harder for that reason too. These tasks had far fewer successful responses in both years, but for our purposes we were interested in what students tried to do and the plausible reasoning behind these attempts. It is often the case that intelligent unsuccessful attempts reveal more about what is needed to understand a concept than is shown by successful attempts.

*Task a*: Represent these numbers on this line: 1.7, 1.71, 1.701, 1.7001



For this task, students were given numbers expressed as decimals and asked to put them on an unscaled numberline. The decimals were chosen to expose classic difficulties with place value, and only 14 students completed it correctly, but for our purposes we were interested in what students did rather than what they failed to do. 41 other students attempted this task. 20 students, half of the failed attempts, took the numbers in the order given and placed them along the line equally spaced. This was the most common approach. The next most common approach, taken by 11 students, was to order the numbers correctly, but space them equally along the numberline, showing knowledge of variation in digits and numbers of decimal places, but uncoordinated with positional variation. Of particular interest were 5 students who understood something about positional variation and how it related to the variation in digits and places, but ordered the numbers incorrectly. This attempt at coordination suggests that covariation is nearly understood, but the underlying sense of numerical value has not been brought into the task. 5 other students ordered the numbers correctly, and knew not to space them equally, but the ways they were placed showed little understanding of relative position. All students who attempted this task had been successful at the earlier task of generating their own numbers and placing them on a line, so we can assume that the coordination task is, for them, not solely syntactic.

*Task b*: Estimate the numbers shown on this numberline in decimal notation:



For this task, some attempts split the unscaled segments into fractions and expressed points as, for example, 1.7, 1.7, or as 1.7 or other simple fractions. One student combined knowledge of decimal notation with this approach and offered labels such as: 1.71½. This suggests knowledge of variation of position, expressed as vulgar fractions, not coordinated with variation of digits in decimal notation. Other students used three decimal places but only a few of these managed to produce plausible estimates. Responses like 1.701, 1.702 for the first two points were fairly common among this group. They showed knowledge of variation of digits but had not coordinated this with variation of position.

Some of the students who did not use more than one or two decimal places in this task did appear to understand further places in *Task a*, so we cannot assume that the use of further places was unavailable.

a ‘covariation’ perspective on representations

The covariation literature points us towards relating change in one variable to change in another, and in the topic we have been investigating this means relating the change in one symbolic variable to the change in another.

Two aspects of the data support the conjecture that covariation augments understanding of representation. Firstly, the difference between the responses of year 7 and 9 students shows something more has to be understood than labelling a scaled numberline segment in order to complete the relationship between decimal numbers and the numberline representation. Placing given numbers correctly on a line, i.e. representing numbers, is harder than interpolating missing values on a scale. The interpolation of missing values could even be a learnt exercise that is unrelated in the students’ mind to decimal number.

Having established that the two-way relationship develops over time, we then established, by identifying the plausible thinking behind failed attempts, that students do indeed have to coordinate positional variation with digital variation and that this is a non-trivial process. From a variation theory point of view we can see that students have not yet discerned all the critical features of number. The representations offer the means to discern critical features, and covariation offers a systematic context for near-simultaneous discernment of their variation. Even when relatively successful, students ignore the underlying value of the numbers they are representing by focusing on matching individual cases rather than relating variation in different dimensions. Fractional representation relates more easily to static, isolated, special cases depicted by chopping segments of the line than to continuous number.

Further research

The research constraints prevented follow-up interviews at this stage, but our intention had only been to confirm some conjectures and set the scene for further investigation. Furthermore, Thompson’s work on covariation with college students (2002, p.203) showed that they typically found it very hard to talk about relations between rates of change although they could operate successfully with them to some extent at the level of examples and algebraic representations. We conjecture that, to get young adolescents to talk about relationships between digital and positional representations, we would have to know more about their likely perceptions in order to prompt them. This research prepares the ground for such enquiries.

Our hypothesis is that learning to coordinate different representations of mathematical variation is not merely a matter of translation, but also of understanding covariation. We further suggest that dynamic approaches can play a substantive supporting role.

References

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Dorfler, W. (2005). Diagrammatic thinking: Affordances and constraints. In M. Hoffmann, J. Lenhard, & F. Seeger, (Eds.), *Activity and sign: Grounding mathematics education*, (pp. 57-66) NY: Springer.

Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*. (pp. 109-122), Mahwah, NJ: Lawrence Erlbaum Associates.

Gibson, J. J., & Gibson, E. J. (1955). Perceptual learning: Differentiation –or enrichment? *Psychological Review, 62*(1), 32-41.

Kullberg, A. (2007). Can lessons be replicated? In J. H.Woo, H.C. Lew, K.S. Park & D.Y. Seo, (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education.* (Vol. 3, pp. 121-128). Seoul: PME

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ.: Lawrence Erlbaum Associates.

Marton, F., Runesson, U., & Tsui, A. B. (2004). The space of learning. In F. Marton & A. B. Tsui (Eds.), *Classroom discourse and the space of learning*. (pp. 1-44) Mahwah, NJ.: Lawrence Erlbaum Associates.

Runesson, U., & Kullberg, A. (in press). Learning from variation. Differences in learners’ ways of experiencing differences. In B. Sriraman, C. Bergsten, S. Goodchild, C. Michelsen, G. Palsdottir, O. Steinthorsdottir & L. Haapasalo, (Eds.), *The Sourcebook on Nordic Research in Mathematics Education.* Charlotte, NC: Information Age Publishing.

Thompson, P. W. (2002). [Didactic objects and didactic models in radical constructivism](http://pat-thompson.net/PDFversions/2002DidacticObjs.pdf)*.* In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing and Modeling In Mathematics Education*. (pp. 191-212) Dordrecht, The Netherlands: Kluwer.

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *Journal of Mathematical Behavior,* *17(2)*, 167-181.

Watson, A. & Mason, J.(2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Lawrence Erlbaum Associates.

Wertheimer, M. (1945). *Productive thinking*. NY: Harper & Row.