VARIATION AND MATHEMATICAL STRUCTURE

Anne Watson and John Mason explore how variation and invariance within examples and exercises can engage learners with mathematical structure.

When examples are selected in order to demonstrate techniques and ideas to students, how are they chosen and what is the effect of that choice? This question has been haunting us for several years and we have learned a lot by thinking about the choices we ourselves make and the choices we see made in other people's lessons. We have recently been finding it very useful to apply variation theory to these questions. This theory has been developed by Ference Marton (2005) and his co-researchers, some of whom are using it to study mathematics lessons and learning. He believes that learning only happens if there is some variation to discern and he sees learning as the discernment of variation. Because mathematical concepts are largely concerned with variables and structures, the theory applies fairly easily to learning about mathematical concepts and techniques, but needs some modification to make it work well. For example, we need to consider invariance as well as variation and it turns out to be important to work out how much or how little variation is necessary for learners to notice what we hope they will notice.

For example, the three diagrams in *figure 1* of straight line graphs all contain graphs of different gradient, but in the first there is too little variation to be of use and it shows special cases only, while in the last there is so much variation that what is noticed most is the invariant point.

The visual impact of these examples is important in helping learners focus on gradient as a concept, which is why it may be important to have several lines on one pair of axes, rather than on separate diagrams, so that the variation is what is discerned.

Does visual impact matter when using symbols, such as when learning about algebraic relationships?

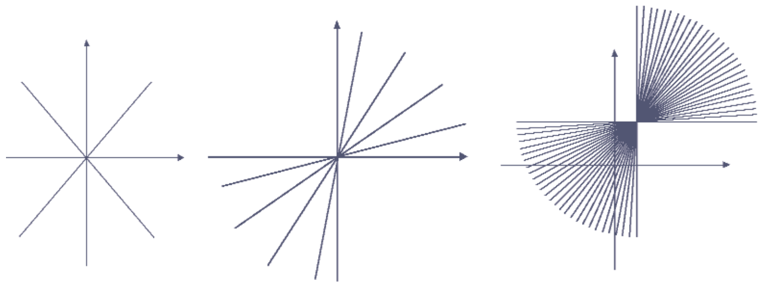

figure 1

Consider this collection of identities:

$$\begin{aligned}(x-2)(x+1) &\equiv x^2 - x - 2\\ (x-3)(x+1) &\equiv x^2 - 2x - 3\\ (x-4)(x+1) &\equiv x^2 - 3x - 4\end{aligned}$$

Lining up the symbols draws attention to the changing coefficients, so that the changes on the right-hand side *look* as if they might be caused by the changes on the left-hand side. Learners observing this sequence as it unfolds before their eyes might develop an automatic expectation of what comes next and might also be explicitly encouraged to have an expectation about what comes before. Perceived variation and invariance generates *expectations* which may be conscious or subconscious. Seeing their expectations confirmed can inspire confidence for learners. If learners think that mathematical examples are fairly random, or come mysteriously from the teacher, then they will not have the opportunity to experience the expectation, confirmation and confidence-building which come from perceiving variations and then learning that their perceptions are relevant mathematically.

It is worth asking, when selecting examples to use in exposition, what possible patterns and expectations learners will be able to develop from the collection of examples offered to them. If there are too many possible patterns to see it is unlikely that expectations will be formed; learners who do develop expectations may be discouraged to find that their minds are not in tune with the teacher's. If there are too few opportunities to develop expectations, then learners become dependent on the teacher to tell them what is going on since they cannot see it for themselves.

For learners to develop well-founded expectations, they need to experience more than one example. We have found it powerful to think in terms of sequences of examples which invite generalisation and which also point to special cases. For example, the quadratic sequence above offers lots of opportunity to explain how the right-hand side might be related to the left and hence to deduce the general method of multiplying brackets, and it also offers the special case of difference between squares by extending the sequence upwards. The use of small integers is not accidental; it is a deliberate choice to make these responses more likely.

Some might say that this sequence does not offer enough variation. For example, it neither varies signs nor coefficients as widely as textbooks commonly do. Our argument is this: at some stage students need to know how to multiply pairs of brackets containing two terms. Indeed they have already done so when learning two-digit multiplication. At some stage they also need to incorporate knowledge about signs and knowledge about multiplying more complicated algebraic terms into this process – but first let's build up fluency and familiarity with the basic process. If this doesn't happen, learners are likely to get stuck on exercises not because they do not know how to 'remove brackets' but because they muddle signs and coefficients.

This may sound as if we are advocating a mastery approach in which learners have to become fluent in one thing before learning another – but we are not. Instead we are arguing for more clarity and honesty about mathematical ideas, particularly about the structure of examples through which learners encounter mathematically significant variation. When teaching multiplying brackets, attention can be focused on the use of variation to reveal the patterns and generalities which result from the techniques. That is the clarity. Then comes the honesty: consider how far and in what way each possible feature can change. In this way learners can be encouraged to develop a sense of the extent of the class of objects. In the case of quadratics, we can play with signs, play with coefficients, play with letters, so that learners know that the following is a quadratic:

$$-0.827 \sin^2 a + \sin a - 7.89$$

Others may say that this approach is too much like 'examples and practice' and that we ought to be advocating development of mathematical thinking throughout mathematics. We would agree that we ought to be nurturing mathematical thinking, but not that this approach promulgates only 'examples and practice'. To make sense of the sequence, learners have to see patterns, form expectations, express these as conjectures about patterns, and generalise – all aspects of the mathematical thinking we want them to develop. But they have to go beyond these processes to form conjectures about the relationships within the sequence and generalise to uncover the underlying procedures which might work in other cases. It is not enough to carry the sequence on downwards and upwards, going with the grain, as it were. They need also to articulate what they think is going on horizontally *across* the grain.

The quadratic sequence, therefore, offers a way to deduce relationships and methods through observing and extending controlled patterns in given examples.

Here are two more examples of how attention to variation can focus learners on essential structures. The first is a sequence of questions to solve for x:

$$\begin{gathered} 2x + 3 > 7 \\ 2x + 3 < 7 \\ 2x + 3 = 7 \end{gathered}$$

The only thing which varies is the relationship symbol and this focuses learners on their meaning rather than on algebraic manipulations. Equality becomes a special case of the relationship between $2x + 3$ and 7. This is in contrast to the typical understanding that inequalities are somehow equalities which have 'gone wrong'.

The second example is a sequence of trigonometric identities:

$$\begin{aligned} \sin^2 x &+ \cos^2 x \equiv \\ 2\sin^2 x &+ 2\cos^2 x \equiv \\ 3\sin^2 x &+ 3\cos^2 x \equiv \\ 4\sin^2 x &+ 4\cos^2 x \equiv \\ &\cdots \\ &\cdots \\ e^x \sin^2 x &+ e^x \cos^2 x \equiv \\ \cos x \sin^2 x &+ \cos^3 x \equiv \end{aligned}$$

The first could be presented graphically, so that learners can see that it always is identical to 1 and they could test values, perhaps within an exploration of Pythagoras' theorem. The next few could be presented as a sequence once the first identity has been established. Learners can react to the visual impact of the layout and see how a family of identities can be generated from the first one. At some point they can be asked what other coefficients could be put in front, developing from positive integers, through fractions to arbitrary numbers. The final two examples might come as a slight shock, and can be checked graphically once an answer has been predicted. They might promote the perception that not just numbers but any function whatsoever can be used as a multiplier. Alternatively, learners can be asked to create for themselves more complicated ways of generating identities by multiplying both sides by any number or any function they like.

An extension of variation theory which we have developed is to focus not only on *dimensions of possible variation* (such as gradient, or coefficients of quadratics, or inequality signs) but also on the *range of change* permitted within each dimension. In the gradient examples, we have stayed with positive gradients and in the quadratics we have stayed with small integers, but as teachers we have to think of going beyond these rather obvious ranges of change so that learners are as familiar with negative gradients and non-integers as possible. In the trigonometry sequence we kept the essential relationship and the layout constant and varied only the coefficients to focus learners on the ways the basic structure might appear in different guises.

This way of thinking about selecting and constructing examples to use in exposition and practice also gives us a way to engage learners in hard mathematical thinking even when being introduced to new procedures and technical aspects of mathematics. Learners are given raw material for conjecturing and generalising instead of merely following our worked examples, because each example only varies in one way from the one before it. For instance, if this sequence is offered:

Divide 60 into the ratios 1:1; 1:2; 1:3; 1:4; …

there is an opportunity to notice and understand the changing nature of the unitary part and hence the nature of ratio; more than if a sequence of highly varied examples is offered.

When using variation to plan tasks, it is usually the case that there are several possible dimensions of variation which could be used. Compare these two exercises:

Expand:	$x(x + 2)$
Remove brackets:	$x(x + 2)$
Multiply:	$x(x + 2)$
Express as powers of x:	$x(x + 2)$

and

Expand:	$x(x + 2)$
	$-x(x + 2)$
	$x(x - 2)$
	$-x(x - 2)$

These two exercises each have the same theme, but in one of them the focus is on instructional language while the other focuses on signs. Choice of dimension of variation can be closely linked to the purpose of setting the exercise – the particular focus of the teacher. Even unintentionally, the dimensions being varied can encourage a particular focus for the learner. Thus a deliberate choice about how to vary the questions or examples implies that the teacher has a very focused purpose in mind, rather than just 'practice' or 'consolidation'.

Careful choice of examples in which certain aspects vary and others remain unchanged can, in these ways, draw learners into engaging with mathematical structure and meaning throughout their mathematics lessons, not just in special lessons or tasks which are designed to encourage thinking. Furthermore, desire and need to cover the curriculum is not upset by this approach, because the mathematics being explored is curriculum content. But what is far more important is that learners' ability to see, expect and think is, through controlled variation, being used fruitfully to learn mathematical methods and concepts.

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Reference

Marton, F. and Tsui, A. (2005) ***Classroom Discourse and the Space for Learning***, Mahwah, NJ: Erlbaum

The Review of the Primary Framework for Teaching Mathematics

The Primary National Strategy is currently undergoing a period of consultation with a view to reviewing the primary framework for teaching mathematics. This is our opportunity as teachers to put our views forward about changes we would like to see and what we would like to keep.

Over the next few weeks the ATM is putting together a position paper in relation to the review. If you would like to contribute to this paper, please post comments on the website noticeboard so that your views can be taken into consideration. The position paper will be published on the ATM website. Go to www.atm.org.uk/policyopinion/

You can also contribute directly to the Primary National Strategy by using the link below
www.standards.dfes.gov.uk/forums/
and then choosing the link 'NEW! Primary National Strategy: Reviewing the Frameworks for teaching literacy and mathematics'.

You might find it helpful to use ideas from the ATM position paper when formulating your own response.