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Using learner generated examples to introduce new concepts

Anne Watson · Steve Shipman

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Abstract In this paper we describe learners being asked to generate examples of new mathematical concepts, thus developing and exploring example spaces. First we elaborate the theoretical background for learner generated examples (LGEs) in learning new concepts. The data we then present provides evidence of the possibility of learning new concepts through a symbiosis of induction and abduction from experience and deduction from the relationships generated in exemplification. In other words, experience can be organised in such a way that shifts of understanding take place as a result of learners' own actions. Actions, in this context, include mental acts of organisational reflection.

Keywords Conceptual learning · Examples · Learner generated examples · Mathematical abstraction · Mathematical generalization

1 Introduction

The studies described in this paper are situated within two theoretical ideas: variation theory, emanating from the work of Ference Marton, and theories associated with learner generated examples, LGEs, (e.g. Watson and Mason 2005). Marton's view is that learning takes place through discernment of variation in near-simultaneous events, the act of discernment itself being an aspect of learning (Marton and Booth 1997). Some of the work of Marton and his associates (e.g. Marton, Runesson and Tsui 2004, p.4) draws our attention to the importance of *perception* of variation.

This view translates easily into mathematics, since mathematical structure manifests itself through relationships among variance/invariance and similarity/difference (e.g. Davis and Hersh 1980, p. 198). Discernment of these relationships leads to defining classes of

A. Watson (✉)
Department of Education, University of Oxford, 15 Norham Gardens, Oxford OX2 6PY, UK
e-mail: anne.watson@education.ox.ac.uk

S. Shipman
Fitzharry's School, Abingdon, UK

mathematical objects, understanding the effects of operations, expressing and manipulating relationships, reification of relationships as new objects, and so on. ‘Object’ here means something that can be thought about: a physical thing, a symbolic representation, or an abstract idea, such as a classification or a relationship, which is represented by symbols or physical artefacts. As mathematicians we also appreciate the importance of *generating* variation. Personal control of variables contributes to perceiving, enacting and understanding the relationships being exemplified. Simon, Tzur, Heinz and Kinzel describe this as a process of reflection on the relationship between activity and its perceived effect (2004). This suggests that if students generate examples, reflection on those examples could, through perceiving the effects of the variations they have made, lead to awareness of underlying mathematical structure. ‘Structure’ here means how elements and properties of mathematical expressions are related to each other.

The idea that LGEs can be used to motivate conceptual understanding is implicit in the writings of mathematicians urging others to create their own examples when interpreting mathematical text (e.g. Halmos 1983; Feynman 1985). In Watson and Mason (2005) we elaborated on this idea as a pedagogic tool. However, there are some difficulties in understanding why it can be effective.

2 Obstacles to the claim that school learners can gain some understanding of new-to-them ideas by generating examples

Firstly, some authors suggest that it is hard to see how a learner can construct objects without already having an idea of what to construct. What is described as the ‘learning paradox’ (Fodor 1980) suggests that learners cannot construct a conceptually richer system than those they already know. However, claiming this is true for mathematics assumes that learning is about becoming acquainted with unfamiliar ideas, whereas in mathematics the methods of enquiry and construction themselves belong to the mathematical canon and allow unfamiliar objects to be made from familiar ones. Indeed, Fodor’s definition of learning (1980, p.149) seems more to do with inductive formation of conjectures than acquisition of facts, so his claim rests on a belief that one cannot hypothesise about ‘richer’ ideas from empirical experience. However, asking learners to construct a function which is not continuous requires knowledge of what ‘continuous’ means, and the ability to break some of the associated conditions. It does not require pre-existing knowledge of actual non-continuous functions from which to hypothesise. Learners may have favourite non-examples (Tsamir, Tirosh and Levenson 2008) but our claim is not that constructing non-examples is easy, merely that it requires nothing more than the ability to manifest one of a range of alternatives to a condition. It does not require higher-order perceptions about relationships such as those that might be needed to make counter-examples (Zaslavsky and Peled 1996). Exercising variation beyond perceived limitations is one way of constructing unknown objects; another is to identify the underlying structures of mathematical representations, foreground and background different aspects of them, and present them in new forms. For example, learners can consider the need for ten number symbols when using powers of ten, and then explore through construction what would happen if we used powers of two—this does not require existing knowledge of binary systems, but does require the opportunity to reflect on individual examples and, through abduction, conjecture about meaning, structure and relationships (Peirce 1931).

Secondly, it is often assumed that learners cannot achieve higher-level understandings empirically, because the actions of mathematical thought required to generate data, perform procedures, and observe similar examples, are not sufficient for conceptualisation. In order to conceptualise there has to be some shift to 'higher mental functioning' which is somehow structured by expert others (Vygotsky 1978 p.57). Schmittau (2003), for example, provides a powerful critique of some current US practice in which learners are encouraged to generate and compare examples to achieve abstract understanding. This traps learners, she claims, into a metonymic world in which their spontaneous conceptions have to be stretched and mangled to cope, often inadequately, with the generalisations their example-generation has produced. However, Davydov himself, on whom Schmittau's work depends, claims that comparison *is* the way to perceive the structures, dependencies and relationships which characterise mathematical abstraction (Davydov 1972, p.93). One way to sort this out is to grasp that what Schmittau is pointing to is the habit of expecting learners to spot patterns and continue them, comparing cases recursively to 'spot' how patterns were generated. Watson has called this 'going with the grain' (2000), (a metaphor from wood-carving in which cutting in the direction of the grain of the wood is superficially easy), to indicate that such term-to-term generalizations may not represent structure. If, however, learners are presented with representations of relationships and have to discern structure for themselves ('going across the grain' and hence revealing the internal structure of the wood) then we have a learning phenomenon which can be described in several traditions. For example, in the Vygotskian tradition this could be described as providing scaffolding, through carefully designed examples, for learners to operate at a higher mental level than they would otherwise, bringing their spontaneous conceptualisations into contact with the formal culture of mathematics.

A further issue is that the literature on using LGEs to engage with new ideas is mostly concerned with advanced learners. For example, Dahlberg and Housman describe how mathematics undergraduates learnt about 'fine functions' by being given a definition to exemplify (1997). They concluded creating examples could be a powerful 'learning event' in which students made real progress in understanding. Hazzan and Zazkis (1997, 1999) asked pre-service teachers to create examples of mathematical objects with various properties and found several ways of proceeding: some used trial and error to create one example, others constructed examples using properties, but often these were trivial or very special cases. In both these studies, construction of examples was a novel process for the students. Watson and Mason (2005) report phenomenological evidence from participants in workshop sessions that exemplification is an effective way to engage with new ideas. Literature about using such an approach with school students is scarce, so it might be assumed that this approach is only suitable for advanced learners.

Three questions emerge:

- Is it possible for learners to create examples of classes of object they have not met before?
- Is it possible for non-advanced students to learn about mathematical objects and relationships by generating their own examples?
- If so, what sort of conditions might contribute to learning?

We[1] use 'example' to mean an instance, illustration, case or element of a mathematical idea, object, process or class (Zazkis and Leikin 2008). Obviously it is possible to create examples while being unaware that to an expert they exemplify ideas which the novice may not have met. The important phrase is 'examples *of*', so that the construction is directed

[1] Anne Watson was the main researcher for the work reported in this paper; Steve Shipman is the teacher involved in Studies 1 and 2, and co-author of this paper.

towards a certain end, and in the process of meeting that end the learner will encounter new-to-them ideas, relationships, and ways of classifying. We use the word 'learn' to mean a process of becoming familiar enough with mathematical ideas to be able to use, manipulate and adapt them, think about them and communicate about them. 'Becoming familiar' includes perception of variation, and, through construction and reflection, appreciation of relationships, and, with deliberate pedagogy, knowledge of conventional mathematical ideas.

In this paper we offer the outcomes of some small studies towards answering the first two questions positively, and offer some insight into the third question.

3 Classroom studies to find out how LGEs of new mathematical ideas are generated

We set up some studies in authentic classroom settings. These are not experiments but are studies of a teacher teaching his usual students the usual topics in their usual lessons but with the use of LGEs as a deliberate new strategy.

3.1 Study 1: LGEs as raw material for understanding structure

In our report of this study we show that shifts towards engagement with structures and relationships *can* occur, unprompted by the teacher, through reflection on examples in an appropriately supportive environment.

The class for the first study was a Year 9 (13/14 year-olds) group in an English comprehensive school. In the UK classes are typically 'set' according to prior attainment and this group was selected from the highest achieving third of the cohort. Steve had been teaching this group for over a year. He also assisted with a parallel class taught by another teacher.

The lesson we shall describe has to be seen in the wider context of Steve's usual teaching strategies with all groups. Steve usually spends about 50% of lesson time leading discussion and exposition, with the starts and ends of lessons involving 'thinking' tasks—tasks which need more adaptive reasoning than recall or fluency. The remaining time consists of 30% small group work and 20% independent research (exploration and book/internet research). Homework is often used as a chance to create questions: each pair of pupils sets a 'hard' question on the current topic that the rest of the class has to solve for the next lesson. The setters have to be prepared to provide their answer. This means there are 15 or 16 questions in total and the pupils are involved in the marking and subsequent discussion.

For Steve, the purpose of competence with techniques is to think about concepts. For this reason, during episodes of technical work, he sometimes gives directed help to learners so that they can all take part in whole class discussions of mathematical ideas. He also employs a 'gossip' approach to classroom knowledge in which ideas are allowed to spread around the classroom during lessons.

He often talks in lessons about mathematical strategies, such as 'using inverses' and 'thinking about special cases' (ideas that will be significant in what follows). Working on special cases and thinking hard about them contradicts a common practice in UK mathematics lessons of generating many cases and looking for patterns so that a generalisation 'jumps out'. *Special* examples do not provide raw material for inductive steps but offer evidence of plausible structures from which meaning can be abduced, either because their speciality shows a degenerate relationship, or their structure offers a generic model (Mason and Pimm 1984).

In this study Steve introduced the Year 9 class he had been teaching longest to the idea of irrational numbers by asking them to generate examples involving roots in ways which brought them in direct contact with the way surds behave. They already knew that '√' signifies 'square root' but this does not imply that they were fluent in its use or meaning. For example, they would have been familiar with statements such as '$\sqrt{36}=\pm 6$' but might themselves have only given the positive root. Learners were told of the existence of a 'new kind of number' of the form $(a+\sqrt{b})$. They were then asked to multiply pairs of such numbers to see if they could 'get rid of the roots' and get integer answers. The concept we hoped they would begin to understand was conjugacy: that $(a+\sqrt{b})\ (a-\sqrt{b})$ would always be an integer when a and b are integers. However, what convinced Steve to try this approach was the expectation that the process of choosing and trying out examples would have intrinsic benefits for their understanding of roots and surds, and how they behave when multiplied, even if 'answers' were not 'discovered'. In other words, purposeful play with LGEs would be worthwhile in itself.

There was no deliberate intention to use the parallel class for comparison, because this was an opportunistic study in an authentic context, however it turned out, fortuitously, that both classes were going to be working on examination-type questions about surds around the same time.

With the study group Steve started by giving some examples of general "grid multiplication" to remind them of the structure. Students were familiar with using this as a structuring device for multiplying two multi-digit numbers and algebraic expressions. It is commonly used in the UK to ensure that all pairs of elements are multiplied; the contents of the cells are then added. After a numerical example he showed them the grid in Fig. 1.

Students called out the contents of the cells and he asked them what the sum of the final cell entries would represent. The aim was to remind them that the grid is a tool to express the product. Learners were then asked to propose grids for $(7+2x)(3-4x)$ and then $(a+b)(c+d)$, both times arriving at a grid similar to that in Fig. 1, and worked through the calculations for each cell and for the overall sum. Steve then reminded them about the meaning of the root sign, and showing a few written examples such as $(2+\sqrt{5})$ merely to state 'this is a new kind of number'. Only the plus sign was used in this introduction, but a negative sign had been used in one of the earlier algebraic examples. He asked them, as a class, to consider how to multiply $(a+\sqrt{b})(c+\sqrt{d})$. They agreed that they could use this grid method. Presenting the grid was an invitation to 'see' the similarity of structure. Some of the students worked out that the bottom right entry would then be $\sqrt{b}\sqrt{d}$, but there was no discussion about $\sqrt{b}\sqrt{d}=\sqrt{(bd)}$. They then were asked to choose their own values of a, b, c and d to achieve an integer solution, without using square numbers for b or d as this would be 'cheating'.

Students worked for about half an hour on A3 blank paper. This encouraged record keeping that allowed many examples to be seen and compared without turning the paper. During this episode Steve spoke with every individual student. Each conversation was

Fig. 1 Grid multiplication

X	6	$+y$
3		
$+y$		

different, depending on their chosen examples, with Steve asking them if anything so far had indicated they were close to useful numbers.

We collected all the written notes learners produced during these lessons and analysed them to see the variety of examples created. For each student the researcher identified variation in the choice of the integers they had employed as they generated examples. The dimensions of variation (Marton et al. 2004) focused on were: choice of integers, nature of integers (small, large, having certain multiplicative properties); relations between chosen integers (consecutive, related multiplicatively); signs (although no one used negative signs). We also analysed the progression of choices to see influences of reflecting on earlier choices. We reached agreement about interpreting the data through discussion, with Steve providing the evidence and classroom context and Anne probing for further information and introducing research perspectives. This led to theoretical and practical agreement on what had happened in the classroom.

The encouragement to record work in whatever way they liked could have led to difficulties in analysis, but our priority in class was learning rather than research. Some students had chosen to arrange their work in one sequence so that chronology was easy to discern; others had worked two-dimensionally, developing a particular kind of variation horizontally and then starting a new kind below it. Others had recorded more randomly but with small connected blocks of work where they had been pursuing a particular dimension of variation. It was possible on nearly all sheets to see from the layout which examples had been tried towards the beginning of the time and which towards the end. There was a tendency to record from the top left of the sheet towards the bottom right. It is not the focus of this paper to comment further on layout, but we were struck that most students appeared to relate layout to organising the variation in their work.

Eighteen of the 30 students tried numbers which suggested some kind of organised thinking, see below. The remaining 12 did not show evidence of systematic choice of numbers that we could discern, even if layout was neat and organised, but still generated several examples which can be seen as constituting ‘practice’ of multiplying surds. All learners experienced such practice during the lesson: none did fewer than four examples and some did many more than this.

The availability of calculators tempted them to change everything into truncated decimals at first—this was no surprise as most of their previous experiences with number had been about getting numerical answers. However, most wrote the cell entries in surd form, and those who wrote out the sum before calculating used surd form (e.g., $12+(4\times\sqrt{65})+(3\times\sqrt{35})+(\sqrt{35}\times\sqrt{65})$). They had been given no advice on using the calculators algebraically retaining the surd in the answer. However, there was written evidence that for many the decimal approach gave way to an understanding that the problem was not really about finding special numbers, but about the algebra of irrational roots. This, of course, is our language and not what learners might say—but in much of their work shifts from ‘trying different numbers’ to ‘trying different structures’ could clearly be seen. Abandoning the calculator and staying with the surd notation as an algebraic object with pencil-and-paper was one sign of this, e.g. writing $\sqrt{4}\sqrt{3} = 2\times\sqrt{3}$; choice of special examples was another, e.g. $(4+\sqrt{5})(5+\sqrt{4})$. Some stopped calculating the total as if they were focusing on elimination rather than summation. It was clear that several students began to treat numbers as generic, e.g. $(8+\sqrt{8})(8+\sqrt{8})$. We also believe that, even when calculation was done, many choices of number were structural in the sense that they may have been chosen because of special relations rather than their own value or randomness, e.g. choosing to use prime numbers only, or choosing to use √2 and √8 in the same example. Some students tried to use letters, one of them wrote: ‘$\surd x\surd y$ has to be an integer; square roots to make whole numbers?’

According to the analysis, at least ten learners made a clear shift to working with structure during the lesson, and a few came close to understanding that the roots needed to be the same in order to stand a chance of 'cancelling each other out'. For example, a few learners working together multiplied $(2+\sqrt{2})$ by itself (Fig. 2).

The appearance of an integer in the bottom right corner of the grid caused frustration rather than pleasure; two students (who may have been sitting near each other) wrote 'wasted integer' next to the '2' in the bottom right. Looking from an advanced perspective of knowing how to 'get rid of the roots' it is hard to see why learners had wanted to cling to them in this example. However, we recall experiencing that same sense of frustration ourselves as learners, not realising that the key idea in the solution is that the top right and bottom left terms have to be additive inverses. Possibly because the students had focused on changing the numbers, and this is how the challenge had been set up, changing the sign was not a dimension of variation which occurred to them. However, one of the powers of the grid layout of multiplication is that when it is used regularly and creatively such changes might come to mind more easily.

To support the claim that students shifted their understanding during their example-generation we show some sequences of their example choices in Table 1. All of these show evidence of systematic searching with some variables held constant. There were others in which no such control could be observed.

In these examples it looks as if the kind of reasoning taking place is not inductive. That is to say, it is not about generating some examples which work and inducing from them a generality which can then be algebraically expressed—the kind of empirical reasoning which Vygotsky claims does not aid abstraction (1986, p.107). Here, however, it looks as if some learners have shifted *themselves* towards working with structure, through special choice of numbers as generic objects, and some have even employed algebra. How can this happen when they do not already know how irrationals behave when multiplied, and when they have not had much experience of those special cases where the product of binomials has only two terms? Examination of the work done and the teaching context suggests some explanations.

In this particular lesson, 'gossip' seems to have helped the spread of ideas, since $(2+\sqrt{2})$ appears in several scripts. That these ideas cropped up in a third of the class although no one had an actual answer suggests that many of Steve's students are genuinely interested in exploration even if they had to get ideas about how to explore from each other.

Choosing likely integers and converting their roots into decimals appeared to become tedious and unproductive, so learners needed more efficient ways to look at the problem, i.e. shortcuts and curtailments, and were probably reflecting on their work with this in mind. It is likely that fortuitous examples were found by some learners which, on reflection in discussion with others, seemed to be more likely candidates for development than others. Steve had given them no input about how $\sqrt{b}\sqrt{b}$ would be the same as b, but because

Fig. 2 $(2+\sqrt{2})^2$

	2	**$\sqrt{2}$**
2	4	$2\sqrt{2}$
$\sqrt{2}$	$2\sqrt{2}$	2

Table 1 Examples of Steve's students LGE generation

LGEs	Interpretative comment
$(7+\sqrt{5})(6+\sqrt{7})$ $(7+\sqrt{10})(6+\sqrt{6})$ $(7+\sqrt{8})(6+\sqrt{4})$ $(7+\sqrt{4})(6+\sqrt{3})$	This sequence appears to show some understanding of controlling variables, but although the third example has the potential to give useful information, the next unhelpfully pairs √4 with √3
$(7+\sqrt{19})(\sqrt{17}+3)$ $(7+\sqrt{18})(\sqrt{18}+3)$ $(7+\sqrt{18})(\sqrt{17}+3)$ $(7+\sqrt{17})(\sqrt{17}+3)$	In earlier work with exemplification, we have seen that certain prime numbers such as 7 and 17 are often selected as if they are neutral or generic. This could show awareness that multiplicative properties might be useful. This student used a calculator for all four examples. This student then stopped–maybe with some sense of matching the roots.
$(4+\sqrt{4})(5+\sqrt{5})$ $(2+\sqrt{2})(2+\sqrt{2})$ $(8+\sqrt{8})(8+\sqrt{8})$	This learner may understand that the challenge is about structure, and is experimenting with this in mind, controlling variation within each example rather than between examples. A calculator was used. The class has not yet discussed that √8 is the same as 2√2.
$(a+\sqrt{2})(b+\sqrt{8})$	This example comes from someone who appears to grasp that specific roots might cancel each other out and tries to construct generalities with them.
$(12+\sqrt{6})(2+\sqrt{3})$ $(12+\sqrt{6})(12+\sqrt{3})$ $(12+\sqrt{12})(\sqrt{12}+12)$	This shows a shift to a structured approach. The student seems to understand that squaring the square root might be helpful. No decimal calculations were recorded. A few students changed the order of terms as in this case; we do not know if commutativity was thought about or not.
$(a+\sqrt{b})(b+\sqrt{a})$	This learner may be attempting to change the order to effect the desired elimination. This example followed several apparently random examples, not of this structure.
$(2+\sqrt{3})(\sqrt{2}+\sqrt{3})$ $(2+\sqrt{3})(3+\sqrt{2})$ $(2+\sqrt{2})(3+\sqrt{3})$	This shows a structural search for some relationships among the chosen integers; only the first one was worked out with a calculator.

students did not think of using the negative sign the explorations described above led more readily to conjectures about $\sqrt{a}\sqrt{b}$ than to conjugacy. About six students wrote statements indicating that they felt they needed square numbers in order to produce integers. Three students produced systematic lists of decimal numbers generated from a√b; two stopped when they reached 3√4. We can speculate that this choice led them to stop this approach because 4 is a square number. One student then went on to write, without comment: √9√1, √9√4, √9√9, √9√16 .... This student also had written elsewhere on the sheet: 1√8 (2√2) and 2√8 (4√2).

Using LGEs can therefore, as we suspected, be motivating and provide an enquiry-generated start to a new topic. The examples many students used showed a growing sense of structure while at the same time providing practice in the basic procedure of surd multiplication and an arena for coming to understand what surds are. Furthermore, many students seemed to use numbers as generic objects, and saw a need for relations among them. This analysis fitted well with the teacher's perspective on the lesson, although his was inevitably more class-focused. But did it help students learn in the sense given earlier? In the lesson following the one just described, Steve showed them briefly how using the negative sign would resolve the matter. Then typical examination questions involving rationals, irrationals and surds were set. When working through questions as a whole class, many students made suggestions about choices of method, such as when it is helpful to eliminate surds (e.g. when they appear in a denominator) and when they are best left in that form (e.g. when they are going to be multiplied by another surd). Steve reports that they referred to 'our' methods and 'our' ideas.

The parallel group had previously been taught the same subject matter by another teacher using a rule-based approach focusing on definition, technique, memorisation and application in given questions. In Steve's perception, this parallel group could cope with familiar questions but struggled with more complicated problems. Rather than suggesting methods based on meaning, their approach was to try to identify which method might be useful from syntax. We claim, therefore, that the LGE process helped the first group learn about the concepts and relationships of surds, and led students to shared ownership over this area of mathematics.

3.1.1 Discussion of first study

This lesson confirms that generating examples can provide a good way to start understanding a new concept, with some caveats. Firstly, there was a goal rather than a directionless exploration, and it was a goal which could be reached with a searching strategy structured by the grid method. Secondly, Steve's usual teaching included attention to structural aspects of mathematics such as inverses and generalisation from features of special cases. We need to know more about whether and how students used these perspectives in their exploration. There is evidence that some students constructed special cases, but their use of inverses seems confined to squares and square roots, rather than the inverse nature of + and −. Systematic example generation, controlling one variable and allowing another to vary in order, was only useful if it happened to produce illuminating examples. Algebraic structures coalesce around the concepts of inverse, identity, operation, combination, commutativity, distributivity, function and so on. Sometimes these are communicated by systematic example-generation and sometimes they are not.

Learners had engaged with new ideas through making their own examples. These examples were associated not only with objects (here surds), but with relationships between objects. Learners had to use multiplication, reflecting on its outcomes, to get a sense of the multiplicative relationships between surds. The fact that none found the conjugate during the lesson does not negate this. The task had drawn them into a 'space of relations[2]', namely that they were not merely generating examples of new-to-them classes of object, but examples which fulfilled certain relational constraints so that they were thinking about getting something specific to happen. Thus they were not only exploring and extending their personal experience of integers, roots, and multiplicative methods (their 'example

[2] Grateful thanks to Elaine Simmt for using this expression.

spaces' of numbers and actions relevant to this task (Watson and Mason 2005)) but also exploring and extending their experience of multiplicative relationships and getting a wider sense of what happens when two of these 'new' kinds of number are related by multiplication. This extends their sense of multiplication; they are no longer in a world where multiplication can be rephrased as 'x lots of y'. We conjecture, therefore, that in a space of relations the objections raised by Schmittau (2003) and described earlier do not arise, because learners are invited to engage with relations between objects from the start, in other words they have to objectify the relationship through generating examples which fulfil it. They are not merely trying to provide and describe examples generated 'with the grain', i.e. generated sequentially according to some repetitive rule, but have to identify and produce relationships between elements of the examples 'across the grain', in relationship to each other, and in doing so will learn more about the properties of the new class of objects through the behaviour in this particular relation.

Davydov's claim, (1972) that generalisation comes from comparisons between examples, i.e. how their differences enable you to see critical common features, rather than seeing properties, relevant or irrelevant, which they happen to have in common, appears to be upheld. There is no smooth path from example-generation to conclusion. The intended salient features are drawn out as 'common' through active comparison of special cases rather than casual pattern-spotting. Indeed, the use of the calculator to express roots as truncated decimals (often given to seven places) discourages a pattern-spotting approach.

Having offered some classroom evidence which supports the claim that LGEs can be used for learning about new mathematical ideas, we have not yet carefully addressed the question of whether this only applies to advanced learners. Some of the students in this class could be described as strong mathematicians, but not all, although they were *relatively* high achievers in the context of that school.

3.2 Study 2: low attaining students can learn by generating examples

The literature on LGEs relates only to groups of learners with high achievement, but our sense of adolescents is that, given a suitable environment, any learner can respond with cognitive maturity 'primarily as a thinking being' (Vygotsky 1986 p.30). In this study, which we report briefly, we explored this idea further, and also sought to learn more about the nature of learning in such environments.

Steve asked a previously low-achieving group of 16 year olds, those who were predicted to get the lowest grades in public examinations, to explore relationships between sides of right-angled triangles. Steve also taught another group in the same year, again one that had previously been taught by someone else, whose previous attainment as measured by national tests was the highest in the school.

The first class was asked to draw ten right-angled triangles, measure the sides as accurately as they could, and name one of their non-right angles θ each time. They had been taught by Steve for most of a year and were by now used to such 'playing'. The 'gossip' method of classroom interaction was exploited successfully during the lesson and all learners were eventually familiar, either by finding it themselves, or by hearing it said by peers or the teacher, with the idea that dividing the lengths of sides would lead to the same results for the same angles. Our analysis of students' work gives no further insights into the process than those developed in study 1. Whole-class discussion at the end of the lesson ensured that knowledge was collectively developed, shared and articulated. In the space of 1 hour sine, cosine and tangent had been found collaboratively to be invariant ratios for fixed angles in 'different' triangles. This finding emerged in class discussion orchestrated by Steve. In the following lesson and homework the same class managed to work through, correctly, 30 typical examination questions designed for higher-achieving learners who were expected to get top grades. This process began with whole class discussion, referring regularly to the learners' own examples from the previous lesson and using their own suggestions.

Steve's other group of 16 year olds had first been taught these trigonometric ideas 2 years before, and were revising them for the upcoming high-stakes examinations. They were unable, in a group discussion, to achieve a comparable level of competence with unfamiliar questions and were also unable to generate their own examples and methods. We believe this was due to earlier acculturation in more exam-focused ways of working with a heavy emphasis on procedural acquisition and competence. In general, these students demonstrated memory by generating formulae from a mnemonic, and rearranging equations according to methods, but did not appear to have the underlying knowledge and intuitions required to be sure of doing this correctly. We are, of course, not making claims about individual learners, but teacher-reports about whole class characteristics have significant value in raising questions for further research.

This comparison shows that classroom norms, expected ways of working, available tools, the use of knowledge distributed in the class, and the nature of teacher input combine with the power of LGEs to promote learning new concepts. The ways purposeful generation is structured, and the roles of variation in exploration and generation, were central to its success. Furthermore, learners who had 'found' trigonometric ratios through example generation which shifted from fairly random trials to deliberate specialisation, claimed ownership of what they had found in a similar way to those in the surd study. Some of them told Steve that they were pleased that 'their' results could help them answer examination questions.

3.2.1 Further teaching experiments

In the year following the successes in these studies, Steve used the same approach to teaching surds for classes with average and below average prior achievement, this time being careful to insert the possibility of using negatives as well as positives. We do not report these fully here, but it is worth noting some of the outcomes. A few learners found conjugacy; for example, one found that $(2+\sqrt{12})$ and $(-2+\sqrt{12})$ produced the desired result. Someone wrote that (s)he had used 'numbers which are close together' and in the written work had used $\sqrt{5}$ and $\sqrt{20}$, suggesting that (s)he meant 'close in meaning' rather than value. They had no difficulty remembering 'our rules' for surds 4 weeks later. Steve also removed the constraint that they must not use square numbers. Four students found for themselves that $\sqrt{a}\sqrt{b}=\sqrt{(ab)}$. It was evident from the quantity of rough work that both classes were active in exploring unknown territory, even when it offered more frustration than success, and that at the very least they all used the root symbol correctly, and practised using it in alternative representations of whole numbers.

3.2.2 Discussion of the studies

Exploring new concepts through example generation is possible even when the generation task is essentially constructive, as in these studies. However, it may be important that the studies hinged on exemplifying relationships, rather than objects. Asking learners to construct objects of which they have no knowledge might be pointless, unless it is possible to do so by adapting existing knowledge to fit new constraints. Also, asking learners to exemplify the next object in a sequence whose generating action is known might not result in learning about underlying structure. In our studies the LGEs required identifying or creating relationships, or deducing structure, or engaging with the semantics of a mathematical expression rather than its syntax. Generating examples does appear to have been powerful in promoting learning, as Dahlberg and Housman conjectured in their work with undergraduates (1997).

These studies support the ideas in Watson and Mason (2005) that LGEs are an appropriate way to introduce new ideas in mathematics lessons, and show further that all learners might be capable of generating examples, and that significant learning can result from the process because learners generate and explore example spaces related to the ideas, in particular spaces of relations between objects.

The importance of normal classroom expectations and teacher guidance cannot be overestimated however.

4 Conclusion

What is required for learning through exemplification is not so much the observation of numerical patterns generated in sequences of different examples, although this observation would tell the learner something about structure, but conjecturing relationships which connect different variables within examples. Making this shift of perception is non-trivial and very sensitive. One way to do it is to discern critical features by comparing similar examples. Another is to conjecture from characteristics of special cases. For example, some low-achieving students 'saw' the identity $\sqrt{a}\sqrt{b}=\sqrt{(ab)}$ because of the examples they had generated using already-square numbers revealed this relationship more clearly than a calculator approach.

The data from all these lessons provide evidence that students *can* learn new concepts through a symbiosis of induction and abduction from experience and deduction from the relationships generated in particular examples. In other words, their experience can be organised in such a way that shifts of understanding take place as a result of learners' own actions, including mental acts of organisational reflection on self-generated examples and example spaces. In the classes described here, this was in part a group endeavour, in that the example spaces generated by the whole class were available for reflection.

We have offered these studies to challenge simplistic interpretations of Vygotsky's claim that abstraction cannot follow from exemplification. In these studies, Steve and the learners were not relying on inductive reasoning to learn about the underlying concepts. They reflected on results, and on the internal structures of examples, in unfamiliar mathematical situations, because these were the normal expectations in their lessons. Example generation provided the raw material for mathematising, and they learnt some new ideas as a result.

We were especially heartened to find that the previously low achieving groups could also make these shifts, because engagement in examples they have created themselves provides 'relevance', 'realism' and emotional connection.

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