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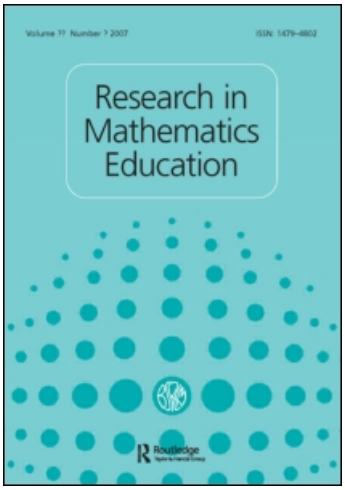

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Shifts of mathematical thinking in adolescence

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Shifts of mathematical thinking in adolescence

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This theoretical paper relates key features of the mathematics adolescents are expected to learn in school to other aspects of adolescent development. Difficulties in mathematical learning at that age include changes in perspective and in the actions that are mathematically productive. Commonly-recommended methods of trying to engage adolescents in mathematics do not necessarily enable students to shift to new perceptions and new ways of constructing mathematical understandings, yet the shifts students need to make are in accord with other aspects of adolescent development.

Keywords: Learning secondary mathematics; adolescence; secondary maths curriculum

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Introduction

Learning secondary mathematics can relate closely to the adolescent project of negotiating adulthood. However, all too often it does not, yet the same kinds of adolescent autonomous thinking which so often lead to disaffection and rejection, not only of mathematics but of school and life more generally, can be embedded and enhanced positively within the teaching and learning of mathematics. I argue that engagement in the new ways of thinking that are appropriate for higher, abstract mathematics is compatible with the difficult changes of adolescence, and that this compatibility is not an automatic property of some of the teaching, curriculum and assessment styles currently recommended for this age group.

During the late 80s and early 90s I was head of mathematics department in a school which served a socio-economically disadvantaged area with low academic achievement. The introduction of a national curriculum in England in 1988 imposed an academic curriculum on students who had not been expected to learn in structured, written, abstract contexts but had flourished in vocational subjects. Nevertheless, in the early days of assessment associated with this curriculum our mathematics results were usually the best of all subjects, and similar to those achieved in creative arts, with nearly 100% of any cohort achieving a mathematics grade and around 35% achieving high grades – much better than some schools with a comparable intake. We did this by nurturing and depending on students' natural propensity to explore and to apply quantitative and spatial reasoning in everyday ways. We had presented mathematics in school as being similar to the ways in which problems arise out of school, and enabled students to see how school mathematics could be relevant to out-of-school contexts. The practices were similar to those reported in Boaler (1997) and indeed we were involved in the same curriculum and

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assessment project as 'Phoenix Park' in her research. But our students did not do as well as those in nearby middle-class schools, and, crucially, their high status success rate fell when the rules for national tests reduced the proportion of the grade that could be gained through portfolio assessment. We did not enable the majority of students to contact the essentially abstract, structural understandings which characterise mathematics in its entirety. In Vygotsky's terms, we failed to support them in engaging with the scientific concepts which can only be taught deliberately (Kozulin 1986, xxxiii). In terms proposed by Inhelder and Piaget (1964), we failed to offer experiences which demanded engagement with abstract mathematical ideas unless these could be accessed structurally through inductive reasoning.

In this paper 'higher' mathematics means ideas typically taught in secondary schools which build on existing understandings of abstract concepts, and which do not relate easily to familiar actions and phenomena, and the introduction of formal ideas to inform the way we observe the world. Higher mathematical ideas cannot be accessed directly by inductive reasoning from, or application of, everyday spatial and quantitative awareness. Instead, higher mathematics offers formal tools whose use offers new ways to act in mathematical and other situations. For example, trigonometric ratios are constant for the same angle in different situations, so can be used as a tool for calculating other measures. Their invariance can only be conjectured through discovery methods, but their formalisation as functions extends their meaning, such as to describe wave behaviour. Another example is the use of symbols to describe consistent numerical relationships. Symbol use can arise through a need to express observed, inductively-reasoned relations, but the formal use of symbols can extend to represent variation and predict relations. Higher mathematics is positioned between 'elementary' and 'advanced'. This distinction highlights its role in providing learners with experience of how elementary concepts can be combined, extended and adapted to develop new ideas and perspectives only accessed through formal learning.

This paper argues two things relevant for current curriculum changes towards better engagement in mathematics: (i) that recommendations based on participation and relevance may not *ensure* learning higher mathematics; and (ii) that teaching higher mathematics requires a focused understanding of the shifts of mathematical attention and perception adolescents have to make.

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Adolescent concerns

Adolescents are broadly concerned with the development of identity, belonging, being heard, being in charge, being supported, feeling powerful, understanding the world, and being able to argue in ways which make adults listen (Coleman and Hendry 1990; Emler 2001; Emler and Reicher 1987)[1]. Adolescents engage with these concerns through interaction with peers. This was regarded by Vygotsky as the "leading activity" of adolescence (Elkonin 1977) – the activity which leads the development direction but is not necessarily the only influence. He saw that peer interaction is the context in which adolescents work out their relationships with others, adults, the world and themselves. They do this by engaging in formal-logical thinking, becoming capable of self-analysis, and analysis of other situations, as internalisations of social consciousness developed with peers (Karpov 2003). This facility is attributed by Piaget partly to development of the cortex, but it is also

clear that maturity alone is not the only factor in social development (Inhelder and Piaget 1958, 337). Experiences of new kinds of situation are also important, in particular experiences which promote new kinds of classification and response (Inhelder and Piaget 1964, 289). Thus, different situations might engender different forms of mature argument, and different kinds of abstraction, by embedding different kinds of classification in adult-novice interaction. Coleman and Hendry point out that adolescence is the time when verbal and kinaesthetic socialised responses to sensory stimuli (1990, 47), which have been adequate at an elementary level, are put aside as inadequate for some abstract tasks. Vygotsky emphasises that the maturing brain is physiologically influential in how identity develops (Vygotsky 1978), but that understanding more complex, abstract ideas is also influenced by the behaviour and interactions of nearby adults. Through affirmation or approbation or otherwise nuanced response, adults mediate the activities of adolescents to influence the development of identity and knowledge.

In summary, adolescent learning is very concerned with the development of self in relation to others, and this entails becoming more able, with help, to:

- deal with unfamiliar situations as well as familiar ones;
- focus on imagined and abstract ideas as well as sensory data;
- act in ways informed by reason as well as by intuition;
- think about social and abstract implications as well as immediate reactions to objects;
- act in ways that are socially mediated rather than driven by immediate responses.

School classrooms are very important places for adolescents, because teacher behaviour dominates, while the majority of those present are adolescents whose behaviour and interactions are constrained by adult goals and expectations – even if this behaviour is oppositional. Teaching which imposes formal methods and classical definitions as starting points for new mathematical ideas, and then pathologises students who cannot perform procedures as "lacking ability or concentration", fails to do justice to the capacity of learners to apply naïve ideas in imaginative ways. In our school, mathematics results matched those of creative arts because we saw this capacity as a proficiency. That same capacity is also manifested in well-known misconceptions which often arise due to application of inappropriate generalities to new situations. Teaching that ignores or negates the way a student thinks imposes mental behaviour which feels unnatural and uncomfortable and undermines students' thoughtful efforts to make sense. Students seen as deficient are often forced to revisit the mathematical sites of their earlier failure by, for example, adding fractions during years 6, 7, 8 and 9, without being given new intellectual tools to use. At best this is marginally productive, and at worst emotionally damaging, because students can become trapped into repeated failure with no way out except to adopt negative behaviour or to accept such treatment compliantly, while hating both mathematics and mathematics lessons. Emler points out that, while failure in school tasks is not necessarily a major source of low self-esteem, it is such if students have emotional investment in being successful (2001). They have to 'not care' to avoid being upset.

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The development of attitudes to authority is exacerbated in adolescence by the density of encounters with authority, and the raising of the stakes in such encounters (Emler and Reicher 1987, 170). In mathematics teaching, authority includes the imposition of non-intuitive ideas, and the stakes are raised through testing. Those whose thinking never quite matches what the teacher expects, but who never have the space, support and time to explore why, can at worst become disaffected, and at best come to rely on algorithms and mnemonics. While all mathematicians sometimes rely on algorithmic knowledge, learners for whom that is the only option are dependent on the authority of the teacher, textbooks, websites and examiners for affirmation. Since a large part of the adolescent project is the development of autonomous identity, albeit in relation to other groups, something has to break this tension – and that can be a loss of self-esteem, rejection of the subject, or adoption of disruptive behaviour (Coleman and Hendry 1990, 155).

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Disaffection in higher mathematics

In a large study of self-reported motivation across three core subjects in secondary schools in Australia, mathematics was the subject in which the class a student was in had most effect on enjoyment, aspirations, engagement and self-concept (Martin and Marsh 2005) (n = 1701). Martin and Marsh claim that this may be because mathematics is a subject in which students typically struggle and which they do not value, so pedagogy is critical. Nardi and Steward, in a self-report study of 70 mid-adolescent students, identified possible pedagogic reasons behind this effect. Mathematics lessons were experienced as tedious and depersonalised; students felt isolated and obliged to rote-learn an 'elitist' subject (2003)[2]. These perceptions were not necessarily associated with unwillingness to engage with mathematics; students often stated that they wanted lessons to be relevant and fun, and wanted to understand what they were doing rather than merely follow rules. In a systematic review of published research about disaffection and motivation in secondary mathematics, Kyriacou and Goulding (2006) summarise the body of similar studies:

> Taken together, these studies point to the importance of basing strategies aimed at increasing the motivational effort of the target group of KS 4 pupils on providing a classroom climate in which (i) the teacher is highly supportive; (ii) the work is both challenging and enjoyable; (iii) there is a high level of cooperation among pupils; and (iv) all the pupils in the class feel equally valued by the teacher. (p. 21)

They also reviewed a range of initiatives that increased motivation and achievement, such as teaching which provided a supportive, participatory classroom climate; or which encouraged students to act like mathematicians; or which focused on raising motivation through ICT use or assessment for learning practices. All these provided evidence for the effectiveness of these characteristics in changing motivation.

In many countries, policy response to this kind of study has been to reconstruct curricula to offer mathematics through contextual problems and exploration, and in which groupwork, self direction and discussion figure centrally. For example, this approach is valued highly in the English curriculum for secondary mathematics (QCA 2008). It has informed its key ideas: competence, creativity, application and

critical understanding, and its approach to engaging all learners in mathematics (QCDA 2009).

These typical responses draw on the socio-cultural nature of classrooms, the power of dialogue to develop thinking, and educational aims such as problem-solving, adaptability and critical citizenship. However, while these responses are important for learners to view mathematics as an arena for shared human communicative endeavour, and thus to aid learning, these foci cannot *on their own* alter the ways in which learners approach mathematical problems. In the next section I shall show what changes of reasoning are necessary to support learning in higher mathematics. In the following sections I shall describe how these changes are compatible with adolescent changes of thinking. Critical shifts towards abstraction do not have to be experienced as reduced algorithms or elitist mysteries – both described as sources of disaffection by Nardi and Steward (2003) – nor do they have to be avoided in attempts to improve the accessibility of mathematics.

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Shifts in higher mathematics

Diagnostic studies (e.g. Hart 1981; Ryan and Williams 2007, appendix 1; Foxman, Martini, Tuson and Cresswell 1980) show consistently that the aspects of mathematics which cause problems for the majority of 14 year olds relate to:

- the need to make shifts of perception and interpretation, such as seeing fractions as objects rather than as a pair of integers;
- the need to become suspicious of additive, linear, and other elementary assumptions and intuitions;
- understanding of new notations, such as interpreting symbolic expressions;
- keeping track of meaning and purpose in multi-stage problems;
- classification problems, such as what technical terms mean and whether categories are exclusive or inclusive.

This overview enables us to analyse the subject matter and the teaching instead of pathologising the learners. To overcome problems with mathematics, therefore, learners need to adopt new kinds of classification, new kinds of perception and interpretation, and new representational tools. These changes of thinking are unlikely to arise in spontaneous contexts through peer-interaction except with particular intervention. Intervention is most likely to come from someone who already knows about, and appreciates the power and current relevance of, a new category of generalisation, or an abstract tool that would not arise in everyday practices. For example, keeping track in complex tasks often requires new recording tools. Students might invent some, but communication might be better using new-to-them conventions which might also offer extensions of meaning. Students engaged in collaborative problem-solving may need to be prompted to use new structural tools, or theorems about properties, or alternative representations. Without such an intervention 'engagement' can be limited to thinking up different inputs to trial, applying rules they hope will be appropriate, and reflecting on their outcomes using their limited experience (Sierpinska 1995). For example, younger secondary students will over-emphasise whether whole-number outcomes of some mathematical action are odd or even because these are easy to spot from visual cues, and recognising them

is habituated in primary mathematics. Shifting learners to thinking about not readily observable, structural characteristics such as the multiplicative structure of a number takes intervention, time, and multiple experiences (Watson 2009; Vergnaud 2009).

Many higher mathematics ideas involve manipulating and adapting ideas which are often in conflict with intuitive notions, sensory responses and earlier school experience (Fischbein 1987). For example, students have to accept that multiplication does not always increase magnitude, so a purely material understanding of multiplication as 'so many lots of' has to be abandoned for something more abstract – the formalisation of scaling. If they do not have a strong sense of continuity, perhaps from earlier experience of comparing measured quantities, their everyday experience of scaling can conflict with their understanding of number (Schmittau 2003).

In higher mathematics, the properties of mathematical objects, implied by internal relationships, are more important than their sensory features (Burger and Shaughnessy 1986; Mason 2003). For example, squares and rectangles are quadrilaterals because they have four sides, and consequently they all have an internal angle sum of 360°, even though they are called by different names and may also look different. Pursuing properties instead of physical appearance leads to the relation of inclusion between squares and rectangles because all their angles are right angles, and squares have an extra constraint, rather than separation because they look different. For many students, the mathematics classroom is a site where natural ways of thinking by generalising from sensory data are frequently overridden by ideas which are not obvious, and may appear to be arbitrary. Without intentional help to shift from relying on sensory impact towards reasoning from properties, students can give up attempts to make personal sense of what they are offered, and instead rely on a disconnected collection of rules and methods.

We can learn more about shifts of perception by looking at students who do this successfully on their own. Students of Grootenboer and Zevenbergen (2007) found the sum of interior angles of an octagon through enquiry, and then generalised to all polygons. They got there by identifying patterns, using examples, constructing generalisations, testing hypotheses and applying intuitive insights, but what was crucial was that *some* learners changed the way they look at examples from looking for characteristics of the relations within the examples, to comparing those relations across examples. This illustrates Vygotsky's contention (1986, 202) that what is required to learn is to recognise entities as instantiations of abstract principles, as concretisations of abstract relations. To learn new abstract ideas, it is the structures of, qualities of, and relations within generalisations that have to be identified and compared, not the instantiations.

The perceptions, interpretations, classifications and representations of higher mathematics are qualitatively different from those of earlier mathematics, for which visual and physical experience can often be a suitable ground, because they relate to relations and properties rather than objects. Merely providing new language, symbols and definitions does not automatically enable learners to look at objects differently. Sometimes the associated language and symbols are familiar, but the interpretation is different. Vergnaud gives two examples in which such shifts of interpretation are necessary (2009, 90 and 93). He looks at the difference between using the concept of symmetry to complete a drawing of a familiar symmetrical object about a vertical line, compared to constructing the image of an irregular triangle in a line which is

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outside, and not parallel to, any features of the triangle. The properties of the objects and the nature of 'symmetry' are very different for learners. In the first there is familiarity and visual sense and in the second there are only formal labels, relations between features, and abstract properties to assist. His second example is the formula $v = a \times h$, which could be seen as a rule to calculate volume, or an equation to enable us to find h given v and a, or a statement about the relation between three variables. He says that these 'epistemological jumps' cause difficulties for learners (91). Other descriptions include 'epistemological break/obstacle' (Bachelard 1938/1980); 'conceptual change' (e.g. Vosniadou 1994) and 'cognitive gap' (e.g. Filloy and Rojano 1987). I call these 'shifts' for two reasons: firstly, that a fluent mathematician might choose between several alternative perspectives on a problem, dependent on the task or on their exploratory needs; second, that there may be ways to 'smooth' the change from one perspective to another. Because mathematics provides its own models and representations, its own 'meta-mathematics' (Otte cited in Sierpinska 2005, 117), it is possible to bridge such differences. Goodwin and Johnson-Laird (e.g. 2005) suggest that the construction of mental models is how we understand relations, relations between relations, and reasoning about them. Learners, therefore, need time and several experiences to become fluent users of new models, or schemata, and to focus on new relations and properties with new symbolic tools. Iconic representations can connect physical and symbolic understandings (Bruner 1966). In mathematics, manipulable objects and visual images appear to be capable of providing bridges to, but not substitutes for, reasoning from properties. Sierpinska (2005), in her studies of learning linear algebra, moves away from the commonly held notion that some mathematical content is inherently difficult towards a view that there are generic shifts of perspective needed for mathematics, such as between analytic and geometric, which can be scaffolded by providing suitable artefacts, tools and symbol systems and encouraging students to experiment with them, including mental experiments (Otte 2005, 15).

I shall show that these shifts to more abstract ways of thinking are typical adolescent cognitive developments.

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Changes in reasoning and learning in adolescence

The literature on adolescent learning, as a particular stage in life, is broadly in agreement about the kinds of problems that adolescents are more likely to be able to solve than younger children. This section is not a systematic review of the literature – that would require a paper on its own. Instead I give examples of studies which illustrate that the capabilities required to learn higher mathematics ideas, as described above, are manifested in studies of older children's thinking, hence learning higher mathematics need not be an alienating experience for learners, and need not be avoided by focusing a curriculum on applications and empirical approaches.

Inhelder and Piaget report a study carried out with students of a range of ages, from very young to late adolescent, in which participants predict which objects would sink and which would float, and give reasons (1964). They found that young children would make simplistic assumptions about size, and older children began to suggest distinctions between the variables of size (a visible characteristic) and mass – which is not visible. In adolescence, participants would find ways to compare the

mass of objects, and use these to make predictions about the amount of water displaced. Older participants were able to coordinate implications about water displacement of two objects, and compare these to make complex predictions about features that could not be measured materially. Younger children could only make hypotheses about objects they could handle or see, but older children could contemplate abstract properties. This study is useful in two ways: firstly, that becoming able to use abstract ideas to *imagine relations in phenomena* and their implications is something that becomes easier with age; secondly, that the properties being used for *classification and generalisation* matter when reasoning about phenomena. Older students were more able, voluntarily, to identify the need to work with an abstract idea, and their capability to do so is likely to have been influenced by their more extensive experience of talk, distinction-making and tool-use. When teachers intervene to introduce 'scientific' classifications that are unlikely to occur spontaneously, relevant abstractions become available to more learners, through talk, than they might see for themselves.

Piaget offers the idea of conflict to explain how learners have to adjust their understandings in the light of new, and contradictory, experience. However, much activity advocated in modern mathematics curricula suggests that inductive reasoning from repeated successful experience can also lead to new understandings. Dixon and Kelley (2007) showed that new theoretical understandings can indeed develop when students have to 'redescribe' empirical phenomena. Subjects aged 8, 12 and 19 had to work out which way to turn a cog at one end of a hidden connected line of cogs so that the final cog fulfilled a particular function. To solve this problem, subjects ascribed properties to each of the cogs, and developed a theorem that alternate cogs would rotate in opposite ways. The process by which they did this involved some initial assumptions in which the older children made fewer errors than the younger, since they had plausible *theorems-in-action* about cog motion (Vergnaud 1994). Three interesting insights into reasoning arose from this work: that an important step was to ascribe properties to cogs which related their sequence position to their behaviour – *a relation was described*; and that the situation had a *predictive purpose* which motivated subjects' enquiry. Another interesting feature of this study was the *use of representation*. The physical situation, and rotation, can be analogically represented in diagrams and gestures without much need for translation so a shift from thinking about movement and position to encapsulating these together in visual or mental form is easy to make.

Two other insights about adolescent thinking, 'relational complexity' and 'chunking' (Halford 1999; Halford, Cowan and Andrews 2007), also relate to mathematics learning. Over many studies, older children appear able to deal with more active items in their working memory, that is up to five objects, variables or pieces of information, than younger children can. In particular, comparisons between binary relations, which are 4-dimensional situations, can first be made in adolescence (Halford 1999, 205). The examples Halford gives are key mathematical ideas, such as distributivity, proportion and equivalence. The mechanism of 'chunking' combines objects and relations into new entities, such as 'difference' being seen as a new object rather than the result of a subtraction. Seeing a mathematical relation as one object, such as a ratio, difference, or function, rather than as a connection between two objects, makes reasoning about the mathematics easier. Halford's work accords with epistemological work on the nature of mathematics, in that it describes a plausible

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cognitive mechanism behind how we encapsulate and reify mathematical ideas. But while he suggests that some mathematical ideas are only understandable when learners are old enough to juggle four objects in working memory, there is evidence from teaching experiments to suggest that some forms of presentation make relations between binary relationships possible for very young students. Proportional capacities seem to be understood by young children who work extensively with comparing quantities of water (Schmittau 1993), and distributivity can be understood by young children who use Cuisenaire rods or algebra tiles to represent relational structures (e.g. Gattegno 1961). One possibility is that the materials encapsulate the relations and hence allow chunking of the lower-order relations, but Schmittau and Gattegno found that reasoning can go beyond the limitations of the physical models and extend into imaginative spaces. Another possibility is that students are reasoning inductively from patterns in phenomena, constructing generalisations which describe the relations between variables, and thus doing more than imitating (Halford and Busby 2007), but not necessarily seeing the particular relation as a new object.

A further development of thinking as children get older is capability to edit out irrelevant factors from situations (Davis and Anderson 1999, 174), but when students do not know what is, or is not, likely to be salient there is a propensity to be misled by surface, or other inappropriate, features. Brown points out that misleading attention to surface features can be due to learners bringing understandings that have worked for them in the past (1989, 372), a habit we see often in mathematics (e.g. Ryan and Williams 2007), but can also be due to lack of knowledge about the representation and how it works. Teachers are the source of knowledge about what is salient, and can construct situations in which students have to focus on new variables and new covariance, from which new relations can be deduced. To know what is relevant and irrelevant includes knowing classes of non-examples of new categories, and possible 'counter-examples' which challenge the conditions for new theorems (Lakatos 1976).

Another way to prompt new conceptualisation is to provide microworlds in which explanation of phenomena operates at new levels because of the context. Pratt and Noss (2002) show how explanations developed in a new-to-students conceptual world of randomness. In a sequence of clinical interviews with early adolescent students, involving a sequence of predictions of on-screen phenomena, they found that, by developing the software to perturb thinking, they could organise the setting so that students eventually developed a new idea, N, which was more sophisticated and abstract than their naïve notions of randomness. Students did not, however, use idea N the next time it was appropriate; it had to be cued by a situation which was structurally similar to that which originally led to its creation. N was a situated abstraction which only gradually, through tasks which coordinated naïve ideas with new situations, became understood in its own terms. This required design of a neighbourhood of nearly-similar situations. It is also noteworthy that the tasks they used had an imaginary purpose, mending a 'broken' artifact in a microworld, rather than being only explanatory.

From the above discussion, we have seen that adolescent learners can, in suitable circumstances: adopt new classifications; re-theorise as a result of conflicting experience; learn from successful repetition; infer structures from multiple instances, including symbolic instances; identify relations; shift between phenomena and

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analogous images; imagine relations when offered new classifications; extend relations into imagined new spaces; handle new or unfamiliar entities if there is an accessible form of representation; and learn about relations between relations. These abilities may depend on: availability of appropriate near-iconic models and images; having attention drawn to salient features; interference or support from prior knowledge; and having a purpose. For these abstract moves, the presence of an adult to offer new classifications of higher-order ideas, to draw attention to salient aspects, and to structure experiences that make these necessary, is critical. The studies above indicate that the following pedagogic actions are entirely consistent with both the nature of higher mathematics and with the learning developments that go on in adolescence:

- create a need to describe a relation;
- give tasks with a predictive purpose so that there is feedback;
- introduce new classifications;
- ask students to imagine invisible relations;
- ensure that students have 'chunked' relevant knowledge;
- help students cope with new levels of relational complexity;
- expect generalisation of experience;
- reduce the possibility of being distracted by irrelevant variation;
- provide appropriate models, images and representations to enable expansion of ideas beyond physical limitations;
- make available a range of standard and non-standard examples, non-examples and counter-examples;
- sequence tasks which require and enable students to adopt higher levels of abstraction.

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Commonly-recommended approaches

I now return to considering teaching which uses adolescent concerns and propensities, such as the use of open-ended tasks, contextual mathematics and collaborative approaches. These address students' perceived needs for participation (with others and with adults), inclusion in what is going on around them, purpose and relevance. It is widely reported that students who have been encouraged to undertake complex explorations and applications of mathematics instead of more formal teaching can do as well in traditional tests as those who have had a more formal diet, and better in non-routine test items (Boaler 1997; Boaler 2006; Watson and De Geest 2005; Senk and Thompson 2003). Most of these studies address teaching and learning in early adolescence by examining changes in social, organisational and emotional aspects of doing mathematics. There is, however, a shortage of studies which (a) effectively separate out the influences of different aspects of pedagogy or (b) address explicitly the development of conceptual understandings on which higher mathematics can be founded, as described above.

Stoyanova (2007) addresses (a) by identifying what aspects of enquiry pedagogies[3] are associated with higher attainment. She evaluates a mathematics curriculum project in which students used tasks which encouraged investigation, problem posing, and other features aimed at harnessing learners' natural enquiry to learn mathematics. Her study is of interest because of its size, 4500 students, and also

because it relates not to a controlled teaching intervention but to what happened in a large scale roll-out, which inevitably led to wide variation in teachers' interpretations of, and practices of, the intended kinds of teaching and enquiry. While it is impossible to say precisely what the teaching is like in these conditions, the results are more relevant than smaller intervention studies for evaluating curriculum change. Teachers knew that students would be assessed on mathematical strategies, application of mathematics and verification of results, and mathematical reasoning, as well as conceptual and procedural knowledge, and had introduced these features into their teaching. The test items related to five content areas: number, space, measurement, chance and data, and algebra. Test items about methods of working were embedded in contexts from these content areas. Items were generally non-routine and often multi-stage, so related closely to the curriculum aims of many countries.

Results of analysing test answers and teacher surveys showed statistically-significant connections between several practices associated with teachers' interpretations of enquiry methods and mathematical achievement (Department of Education and Training 2004). Relative success in the test was related to the use in lessons of some, but not all, enquiry methods, and different methods were differently effective, in test outcome terms, for different age groups. I only report on year 10 (equivalent to English year 11), since younger students are not the focus of this paper. There was a significant association between those who had experienced the use of *problem-posing*, *checking* by alternative methods, asking *'what if..?'* questions, giving *explanations, testing conjectures*, checking answers for *reasonableness*, splitting problems into *subproblems, looking back* over work, encouraging *persistence,* and higher test achievement. In contrast, generic *problem-solving strategies*, *making conjectures, being told* what to do when stuck, and *sharing strategies*, were not significantly associated with particular levels of achievement, and *use of real contexts* was negatively associated. The effective pedagogic features have in common that they give the adolescent some authority in mathematical work: looking back and checking their own work in various ways, giving explanations, asking new questions, testing hypotheses, and problem-posing. Comparing these aspects to adolescent concern for being in charge, feeling powerful, understanding the world, and being able to argue in ways which make adults listen, suggests that the appeal of such methods to general aspects of adolescence makes a difference. But we can probe further than these typical findings; for example, 'making conjectures' was *not* associated with higher achievement, but 'testing conjectures', which involves modifying ideas in response to feedback, was. The effective features all engaged adolescents in exercising power in relation to *new* mathematical experiences, new forms of mathematical activity, and being asked to use and express these, and to display authority in doing so, rather than rehearsing old forms of activity. Some of the non-associated features reduce power and control: constraining students to use particular strategies, not letting them get 'unstuck' for themselves. Others were about students voicing their thoughts: making conjectures without the associated testing which gives feedback, and sharing strategies with peers rather than using them in dialogue with an expert. Both of these generate participation in the sense of vocally contributing their ideas for others to hear, and hearing others' ideas, but neither was associated with higher achievement. Perhaps this is because merely saying what they already think, and having it accepted

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by others without critical feedback, may limit students to their existing ways of thinking.

As well as features of pedagogy, higher test outcomes were also strongly related to being given mathematical work which was perceived as 'advanced' and to being taught by more qualified teachers, so another challenge is to be explicit about the role of the teacher and the nature of subject input in enquiry learning. This is a critical aspect of teaching which is often under-reported. In Senk and Thompson's collection of evaluations of curriculum reform (2003), studies rely on reports of general pedagogy and reference to published materials; some studies report tasks but not how teachers intervened with students to help them engage on new levels of conceptualisation. Without such reports, we do not know if students are doing better through applying known procedures in new contexts, developing their understanding of elementary concepts, employing inductive or ad hoc reasoning with empirical data, or engaging in higher mathematics such as conceptualising functions and reasoning about relations. In other words, we need to know more about the nature of adult intervention, possibly along the lines of the interventions listed at the end of the previous section.

Curriculum documents from several countries show that 'relevant contexts' are to be used as motivational devices or contexts for problem solving, but in Stoyanova's study, success in year 10 was negatively associated with use of 'real' contexts[4]. The generalities with which students engage in 'real' contexts are not necessarily mathematical, since the questions are grounded in other generalities. For example, students who have engaged in reading a graph of the decay of radioactive waste are likely to be thinking more about the effects of radioactivity than about exponential functions. To engage with new mathematical ideas, students have to reclassify the components of the problem according to mathematical concepts, and reclassification requires some kind of intervention (Sierpinska 1995; Stech 2007). Everyday problems are resolved using everyday and ad hoc methods and do not provide sufficient basis for the creation of new mathematical knowledge; mathematics, in contrast to everyday reasoning, is about representation and generalisation (Sierpinska 1995, 14). A role, therefore, for teaching is to facilitate students' overcoming of the inherent difficulty of 'lifting' the ways we think in one context and applying them in other contexts (Freudenthal 1973, 130). Mathematics is not essentially empirical. Its source of empowerment is in its abstractions, its reasoning, and its hypotheses about objects which only exist in the mathematical imagination, and empirical exploration does not necessitate the development of analytical and deductive methods.

The approach to this issue taken in the Realistic Mathematics movement has its roots in Freudenthal's understanding of mathematics as a human activity, and Treffers' accompanying descriptions of 'vertical' mathematisation, meaning reorganisation within the mathematical system (1987). A Vygotskian view would be that this shift necessarily disrupts previous notions, challenges intuitive constructs, and offers new ways of thinking appropriated by learners as tools for new kinds of action in new situations. A Piagetian view would be that the learner has to make new kinds of classification which may be based on properties which transcend the distinctions the learner currently makes (Inhelder and Piaget 1964, 289). Both views emphasise the need for multiple experiences over time to achieve 'interiorisation' (Vergnaud 2009). Without specific attention to these shifts, mathematical learning may not progress, and the introduction of 'real' and 'relevant' contexts to motivate students

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to learn mathematics could even limit access to formal mathematical ideas. Similarly, if not informed by meaningful interaction with someone who knows more mathematics than they do, who can see how new formal mathematical ideas might be brought to bear on a task, students whose learning depends mainly on collaborating with peers may not encounter new conceptualisations.

Mathematical shifts and adolescence

These cognitive shifts are not in opposition to other aspects of the adolescent project, and do not require the adult to be in authority over the student. The authority of mathematics does not reside in teachers, textbook writers, and inductive situated 'truths', but in the ways in which minds work with mathematics itself (Freudenthal 1973, 147; Vergnaud 1997). For this reason, abstract mathematics, like some of the creative arts, can be an arena in which the adolescent mind can have some control, can validate thinking, and can appeal to a constructed personal, authority.

Shifts towards seeing abstract patterns and structures within a complex world are seen as typical of adolescent development by both the Piagetian and Vygotskian schools (Coleman and Hendry 1990, 47). Shifts from proximal, ad hoc, and sensory methods of solution to abstract concepts are hard to make and need deliberate support (Bachelard, 1938/1980) – indeed this is what is at the heart of Vygotsky's insistence that talk with *knowledgeable* others is a necessary aspect of learning scientific concepts (1978, 131). The adolescent has to be helped to learn, as with other abstract understandings, when it is appropriate to shift between approaches. Importantly, the shifts necessary for mathematical understanding described above are particular versions of the more general shifts in adolescent cognition described at the beginning of this paper. Here, they are elaborated for mathematical learning in the light of the above arguments:

- from dealing with what is familiar to what is unfamiliar, using new tools, new classifications and ways of seeing;
- from tangible, observable, features to imagined, abstract aspects, brought into their experience by adult-constructed language and tasks;
- from sensory, intuitive and quasi-intuitive responses to reasoned and reflexive responses;
- from immediate reactions to mediated reactions;
- from focusing on objects to focusing on relations between objects and the implications of these;
- from understanding relations among objects near-to-hand to extending these in imagination beyond what can be seen.

The expansions of epistemological activity embedded in higher mathematics are *de facto* similar to the ways in which adolescents learn to negotiate with themselves, authority, and the world.

Conclusion

I have examined the ways of thinking that characterise higher mathematics, and compared these to the cognitive possibilities of adolescence. I have shown that these

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are compatible with the emotional and social concerns of adolescence. I have, therefore, laid out the groundwork to support my opening statement, that learning higher mathematics can relate closely to the adolescent project of negotiating adulthood. I have shown that teaching approaches which attend mainly to emotional and social aspects of adolescence can not necessarily provide the new ways of thinking required for mathematics. We need to be more explicit about the nature of such shifts. We need to know more about well-designed pedagogical environments which make *necessary* the construction and adoption of the tools of higher mathematics, and which attend explicitly to the nature of the higher concepts. As a field, we need research reports about successful teaching of adolescents that include detailed information about how teachers support the particular shifts of mathematical understanding that secondary students need to make in order to understand higher concepts.

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Notes

1. This view of adolescence may be dependent on culture, but I adopt it here because it is relevant to the UK culture in which I work, and also to other Western European and North American dominant cultures.
2. 2007 TIMSS results show a 25 percentage point drop in enjoyment of the subject in the UK
3. What is meant by 'enquiry' is encapsulated in the italicised aspects below on which teachers were questioned for evaluative purposes.
4. Success in year 3 was positively correlated with use of 'real contexts' however. This could be because elementary mathematical ideas are often learnt as successful formalisations of informal ideas, whereas secondary school mathematical ideas are more often formal mathematical ideas which do not easily relate to experience and intuition (Nunes, Bryant and Watson, 2009).

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