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ICMET'13

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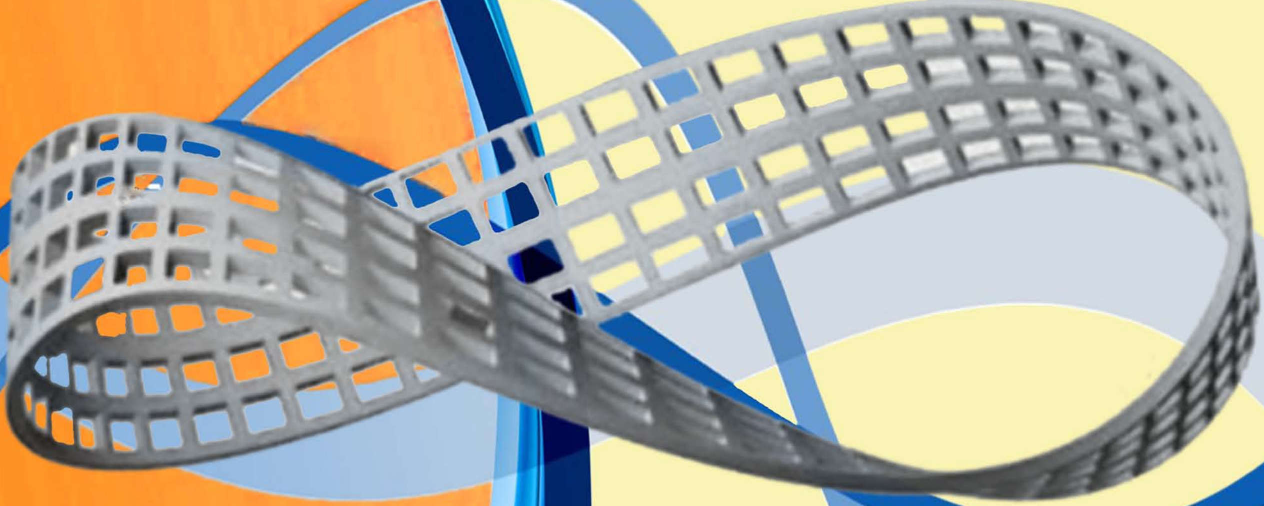

MOHANDAS COLLEGE OF ENGINEERING AND TECHNOLOGY

(An ISO 9001:2008 Certified Institution, Affiliated to the University of Kerala)

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ANAD, NEDUMANGAD, THIRUVANANTHAPURAM 695 544

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FOREWORD

I am happy that the Department of Mathematics took the initiative in organizing the International Conference on Mathematics Education and Mathematics in Engineering and Technology and much more when it was on the occasion of commemorating the fiftieth death anniversary of the founder Sri V. Ṇ Gangadhara Panciker. This is the first international conference at MCET and I congratulate the General Convener and Head of the Department of Mathematics Dr. Mallayya and all the faculty members of the department for taking active interest in organizing and making all arrangements with utmost care for the successful conduct of the event. Mathematics is the backbone of all streams in Engineering, Science and Technology and recent developments and innovations in mathematics have triggered vast technological advances. A sound knowledge of mathematics from basic to advanced level is essentially needed for understanding and assimilating any technological process. In this context the initiative taken by the department of mathematics to organize the first international Conference at MCET on 'Mathematics Education and Mathematics in Engineering and Technology' is quite apt.

Active participation of delegates and distinguished experts from reputed institutions in India and other parts of the world like Greece, Manchester, California, Oxford, Hungary, South Africa, Turkey, Florida, Ohio, New Zealand, and Nepal has made the conference a great success. I have great pleasure in presenting the Proceedings of ICMET'13 with papers covering a broad spectrum of topics relating to mathematics education and mathematics with applications in Engineering, Science and Technology and I hope that it will remain a valuable document for enthusiasts in future. I express deep gratitude to all the contributors to this volume and all the delegates who enriched knowledge with their presentations and active participation, and made the four day academic meet a memorable event in the history of MCET.

Dr Ashalatha Thampuran
Director, MCET

PREFACE

It is a matter of immense pleasure for the Department of Mathematics to take the lead in organizing the first international event in MCET. Taking the vast importance and inevitable role of mathematics in all streams of Engineering, Science and Technology, an International Conference on Mathematics Education and Mathematics in Engineering and Technology (ICMET'13) was organized in MCET from 17-20 Dec 2013. It was organized on the occasion of commemorating the fiftieth death anniversary of Sri. V. Ṇ Gangadhara Panicker, the noble soul behind this fast progressing institute of technology. The event also celebrated the International Year of Statistics with talks by eminent Professors of Statistics.

The conference was cosponsored by KSCSTE and technically supported by Kerala Mathematical Association (KMA), Indian Society for Technical Education (ISTE Kerala Section) and Indian Society for Industrial and Applied Mathematics (ISIAM) and I express heartfelt thanks for their valuable support.

While celebrating the International Year of Statistics, it is a great honour for us to have the conference inaugurated by Dr. Andreas N. Philippou, an eminent Professor of Statistics from University of Patras in Greece and former Education Minister of Republic of Cyprus. With 20 invited talks including key note addresses by Dr. Philippou and Dr E. Krishnan (visiting faculty IISER TVM) and 26 papers the conference provided a broad platform for powerful intellectual stimulation for 150 delegates from various parts of India and abroad to interact and share mathematical knowledge updating recent advances and innovations in the field. I am happy to know that the delegates who arrived from far and wide had a nice and fruitful time and carried home pleasant memories of the academic meet. We are deeply indebted to all the esteemed speakers and delegates for making the conference a success with their pleasant participation, keen interest, and valuable presentations.

I gratefully acknowledge the wholehearted support, help and advice provided by the Hon'ble Chairman Sri. G. Mohandas, Secretary Mrs Rani Mohandas, Treasurer Sri Krishna Mohan and Administrative Secretary Sri Nandagopal. Words are not enough to express heartfelt gratitude to the Respected Director Dr. Ashalatha Thampuran for the motivation, guidance, and unstinting wholehearted support at each and every stage of this venture. I also thank the Principal Dr Ibrahimkutty, all Heads of departments, Faculty members, Advisory Committee and Organizing Committee members, Colleagues, Students, Volunteers and all sponsors for their valuable support. Heartfelt thanks are due to Dr George Gheverghese Joseph (Manchester, Academic Convener ICMET) and Dr C Satheesh Kumar (Kerala University) for their sincere efforts in planning and organizing the scientific programme. I also place on record our deep sense of gratitude to one of the esteemed delegates Dr Premjit Singh (Athens, Ohio, USA) for instituting an annual scholarship in MCET in the name of her mother Mrs. Kaushalya Vati to encourage the education and empower a needy and deserving girl student of MCET.

I sincerely acknowledge the valuable technical help received from Sri P.R Krishnakumar (Chief Librarian, MCET), Sri Madhav Sankar Ṛ Warrier (B Tech Student, MCET), and Sri Rahul R.M (P.G Student, Kerala University) at various stages in bringing out this volume.

Dr V. Madhukar Mallayya
General Convener, ICMET'13
Head, Department of Mathematics

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Proceedings, ICMET' 13 (17-20 Dec 2013)

Distributions of order k with applications

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(Keynote Speaker)

Abstract : The distributions of order k are infinite families of probability distributions indexed by a positive integer k, which reduce to the respective classical probability distributions for $k = 1$, and they have many applications in Statistics, Engineering, Meteorology, etc. A few of the most applicable ones, namely the geometric, the negative binomial, the binomial, and the Poisson distributions of order k, are briefly discussed presently and an application is given in Reliability.

1 The Geometric Distribution of Order k

For any positive integer k, denote by T_k the number of independent trials with success probability p until the occurrence of the k^{th} consecutive success, and set $q = 1 - p$. Philippou and Muwafi (1982) found that, for $n = k, k+1, \cdots$,

$$f_k(n) = P(T_k = n)P^n \sum \binom{n_1 + \cdots + n_k}{n_1, \cdots, n_k} \left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \tag{1.1}$$

and 0 otherwise, where the summation is taken over all k-tuples of non-negative integers $n_1, n_2, \cdots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - k$.

The proof is based on the observation that a typical element of the event $(T_k = n)$ is an arrangement

$$A = x_1, x_2, \cdots, x_{n_1+n_2+\cdots+n_k} SS \cdots S(kS's) \tag{1.2}$$

such that n_1 of the xs are $E_1 = F, n_2$ of the xs are $E_2 = SF, \cdots, n_k$ of the xs are $E_k = SS \cdots SF(k-1Ss)$, and $n_1 + 2n_2 + \cdots + kn_k = n - k$. Fix $n_1, \cdots, n_k$. Then the number of the As is

$$\binom{n_1 + \cdots + n_k}{n_1, \cdots, n_k} \tag{1.3}$$

and each one has probability

$$\begin{aligned} P(A) &= [P(E_1)]^{n_1}[P(E_2)]^{n_2} \cdots [P(E_k)]^{n_k} P(SS \cdots S)(kS's) \\ &= q^{n_1}(pq)^{n_2} \cdots (p^{k-1}q)^{n_k} p^k = p^n \left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \end{aligned} \tag{1.4}$$

Therefore

$$P(\text{ all } A's : n_i \geq 0 \text{ and fixed, } 0 \leq i \leq n) = \binom{n_1 + \cdots + n_k}{n_1, \cdots, n_k} P^n \left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \tag{1.5}$$

But the non-negative integers $n_1, n_2, \cdots, n_k$ may vary subject to the condition $n_1 + 2n_2 + \cdots + kn_k = n - k$, and this completes the proof of (1.1)

Is $f_k(n)$ a proper probability mass function?

The answer was given by Philippou, Georghiou and Philippou (1983) who employed the transformation

$$n_i = m_i (1 \le i \le k), \qquad n = m + \sum_{i=1}^{k} (i-1) m_i \tag{1.6}$$

and the multinomial theorem to show that

$$\sum_{n=k}^{\infty} f_k(n) = \sum_{n=k}^{\infty} P(T_k = n) = 1 \tag{1.7}$$

They named the distribution of T_k the geometric distribution of order k with parameter p, since for $k = 1$ it reduces to the geometric distribution with pmf.

$$f_1(n) = q^{n-1} p \qquad n \ge 1 \tag{1.8}$$

They also employed the transformation (1.6) to obtain the probability generating function (pgf) of T_k, say $g_k(s)$ and hence its mean and variance:

$$g_k(s) = \frac{(1 - ps) p^k s^k}{1 - s + qp^k s^{k+1}} \qquad |s| \le 1 \tag{1.9}$$

$$\mu_k = \frac{1 - p^k}{qp^k} \text{ and } \sigma_k^2 = \frac{1 - (2k+1) qp^k - p^{2k+1}}{q^2 p^{2k}} \tag{1.10}$$

A different derivation of $g_k(s)$ was first given by Feller (1968) who used the method of partial fractions to derive the surprisingly good approximation

$$P(T_k > n) \simeq \frac{1 - ps_0}{(k + 1 - ks_0) q s_0^{n+1}} \qquad n \ge k \tag{1.11}$$

where s_0 is the unique positive root of $p^k s^k / g_k(s)$ Relation (1.1) implies

$$P(T_k = n | p = 1/2) = \frac{F_{n-k+1}^{(k)}}{2^n} \qquad n \ge k \tag{1.12}$$

and

$$P(T_2 = n | p = 1/2) = \frac{F_{n-1}}{2^n}, \qquad n \ge 2 \tag{1.13}$$

where F_n is the n^{th} Fibonacci number, and $F_n^{(k)}$ is the n^{th} Fibonacci number of order k, since $F_n^{(2)} = F_n$ and

$$F_{n+1}^{(k)} = \sum \binom{n_1 + \cdots + n_k}{n_1 \cdots, n_k} \qquad n \ge 0 \tag{1.14}$$

where the summation is taken over all k-tuples of non-negative integers $n_1, n_2, \cdots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n[30, 38]$.

Alternative simpler formulas for calculating $f_k(n)$ have been found. The following recurrence, for example,

$$f_k(k) = p^k, f_k(n) = qp^k, \qquad k+1 \leq n \leq 2k$$
$$f_k(n) = f_k(n-1) - qp^k f_k(n-1-k), \qquad n \geq 2k+1 \tag{1.15}$$

due to Philippou and Makri (1985), is very efficient.

2 Negative Binomial Distributions of Order k

Assuming that $X_1, \cdots, X_r$ are independent random variables distributed as geometric of order k with parameter p, and setting $T_{rk} = \sum_{j=1}^{r} X_j$, the latter authors showed that, for $n \geq rk$

$$\begin{aligned} f_{r,k}(n) &= P(T_{r,k} = n) \\ &= P^n \sum \binom{n_1 + \cdots + n_k + r - 1}{n_1 \cdots, n_k, r-1} \left(\tfrac{q}{p}\right)^{n_1 + \cdots + n_k} \end{aligned} \tag{2.1}$$

and 0 otherwise, where the summation is taken over all k-tuples of non-negative integers $n_1, n_2, \cdots, n_k$ such that $n_1 + 2n_2 + . \cdots + kn_k = n - rk$.

By means of the transformation (1.6) and the multinomial theorem,

$$\sum_{n=k}^{\infty} f_{r,k}(n) = \sum_{n=k}^{\infty} P(T_{r,k} = n) = 1 \tag{2.2}$$

They named the distribution of $T_{r,k}$ negative binomial distribution of order k with parameters r and p, since for $k = 1$ it reduces to the negative binomial distribution with pmf

$$f_{r,1}(n) = \binom{n-1}{r-1} p^r q^{n-r}, \qquad n \geq r \tag{2.3}$$

The pgf, mean and variance of $T_{r,k}$ are

$$g_{r,k}(s) = \left[\tfrac{(1-ps)p^k s^k}{1-s+qp^k s^{k+1}}^{r}\right] \qquad |s| \leq 1| \tag{2.4}$$

$$\begin{aligned} \mu_{r,k} &= r\frac{1-p^k}{qp^k}, \\ \sigma^2_{r,k} &= r\frac{1-(2k+1)qp^k - p^{2k+1}}{q^2 p^{2k}} \end{aligned} \tag{2.5}$$

Simpler formulas for calculating $f_{r,k}(n)$ exist. Philippou and Georghiou (1989), for example, derived the following efficient recurrence for $f_{r,k}(n)$ which generalizes the above mentioned one for $f_k(n)$.For $n \geq rk+1$

$$f_{r,k}(n) = \frac{q/p}{n-rk} \sum_{j=1}^{k} [n - rk + j(r-1)] p^j f_{r,k}(n-j) \tag{2.6}$$

Denote now by $T^G_{r,k}$ the waiting time until the $r-th$ occurrence of a success run of length greater than or equal to k in iid trials with success probability p. Then it may be easily shown that

$$T^G_{r,k} = T_{r,k} + T^*_{r-1,1} \tag{2.7}$$

where $T_{r,k}$ is as above and $T^*_{r-1,1}$ is distributed as $T_{r-1,1}$ with p and q interchanged. Therefore,

$$f^G_{r,k}(n) = P(T^G_{r,k} = n) = \sum_{x=0}^{n} f_{r,k}(n-x) f^*_{r-1,1}(x)$$
$$= \sum_{x=0}^{n} f_{r,k}(n-x) \binom{x-1}{r-2} p^{r-1} q^{n-r+1}, \quad n \geq r(k+1)-1, \tag{2.8}$$

and 0 otherwise, where $f_{r,k}(n-k)$ is given by either (2.3) or (2.6).

For an alternative formula, simpler than (2.8), we refer to Museli (1996). It may be obtained from the pgf $g^G_{r,k}(s) g^*_{r-1,1}(s)$ of $T^G_{r,k}$, where $g^*_{r-1,1}(s)$ is the pgf of $T^*_{r-1,1}$. Clearly, $T_{r,k}$ is the waiting time until the r-th non-overlapping occurrence of a success run of length k. Ling (1989) introduced another class of very applicable negative binomial distributions of order k, by deriving two recurrence relations for the pmf of the waiting time $W_{r,k}$ until the $r-th$ overlapping occurrence of a success run of length k in iid trials. One of them is for $n \geq k+r-1$

$$P(W_{r,k} = n) = \sum_{j=1}^{r} \binom{r-1}{j-1} p^{r-j} q^{j-1} P(T_{j,k} = n-r+1) \tag{2.9}$$

He also derived its probability generating function, mean and variance. The mean is

$$E(W_{r,k}) = \frac{rq + p - p^k}{qp^k} \tag{2.10}$$

Another formula for the pmf of $W_{r,k}$ analogous to (2.1), was derived by Tripsiannis and Philippou (1997).

3 The Poisson Distribution of Order k

Assuming that $rq \to \lambda (\lambda > 0)$ as $r \to \infty$ and $q \to 0$, Philippou, Georghiou and Philippou (1983) showed that, for $x = 0, 1, 2, \cdots$

$$\lim_{r\to\infty} P(T_{r,k} - rk = x) = e^{-k\lambda} \sum \frac{\lambda^{x_1+\cdots+x_k}}{x_1! \cdots x_k!} = f_k(x;\lambda) \tag{3.1}$$

and 0 otherwise, where the summation is taken over all k-tuples of non-negative integers $x_1, x_2, \cdots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = x$. By means of the transformation (1.6) and the multinomial theorem,

$$\sum_{n=k}^{\infty} f_k(x;\lambda) = 1 \tag{3.2}$$

he distribution with probability mass function $f_k(x;\lambda)$ has been called the *Poisson distribution of order k with parameter* λ , since for $k = 1$ it reduces to

$$f_1(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \cdots, \tag{3.3}$$

which is the pmf of the Poisson distribution

The pgf, mean and variance of $f_k(x;\lambda)$ are

$$g_k(s) = exp - \left[\lambda \left(k - \sum_{i=1}^{k} s^k\right)\right] \qquad |s| \leq 1 \tag{3.4}$$

$$\mu_k = \frac{k(k+1)}{2}\lambda, \qquad \sigma_k^2 = \frac{k(k+1)(2k+1)}{6}\lambda \tag{3.5}$$

See also Aki et al. (1984), Philippou (1983b, 1986), and Balakrishnan and Koutras (2002) for the geometric, negative binomial and Poisson distributions of order k mentioned above.

4 An Open Problem

Denote by $m_{k,\lambda}$ the mode(s) of $f_k(x;\lambda)$, i.e. the value(s) of x for which $f_k(x;\lambda)$ attains its maximum. It is well known that

$$m_{1,\lambda} = \lambda \text{ and } \lambda - 1 \text{ if } \lambda \in \mathbb{N} \text{ and } m_{1,\lambda} = \lfloor \lambda \rfloor \text{ if } \lambda \notin \mathbb{N} \tag{4.1}$$

The problem of deriving $m_{k,\lambda}$ for $\lambda > 0$ and $k \geq 2$ was posed by Philippou (1983) and remains open since then, apart from the following partial advancements.

Luo (1987) obtained a lower bound for $m_{k,\lambda}$. Georghiou, Philippou and Saghafi (2013) established lower and upper bounds and showed that

$$m_{k,\lambda} = \frac{\lambda k(k+1)}{2} - \lfloor \frac{k}{2} \rfloor \qquad 2 \leq k \leq 5, \qquad \lambda \in \mathbb{N} \tag{4.2}$$

Furthermore, in a recent note submitted for publication, Philippou (2013) showed that

$$m_{k,\lambda} = 0, \qquad k \geq 1, \qquad 0 < \lambda < 2/k(k+1) \tag{4.3}$$

. He also showed that

$$m_{2,\lambda} = 0, 0 < \lambda \leq -1 + \sqrt{3} \qquad m_{2,\lambda} = 2, -1 + \sqrt{3} \leq \lambda < 1 \tag{4.4}$$

5 Binomial Distributions of Order k

Denote by $N_{n,k}$ the number of non-overlapping success runs of length k in $n(n \geq 1)$ independent trials with success probability $p(0 < p < 1)$. The asymptotic normality of a normalized version of $N_{n,k}$ was established by von Mises (see Feller 1968, p 324, where a simpler proof is presented).

Its exact pmf was obtained by Hirano (1986) and Philippou and Makri (1986), who named the distribution *binomial distribution of order k with parameter vector (n, p).* They found that, for $x = 0, 1, [n/k]$

$$P(N_{n,k} = x) = p^n \sum_{i=0}^{k-1} \sum \binom{x_1 + \cdots + x_k + x}{x_1, \cdots, x_k, x} \left(\frac{q}{p}\right)^{x_1 + \cdots + x_k} \tag{5.1}$$

where the inner sum is taken over all k-tuples of non-negative integers $x_1, x_2, \cdots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k + i = n - kx$.

Clearly,

$$P(N_{n,1} = x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, \cdots, n. \tag{5.2}$$

A simpler formula than (5.2), in terms of binomial coefficients, was derived by Godbole (1990). Aki and Hirano (1988) established recurrence relations for calculating binomial probabilities of order k and found the mean of $N_{n,k}$. For $n \geq k$

$$E(N_{n,k}) = \sum_{j=1}^{\lfloor n/k \rfloor} [(n-jk+1)p^{jk} - (n-jk)p^{jk+1}] \tag{5.3}$$

Antzoulakos and Hadjiconstantinidis (2001) derived the rth factorial moment of $N_{n,k}$.

Another class of binomial distributions of order k was introduced by Ling (1988) who considered the number $M_{n,k}$ of overlapping success runs of length k in n iid trials . Its pmf is for $x = 0, 1, \cdots, n-k+1$

$$P(M_{n,k} = x) = p^n \sum_{i=0}^{n} \sum \binom{x_1 + \cdots + x_n}{x_1, \cdots, x_n} \left(\frac{q}{p}\right)^{x_1+\cdots+x_n} \tag{5.4}$$

and 0 otherwise, where the inner sum is taken over all n-tuples of non-negative integers $x_1, x_2, \cdots, x_n$ such that $x_1 + 2x_2 + \cdots + nx_n + i = n$, and $max\{0, i-k+1) + x_{k+1} + 2x_{k+2} + \cdots + (n-k)x_n = x$.

Its mean is

$$E(M_{n,k}) = (n-k+1)p^k, n \geq k \tag{5.5}$$

Makri and Philippou (1994) introduced and studied non-overlapping and overlapping circular success runs of length $kN^c_{n,k}$ and $M^c_{n,k}$ in n iid trials and derived their pgfs and means. Their means are

$$E(N^c_{n,k}) = nqp^k \left(\frac{1 - p^{\lfloor \frac{n-1}{k} \rfloor k}}{1-p^k}\right) \left\lfloor \frac{n}{k} \right\rfloor p^n, \quad n \geq k \tag{5.6}$$

$$E(M^c_{n,k}) = np^k, n \geq k \tag{5.7}$$

See, also, Charalambides (1994) and Koutras et al. (1994).

Hirano and Aki (1993) studied the number $G_{n,k}$ of success runs of length greater than or equal to k deriving its pgf, mean and variance (see also Mood (1940)). Its mean in the iid case is

$$E(G_{n,k}) = [(1 + (n-k)q]p^k, n \geq k \tag{5.8}$$

For $x = 0, 1, \cdots, [(n+1)/(k+1)]$, Museli (1996) derived the following simple formula for its pmf in the iid case

$$P(G_{n,k} = x) \sum_{m=x}^{\lfloor \frac{n+1}{k+1} \rfloor} (-1)^{m-x} \binom{m}{x} p^{mk} q^{m-1} \left[\binom{n-mk}{m-1} + q\binom{n-mk}{m}\right] \tag{5.9}$$

Makri, Philippou and Psillakis (2007a) derived another simple formula for the pmf of $G_{n,k}$ and gave an alternative simpler derivation of its mean. They also studied the number $G^c_{n,k}$ of circular success runs of length greater than or equal to k, deriving its pgf and mean. Its mean is

$$E(G^c_{k,k}) = p^k, \qquad E(G^c_{n,k}) = p^n + nqp^k, \qquad n \geq k+1 \tag{5.10}$$

Han and Aki (2000) introduced the number of $l-$ overlapping success runs of length k in n trials and derived its pgf in the iid case, as well as when the trials constitute a higher order Markov chain. See also Aki and Hirano (2000) and Antzoulakos (2003).

Makri and Philippou (2005) reconsidered the number of l-overlapping success runs of length k in n Bernoulli trials, say $N_{n,k,l}(0 \leq l \leq k-1)$, and derived its pmf and its mean. The mean is

$$E(N_{n,k,l}) = p^l \sum_{j=1}^{\lfloor \frac{n-1}{k-l} \rfloor} \{1 + q[n - l - j(k-l)]\} p^{j(k-l)}, \quad n \geq k \tag{5.11}$$

which reduces to (5.3) and (5.5) for $l = 0$ and $k-1$, respectively. They also derived the pmf and the mean of the circular number of l-overlapping success runs of length k in n Bernoulli trials, say $N^c_{n,k,l}(0 \leq l \leq k-1)$. The mean is

$$E(N^c_{n,k,l}) = nqp^k \left(\frac{1-p^{\lfloor \frac{n-1-l}{k-1} \rfloor (k-l)}}{1-p^{k-1}} \right) + \lfloor \frac{n}{k-1} \rfloor p^n, \qquad n \geq k \tag{5.12}$$

which reduces to (5.6) and (5.7) for $l = 0$ and $k-1$, respectively.

For further generalizations of this section and related work, we refer to Sen etal. (2006), Makri etal. (2007a, 2007b), Eryilmaz (2008), and Dafnis etal. (2010, 2012).

6 An Application in Reliability - Exact Formulas for Linear and Circular Systems

A linear (circular) $m-$consecutive $-k-$ out of $-n : F$ system, due to Griffith (1986), consists of n components ordered on a line (on a circle). The system fails if and only if there are at least m non-overlapping runs of k consecutive failed components.

Makri and Philippou (1996) derived four exact formulas for its reliability $R_l(p; m, k, n)$ ($R_c(p; m, k, n)$), as a direct consequence of the binomial (circular binomial) distribution of order k. In fact, if $N^*_{n,k}$ is distributed as binomial of order k with parameters n and q, then

$$\begin{aligned} R_l(p; m, k, n) &= \sum_{x=0}^{m-1} P(N^*_{n,k} = x) \\ &= \sum_{x=0}^{m-1} q^n \sum_{i=0}^{k-1} \sum \binom{x_1 + \cdots + x_k + x}{x_1, \cdots, x_k, x} \left(\frac{p}{q}\right)^{x_1 + \cdots + x_k} \end{aligned} \tag{6.1}$$

where the inner sum is taken over all k-tuples of non-negative integers $x_1, x_2, \ldots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k + i = n - kx$, by the definition of the reliability of the system and (5.5). See also Philippou (1986) for $m = 1$.

The formula for the reliability of the circular system stated next is not given explicitly since we have not stated the respective formula for the circular binomial distribution of order k in Section 5. We note, however, that if $N^*_{n,k}$ is distributed as circular binomial of order k with parameters n and q, then

$$R_c(p; m, k, n) = \sum_{x=0}^{m-1} P(N^{c*}_{n,k} = x) \tag{6.2}$$

by the definition of the reliability of the system.

For related reliability results until the time of their publication, we refer to the paper of Philippou (1986) and the book of Kuo and Zuo (2003).

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Fibonacci numbers, probability, and gambling

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Abstract : The Fibonacci and Lucas numbers are briefly introduced and their relationship to the golden mean and the geometric distribution of order k is presented. Three gambling systems are also touched upon, as well as the odds in some odd-even games.

1 Introduction

Leonardo Fibonacci (c.1175-c.1240) was born in Pisa, Italy, but had a Muslim teacher in sea side Bugia of what is to day Algeria when his father was a customs officer there. He studied and travelled extensively in the Mediterranean, becoming one of the best mathematicians of the Middle Ages [Eves (1990)]. In 1202 he introduced the Hindu-Arabic numerals to Europe with his book Liber Abaci (Book of *Calculation*), which also includes his rabbit problem.

The numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$, formally defined by the recurrence relation

$$F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n, \qquad n \geq 1, \tag{1.1}$$

have been named Fibonacci numbers by nineteenth century French mathematician Edouard Lucas to honor Fibonacci.

We now know, however, that the Fibonacci numbers have been known before Fibonacci by Indian scholars who had been interested in rhythmic patterns formed from one-beat and two-beat notes or syllables.

The astronomer Johann Kepler rediscovered them in 1611, and since then several renowned mathematicians, including J. Binet and E. Catalan, have dealt with them. Lucas studied Fibonacci numbers extensively, and the sequence $1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \cdots$, formally defined by the recurrence

$$L_1 = 1, L_2 = 3, L_{n+2} = L_{n+1} + L_n, \qquad n \geq 1, \tag{1.2}$$

bears his name.

During the twentieth century, interest in the Fibonacci numbers and their applications rose rapidly. In 1961 Vorobyov published his *Fibonacci Numbers*, and Hoggatt, Jr., followed in 1969 with *Fibonacci and Lucas Numbers.* Meanwhile, in 1963, Hoggatt, Jr. and his associates founded *The Fibonacci Association* and started publishing The *Fibonacci Quarterly.* Finally, in 1984, the First International Conference on Fibonacci Numbers and Their Applications was held in Patras, Greece, and the Proceedings were published by Reidel. The Second was held in Santa Clara, California, in 1986, the Third in Pisa,

Italy, in 1988 (one every two years in Europe and the USA, respectively), and so on. The Proceedings have been published by Kluwer until 2004 and by a new publisher thereafter. See, for example, Philippou, Bergum and Horadam (1986, 1988).
The Fibonacci numbers appear in sunflowers, pineapples, and phyllotaxis. They appear in geometry and architecture, in computer science and probability theory, in gambling. The scrambled version $13, 3, 2, 21, 1, 1, 8, 5$ of the first eight of them appears in *The Da Vinci Code*, a novel by D. Brown (2003) which is also a well known Hollywood film.

2 The Rabbit Problem

The rabbit problem which appears in *Liber Abaci* is trivial and may be stated as follows. On January 1, there is a pair of adult rabbits in an enclosure. This pair produces one pair of baby rabbits on February 1^{st}, and one pair of baby rabbits on the first day of each month thereafter. Each baby pair grows to be an adult pair in one month, and produces a baby pair on the first day of the third month of their life as well as on the first day of each month thereafter. Find the number of pairs of rabbits in the enclosure a year later on January 1^{st} after the day's births.
Counting them, or otherwise, the number of pairs of adult rabbits is $A_{13} = 233 = F_{13}$, the number of pairs of baby rabbits is $B_{13} = 144 = F_{12,}$ and the total number of pairs of Rabbits is $T_{13} = A_{13} + B_{13} = 377 = F_{14} = F_{12} + F_{13.}$

3 The Golden Mean (or Golden Ratio, Golden Section, Divine Proportion)

According to Euclid's Elements ($\Sigma\tau o\iota\chi\tilde{\iota}\alpha$ in Greek) "A straight line is said to have been *cut in extreme and mean ratio* when, as the whole line is to the greater segment, so is the greater to the less."
Suppose we want to divide a line segment AB into two line segments AS and SB so that

$$\frac{AB}{AS} = \frac{AS}{SB}$$

Setting $x = AS/SB$, we get

$$x = \frac{AS + SB}{AS} = 1 + \frac{1}{\frac{AS}{SB}} = 1 + \frac{1}{x} \text{ iff } x^2 - x - 1 = 0$$

The equation $x^2 - x - 1 = 0$ has the roots

$$\alpha = \left(\frac{1+\sqrt{5}}{2}\right) = 1.61803\ldots, \quad \beta = \left(\frac{1-\sqrt{5}}{2}\right) = -0.61803\ldots$$

The first, denoted here by α, even though it is usually denoted by ϕ in honor of the Greek sculptor Phidias ($\phi\varepsilon\iota\delta\tilde{\iota}\alpha\zeta$ in Greek), has been called golden section, golden ratio, golden mean, or divine proportion, since it is associated with our perception of beauty. The numbers α and β are irrational.
A geometric construction of S given the line segment AB can be done as follows.

1. Construct a perpendicular BC at point B, with BC half the length of AB. Draw the hypotenuse AC.

2. Draw an arc with center C and radius CB. This arc intersects the hypotenuse AC at point D.

3. Draw an arc with center A and radius AD. This arc intersects the original line segment AB at point S. Point S divides the original segment AB into line segments AS and SB with lengths in the golden ratio.

4 Two Results Relating the Fibonacci Numbers and the Golden Mean

Proposition 4.1. The ratios of consecutive Fibonacci numbers converge to the golden mean or divine proportion

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha = \left(\frac{1+\sqrt{5}}{2}\right) = 1.61803\cdots$$

Proof. Let $x_n = F_{n+1}/F_n, n \geq 1$ Then

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3/2, \quad x_4 = 5/3 \quad x_5 = 8/5 \quad x_4 13/8, \cdots$$

which indicates (and it may be proven so) that x_n converges to a positive number, say x. Consequently,

$$\begin{aligned} x = \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \left(\frac{F_n + F_{n-1}}{F_n}\right) \text{ by (1.1)} \\ &= \lim_{n\to\infty} \left(1 + \left(\frac{1}{x_{n-1}}\right)\right) = 1 + \frac{1}{x} \text{ if and only if } x^2 - x - 1 = 0 \end{aligned}$$

which establishes the proposition,since the positive root of $x^2 - x - 1 = 0$ is α.
The Binet formulas to which we turn now provide the following closed expressions for the Fibonacci and Lucas numbers

□

Proposition 4.2 (Binet Formulas)**.** Let α be the golden mean and $\beta = 1 - \alpha$. Then

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1, \quad L_n = \alpha^n + \beta^n, \quad n \geq 1$$

Proof. Since α and β are roots of $x^2 - x - 1 = 0$, it follows that

$$\alpha^2 = \alpha + 1, \qquad \beta^2 = \beta + 1 \tag{4.1}$$

Multiplying each side of the two equations by α^n, β^n, respectively, we get

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n, \qquad \beta^{n+2} = \beta^{n+1} + \beta^n \tag{4.2}$$

which imply

$$\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta} = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} + \frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}, \qquad n \geq 1, \tag{4.3}$$

and

$$\alpha^{n+2}+\beta^{n+2} = \alpha^{n+1}+\beta^{n+1}+\alpha^{n}+\beta^{n}, \qquad n \geq 1, \tag{4.4}$$

We show now the first Binet formula. Setting $U_n = (\alpha^n - \beta^n)/(\alpha-\beta)$, $n \geq 1$ it suffices to show that U_n satisfies the defining relationship of F_n. It does, since $U_1 = (\alpha-\beta)/(\alpha-\beta) = 1$, $U_2 = \alpha+\beta = 1$ and $U_{n+2} = U_{n+1} + U_n$, $n \geq 1$ by (4.3)

We next show the second. Setting $V_n = \alpha^n + \beta^n$, $n \geq 1$,it suffices to show that V_n satisfies the defining relationship of L_n. It does, since $V_1 = \alpha + \beta = 1$, $V_2 = \alpha^2 + \beta^2 = \alpha + \beta + 2 = 3$ by (4.1) and $V_{n+2} = V_{n+1} + V_n$, $n \geq 1$ by (4.4) □

5 A Few Fibonacci Identities

The following identities may be established by induction on n.

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \qquad n \geq 1 \tag{5.1}$$

$$\sum_{i=1}^{n} F_i^2 = F_n Fn + 1 \qquad n \geq 1 \tag{5.2}$$

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad n \geq 1 \tag{5.3}$$

$$F_{n+1} = \sum_{i=0}^{[n/2]} \binom{n-i}{i} \qquad n \geq 1 \tag{5.4}$$

$$F_n L_n = F_{2n} \qquad n \geq 1 \tag{5.5}$$

We give an even easier proof of the first.

Proof. of (5.1) By means of the definition of the Fibonacci numbers, we have for $n \geq 1$

$$\sum_{i=1}^{n} F_1 = \sum_{i=1}^{n} (F_{i+2} - F_{i+1} = (F_3 - F_2) + (F_4 - F_3) + \cdots + (F_{n+2} - F_{n+1}) = F_{n+2} - 1$$

□

6 Infinite Series and Fibonacci Numbers

$$\sum_{i=2}^{\infty} \frac{(-1)^i}{F_i F_{i-1}} = \alpha - 1 \tag{6.1}$$

$$\sum_{i=0}^{\infty} \frac{1}{F_{2^i}} = \frac{7-\sqrt{5}}{2} \tag{6.2}$$

Proof. The second is due to Millin (1974), a high school student at that time (see also Good (1974)). We prove the first. In fact, we have by (5.3) and Proposition (4.1) as $n \to \infty$ □

$$S_n = \sum_{i=2}^{n} \frac{(-1)^i}{F_i F_{i-1}} = \sum_{i=2}^{n} \frac{F_{i-1}F_{i+1} - F_i^2}{F_i F_{i-1}} = \sum_{i=2}^{n} \left(\frac{F_{i+1}}{F_i} - \frac{F_i}{F_{i-1}} \right) = \frac{F_{n+1}}{F_n} - 1 \to a - 1$$

7 Probability and Fibonacci Numbers - The Geometric Distribution of Order k

Toss a fair coin until a head (H) appears two consecutive times. Denote by E_n the event that this will happen at the $n - th$ tossing. What is the probability of E_n, say $P(E_n)$? What if the coin is not fair? What if the unfair coin is tossed until H appear k consecutive times?
For $n = 2$, $E_2 = \{HH\}$. Therefore

$$P(E_2) = P(\{HH\}) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \frac{1}{2^2} = \frac{F_1}{2^2}$$

For $n = 3, E_3 = \{THH\}$. Therefore

$$P(E_3) = P(\{THH\}) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \frac{1}{2^3} = \frac{F_2}{2^3}$$

For $n = 4, E_4 = \{HTHH, TTHH\}$. Therefore

$$P(E_3) = P(\{HTHH\}, \{TTHH\}) = \frac{1}{2^4} + \frac{1}{2^4} = \frac{F_3}{2^4}$$

In general, it can be proven that

$$P(E_n) == \frac{F_{n-1}}{2^n} \qquad n \geq 1$$

In fact, much more can be shown, the following.

Theorem 7.1 (Philippaou and Muwafi(1982))**.** For any positive integer k, denote by T_k the number of independent trials with success probability $p(0 < p < 1)$ until the occurrence of the k^{th} consecutive success, and set $q = 1 - p$. Philippou and Muwafi (1982) found that, for $n = k, k+1, \cdots$,

$$f_k(n) = P(T_k = n) = P^n \sum \binom{n_1 + \cdots + n_k}{n_1, \cdots, n_k} \left(\tfrac{q}{p}\right)^{n_1 + \cdots + n_k}$$

and 0 otherwise, where the summation is taken over all k-tuples of non-negative integers $n_1, n_2, \cdots, n_k$ such that $n_1 + 2_{n2} + \cdots + kn_k = n - k$.

Proof. It is based on the observation that a typical element of the event ($T_k = n$) arrangement is an

$$A = x_1 x_2, \cdots, x_{n_1+n_2+\cdots+n_k} SS, \cdots S(kS's),$$

such that n_1 of the $x's$ are $E_1 = F$, n_2 of the $x's$ are $E_2 = SF, \cdots, n_k$ of the $x's$ are $E_k = SS\cdots SF(k-1S's)$, and $n_1 + 2n_2 + \cdots + kn_k = n-k$. Fix $n_1, \cdots, n_k$. Then the number of the $A's$ is

$$\binom{n_1 + \cdots + n_k}{n_1, \cdots, n_k}$$

and each one has probability

$$\begin{aligned} P(A) &= [P(E_1)]^{n_1}[P(E_2)]^{n_2}\cdots[P(E_k)]^{n_k}P(SS\cdots S)(kS's) \\ &= q^{n_1}(pq)^{n_2}\cdots\left(p^{k-1}q\right)^{n_k}p^k \\ &= p^n\left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \end{aligned}$$

Therefore,

$$P(\text{ all A's:})n_i \geq 0 \text{ and fixed}, 0 \leq i \leq n) = \binom{n_1 + \cdots + n_k}{n_1, \cdots, n_k} P^n\left(\frac{q}{p}\right)^{n_1+\cdots+n_k}$$

But the non-negative integers $n_1, n_2, \cdots, n_k$ may vary subject to the condition $n_1 + 2n_2 + \cdots + kn_k = n-k$, and this completes the proof of the theorem. ☐

To Theorem (7.1), we have the following corollary.

Corollary 7.1 (Philippou and Muwafi (1982))**.** . Let T_k be as in Theorem (7.1) and assume that $p = 1/2$ Then

$$P(T_k = n) = \frac{F^{(k)}_{n-k+1}}{2^n} \qquad n \geq k$$

and

$$P(T_2 = n) = \frac{F_{n-1}}{2^n} \qquad n \geq 2$$

where F_n is the n^{th} Fibonacci number, and $F_n^{(k)}$ is the nth Fibonacci number of order k.

Proof. It follows from Theorem (7.1) for $p = 1/2$, since $F_n^{(2)} = F_n$, and

$$F^{(k)}_{n+1} = \sum \binom{n_1 + \cdots, n_k}{n_1, \cdots, n_k} \qquad n \geq 0$$

where the summation is taken over all k-tuples of non-negative integers $n_1, n_2, \cdots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n$ [Philippou and Muwafi (1982), Philippou (1983)]. The above formula for $F^{(k)}_{n+1}$ generalizes (5.4)

Is $f_k(n)$ a proper probability mass function?

The answer is yes, by means of the transformation

$$n_i = m_i (1 \leq i \leq k),$$
$$n = m + \sum_{i=1}^{k}(i-1)m_i$$

and the multinomial theorem. ☐

Theorem 7.2 (Philippou, Georghiou and Philippou (1983)). Let $f_k(n)$ be as in Theorem (7.1) Then

$$\sum_{n=k}^{\infty} f_k(n) = \sum_{n=k}^{\infty} P(T_k = n) = 1$$

The probability generating function (pgf) of T_k, say $g_k(s)$ and hence its mean μ_k and variance σ_k^2 are

$$g_k(s) = \sum_{n=k}^{\infty} s^n f_k(n) = \frac{(1-ps)p^k s^k}{1 - s + qp^k s^{k+1}}, \qquad |s| \leq 1$$

$$\mu_k = \frac{1-p^k}{qp^k}, \quad \sigma_k^2 = \frac{1-(2k+1)qp^k - p^{2k+1}}{q^2 p^{2k}}$$

They named the distribution of T_k the geometric distribution of order k with parameter p, since for $k = 1$ it reduces to the geometric distribution with pmf

$$f_1(n) = q^{n-1}p, \qquad n \geq 1$$

A different derivation of $g_k(s)$ was first given by Feller (1968). Alternative simpler formulas for calculating $f_k(n)$ have been found. The following recurrence, for example, is very efficient.

Theorem 7.3 (Philippou and Makri (1985)). let $f_k(n)$ be the probability mass function of T_k. Then

$$f_k(k) = p^k, f_k(n) = qp^k, \qquad k+1 \leq n \leq 2k$$
$$f_k(n) = f_k(n-1) - qp^k f_k(n-k-1), \qquad n \geq 2k+1$$

8 Probability, Fibonacci, and Gambling

There are two well publicized cases of individuals who beat the casinos and won themselves quite a lot of money. There are also many who won once or a few times. Millions, however, lose on the average every day. That is why casinos exist and thrive. The following three systems of gambling enjoy some popularity among gamblers.

(a) **Double your bet system.** In any game of chance you start by betting (say on red in American roulette) a certain amount of money, say A dollars. If you win, you stop. If you lose, you double your bet by wagering 2A dollars. If you win, you stop. If you lose, you bet 4A dollars, and so on. It appears that the system is unbeatable earning you A dollars when you win and stop, but it is not. Pretty soon you may end up not being able to double your bet for lack of money, or you may reach the upper limit posed by the casino. Either way you are a loser.

(b) **The Fibonacci system.** It resembles the double your bet one, but it is less aggressive. In any game of chance you start by betting (say on red in the American roulette) a certain amount of money, say A dollars. If you win, you stop. If you lose, you bet A dollars again. If you gain, you are at the beginning. If you lose, you bet 2A dollars. If you win, you are at the beginning. If you lose, you bet 3A dollars, and so on, moving one step to the right through the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$, when losing and two steps to the left when winning. As in the double your bet system, you may end up pretty soon without money. Moreover, one win is not sufficient to win you money, unless this happens at the first time.

(c) **The d'Alembert system.** According to this system, in any game of chance (say the roulette) you increase or decrease the amount of your bet in a round, depending on whether you lost or won the previous round. The "logic" is that if you lose one round, you are more likely to win on the next, and you should increase the amount of your bet. But this is completely erroneous. In a game of chance, say the roulette, the outcome of any spin is independent of the outcomes of previous spins. The odds at the roulette are always the same.

9 The Odds in Some Odd-Even Games [Schuster and Philippou (1975)]

In tossing an unbiased six-sided die until a six appears, is the best bet "odd" $= 1, 3, 5, \cdots$ or "even" $= 2, 4, 6, \cdots$? What if the die is biased ?

The answer is given in the following

Theorem 9.1 (Bernoulli Odd-Even Game)**.** In an odd-even game of counting independent Bernoulli trials with constant positive success probability $p(q = 1 - p)$ until the occurrence of success, the best bet is "odd", since

$$p\,(\text{“even”}) \;=\frac{q}{q+1}, \qquad p(\text{“even”}) \;- p(\text{“odd”}) = \frac{q-1}{q+1}$$

In counting the number of phone calls arriving at the switch board of Mohandas College of Engineering and Technology (or Kerala University or All Saints College) on December 16, 2013, is the best bet "odd" $= 1, 3, 5, 7, \cdots$ or "even" $= 0, 2, 4, 6, \cdots$?

The answer is given in

Theorem 9.2 (Poisson Odd-Even Game)**.** In an odd-even game of counting the number of occurrences of an event in positive time t following the Poisson distribution with mean rate λt the best bet is "even", since

$$P(\text{“even”}) = \left(\frac{1 + \mathrm{e}^{-2\lambda t}}{2}\right), \quad P(\text{“even”}) - P(\text{“odd”}) = \mathrm{e}^{-2\lambda t}$$

$$P\;(\text{“even”}) \;= \left(\frac{1 + \left(\frac{q-1}{q+1}\right)^r}{2}\right) \qquad \mathrm{P}(\text{“even”}) - P(\text{“odd”}) = \left(\frac{q-1}{q+1}\right)^r$$

Let one think that life is always so simple, consider the following generalisation of 1, which is a Sucker bet.

In tossing an unbiased six-sided die until a "six" appears r times, is the best bet "odd$''$ $= 1, 3, 5, \ldots$ or "even$''$ $= 2, 4, 6, \ldots$?. What if the die is biased ? The answer is based on,

Theorem 9.3 (Negative Binomial Odd-Even Game)**.** In an odd-even game of counting the number of successes in n independent Bernoulli trials with positive success probability p until the occurence of the r^{th} success, the best bet is "odd" or "even" providing r is odd or even respectively since,

$$P(\text{"even"}) = \left(\frac{1+(q-p)^n}{2}\right) \qquad P(\text{"even"}) - P(\text{"odd"}) = (q-p)^n$$

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Meaningful Mathematics

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(Keynote Speaker)

1 Introduction

The historical development of mathematics shows its dual nature as an abstract system of logic and as a practical science applied to concrete problems. Most of the abstract ideas in mathematics have their roots in physical problems and many purely aesthetic developments of mathematics have later found to have practical applications. Unfortunately, the mathematics curriculum from schools to universities in our country is heavily biased towards pure mathematics, with little emphasis on past needs and current applications. This makes mathematics a meaningless manipulation of symbols for the students of pure mathematics and a mechanical application of formulas for the students of other sciences and technology. In this talk, I try to trace the historical development of some of the ideas in mathematics, emphasizing the dialectics between theory and application and also indicate how this can be used for a more meaningful pedagogy of mathematics, taking geometry as an example.

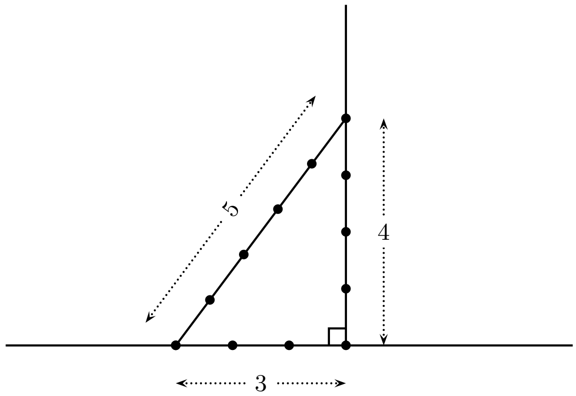
2 The nature of mathematics

Mathematics originally was a study of various measurements and the relations between such measurements, using numbers. This view of mathematics prevailed as late as the eighteenth century, as evidenced by the words of Euler:

> *Mathematics, in general, is the science of quantity; or, the science which investigates the means of measuring quantity*

But even from olden times we note the study of physical measurements evolving into the study of pure numbers *per se*.

For example, the need for checking perpendicularity, of poles for a tent or walls of a building must have been felt at least as early as the sixth millennium BC, when man started to construct his own dwellings, perhaps as a result of large scale agriculture. Even as early as the second millennium BC, it has been known in various parts of the world that perpendicularity can be quantified using the three numbers 3, 4, 5:



This must have led to the investigation of other number triples which can characterize perpendicularity, ultimately resulting in the Pythagoras Theorem, which is *proved* in Euclid's *Elements*, of the third century BC. But then apart from the utility of such numbers in determining perpendicularity, there must have been number-theoretic investigations of such such number triples, since we find a characterization of such triples in Diophantus' *Algebra*, in the third century CE. We then have Fermat making his famous conjecture in the seventeenth century that no cubes or higher powers of two integers add up to a like power of an integer. And many mathematicians all over the world devoting their lifetime to solve this problem for well over three centuries with Andrew Wiles succeeding in 1994.

Thus we see a physical problem translated into numbers and then moving on to purely number theoretic problem of little or no practical significance. Studies on this problem has given to highly abstract theories such as that of elliptic curves, but then we now hear of elliptic curve cryptography, a practical application in computers.

Other results in number theory also have found applications in computer related cryptography. For example, consider the following theorem of Euler, which generalizes a result of Fermat:

Theorem 2.1. If n is a natural number and a has no common factor with n, then $a^{\phi(n)}$ divided by n gives remainder 1

Here $\phi(n)$ denotes the number of natural numbers less than n and having no common factor with n, other than 1. This theorem is the basis of what is known as the RSA cryptosystem, invented by Rivest, Shamir and Adleman in 1977. To explain this, suppose a number m (message) is to be send from a computer to a computer B. To do this in such a way that no other computer listening in can get m, it has to be encrypted. This is done as follows:

(a) B computes the product $n = pq$ of two *large* primes p and q

(b) And two numbers e (encryption) and d (decryption)such that

    (a) e and d are smaller than n and have no common factors (other than 1) with n

    (b) The product ed divided by $\phi(n)$ gives remainder 1

(c) Sends n and e to A

(d) A computes the remainder c (cipher) on dividing m^e by n and sends to B

(e) To retrieve m, B computes the remainder on dividing c^d by n

This works because, by construction

$$de = k\phi(n) + 1$$

so that

$$(m^e)^d = (m^{\phi(n)})^k m$$

Now since c is the remainder on dividing m^e by n, it follows that c^d divided by n gives the same remainder as the remainder got on dividing $(m^e)^d$ by n; and by the above equation, this is the remainder on dividing $(m^{\phi(n)})^k m$ by n. By Euler Theorem, the remainder on dividing $(m^{\phi(n)})^k$ by n is 1 and so the remainder in question is just m.

The effectiveness of this idea depends on the following. For very large numbers n, it is not easy to compute $\phi(n)$, even for powerful computers. But then if $n = pq$, where p and q are primes, it is not difficult to show that $\phi(n) = (p-1)(q-1)$. So, in the computer B can easily compute $\phi(n)$. But a third computer listening in, gets only the numbers n, e and c. To get m, it has to find d and for this it has to compute $\phi(n)$. But then this requires the factors p and q of n, which is not easy to compute. For example, to generate a 140-digit number as a product of two primes requires only a few seconds, even for a personal computer; but to factorize it back may require hours or even days, even for a super computer.

Mathematicians of the early days like Fermat, Euler or Gauss were concerned with both mathematical problems arising from physical situations and abstract mathematical problems arising from number theory and pure geometry. However By nineteenth century, the division between pure and applied mathematics became rather sharp. Thus we have L. E. Dickson, a famous number theorist of the nineteenth century exclaiming:

> *Thank God, number theory is unsullied by any application!*

But then this is what Donald E. Knuth, a mathematician who is more famous as a computer scientist, says about number theory in the twentieth century:

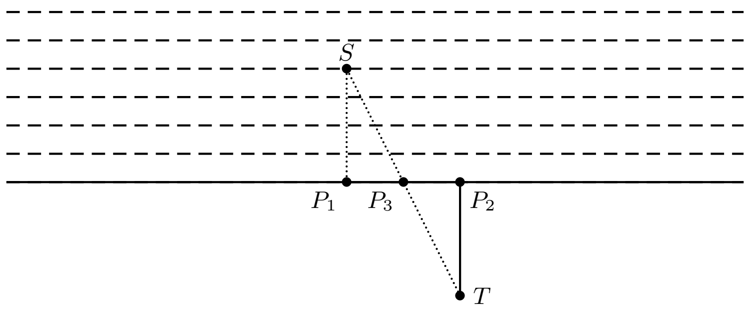
> *Virtually every theorem in elementary number theory arises in a natural, motivated way in connection with the problem of making computers do high-speed numerical calculations*

As mentioned above, the dialectics of the evolution of mathematics is such that what is originally seen as pure mathematics often leads later to unforeseen applications.

3 Teaching geometry

History often provides a good framework for teaching mathematics, starting with the original physical problem which mothered a concept, its evolution into a purely mathematical concept and then its later application in new physical contexts. This may provide the much needed motivation for studying the concept. Such a scheme also helps to maintain continuity of the mathematics curriculum across different stages of education and to establish connections both within different topics of mathematics and with other subjects of the curriculum. In this section I briefly sketch how this can be done in teaching geometry.

Let us start with the notion of congruency of triangles. We start with a tale of Thales in the sixth century BC in Greece. It is told that he was asked by the king to measure the distance of a ship moored at sea from the shore. This is how he is said to have done it. He first erected a pole at water's edge directly in line with the ship. He erected another pole on the shore some distance away from the first and finally a third pole exactly at the middle of the first two. Then he walked back from the second pole, perpendicular to the shore and keeping the ship in sight. He marked the position where this pole came into the line of sight with the ship.



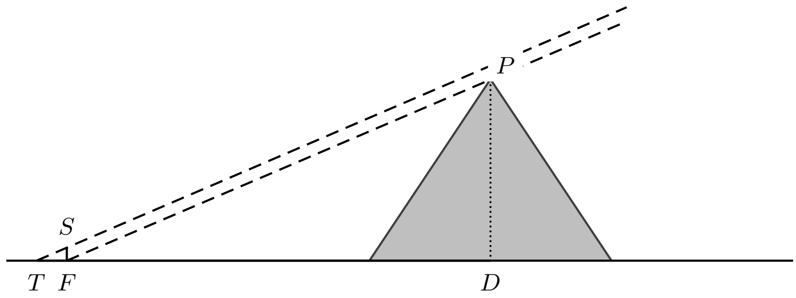
In the figure above, S is the ship, P_1, P_2, P_3 are the poles in order and T is the final position of Thales. Thus Thales ingeniously flips the triangle SP_1P_3 at sea onto the triangle TP_2P_3 on land, so that he can measure SP_1 as TP_2.

This tale can induce a discussion on the conditions under which the sides and angles of two triangles, thereby leading to the idea of congruency. After establishing the fact two triangles with equal lengths for sides also have their angles equal, we can discuss the rigidity of triangular frames; a quadrilateral frame for example, can be deformed without bending the sides, but not a triangle:

And this is the reason why bridges, towers and roofs are built with triangular frames,

The fact that triangles with sides of same length have angles of the same size leads to the theoretical question whether the converse is true. On seeing this is not so, the problem is to see what the relation between the sides is in this case. The idea that right triangles with the same angles have sides of the same ratio can be illustrated again through the tale of another exploit of Thales. It is said that Thales calculated the height of an Egyptian pyramid by measuring its shadow against the show of his staff. This is how Plutarch, a Greek historian of the first century CE recounts the tale:

> *without trouble or the assistance of any instrument merely set up a stick at the extremity of the shadow cast by the pyramid and, having thus made two triangles by the impact of the sun's rays, ... showed that the pyramid has to the stick the same ratio which the shadow has to the shadow*

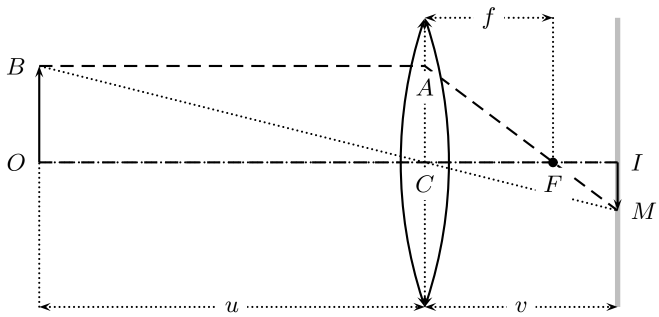


This means that in the figure above, $PD : SF = DT : TF$ and so

$$PD = \frac{DT}{TF} \times SF$$

A discussion on this leads to the idea that any triangles with the same three angles have sides of the same ratio and thus to the idea of similarity.

As a practical application of the idea of similarity, we can derive a result in optics describing the relation between focal length of a lens and the distances from it to the object and image:



In the figure above, ΔCIM and ΔCOB are similar so that $\frac{v}{u} = \frac{IM}{OB}$ and ΔFIM and ΔFCA are similar, so that $\frac{v-f}{f} = \frac{IM}{AC} = \frac{IM}{OB}$. Equating the expressions on the right side of these equations and simplifying, we get

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

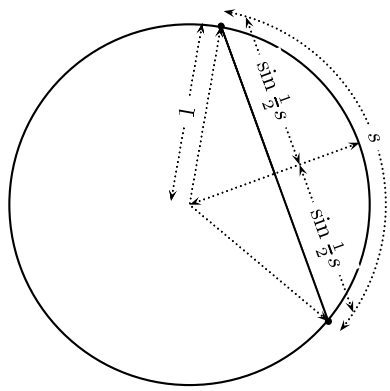
The facts that the sides of a triangle determine its angles and angles of a triangle determine the ratio of its sides leads to the theoretical question of actually determining these. This leads to trigonometry. After introducing the various trigonometric ratios, it can be shown that the angles are determined by the sides according to the rules

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

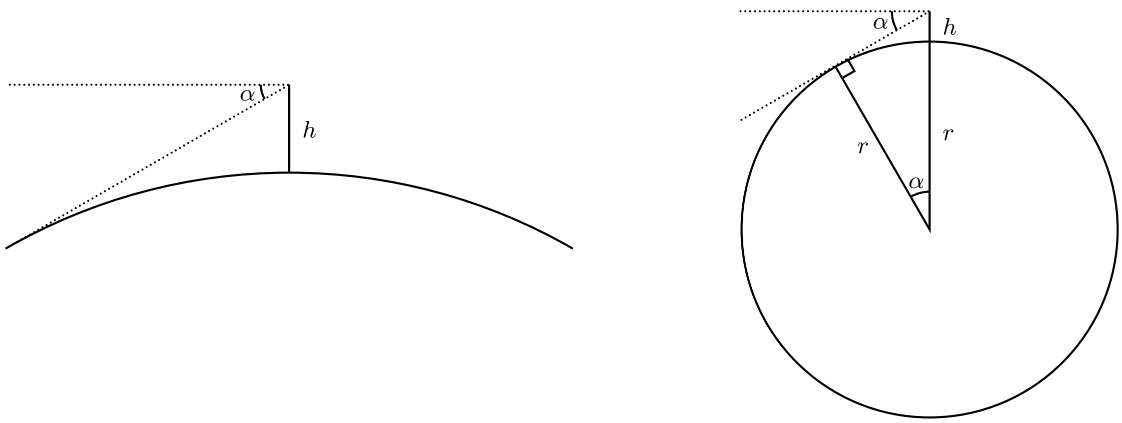
and that the angles determine the ratio of the sides as

$$a : b : c = \sin A : \sin B : \sin C$$

It is also interesting to note that the concept of the sine of an angle arose in astronomy, not for measurement of triangles, but for determining the chord of a circle in terms of the arc subtending it:



Using this idea, the Persian scholar Abu Rayhan al-Biruni computed the radius of the earth in the first millennium CE by first computing the height of a mountain using the angle of elevation and then measuring angle of depression from the top of the mountain to the horizon:

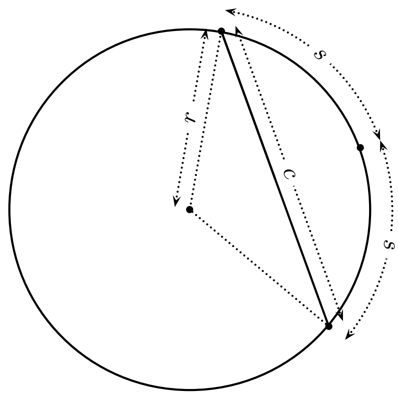


From the figure on the right, we find that $\cos \alpha = \frac{r}{r+h}$ and so

$$r = \frac{h \cos \alpha}{1 - \cos \alpha}$$

A modern application of trigonometry on computing the height of lunar mountains using satellite photographs can be found at `http://stupendous.rit.edu/classes/phys236/moon_mount/moon_mount.html`

A discussion on trigonometry must include the question of how these values are actually computed. Some of these values can be computed using results from Euclidean geometry, as Claudius Ptolemy of Egypt did in the second century CE. But a generic method was found by Madhavan of India in the fourteenth century:



$$\tfrac{1}{2}c = s - \left(\frac{s^3}{(2^2+2)r^2} - \left(\frac{s^5}{(2^2+2)(4^2+4)r^4} - \left(\frac{s^7}{(2^2+2)(4^2+4)(6^2+6)r^6} - \cdots\right)\right)\right)$$

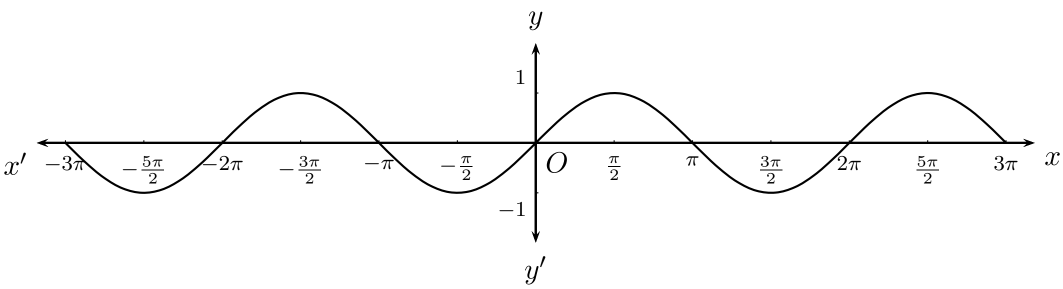
Translated into modern terminology, this gives

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots$$

which was rediscovered by Newton in the seventeenth century.

An interesting offshoot of Madhavan Series is that it converges for all real values of x, whereas, $\sin x$ defined using circles is meaningful only for $0 \leq x \leq \frac{1}{2}\pi$. Thus from its origin as a geometric measurement, sine becomes an algebraic (or analytic) function defined on real numbers.

The invention of analytic geometry in the seventeenth century gives a geometric form to the sign function again, as a wave:



This has made the sine function particularly useful in the study of periodic phenomena in physics, such as simple harmonic motion. Again, the studies on vibrating strings and transmission of heat in the nineteenth century led to the theory of approximating all periodic functions in terms of the sine and cosine series and to Fourier Series.

The invention of analytic geometry is a revolution in mathematics, providing a translation scheme between algebra and geometry. The problems of determining tangents to curves and computing areas bounded by curves contributed much to the development of calculus and later to mathematical analysis. It is rather unfortunate that most books on analysis for college courses ignore the geometric motivation of the ideas involved. For example, almost all of them take the so called *least upper bound axiom* for completeness of the real numbers, instead of the original Dedekind axiom:

if the set of real numbers is split into two non-empty sets such that every number in one set is less than every number in the other, then either the first set contains a least number or the second set contains a largest number

which can be easily understood as a rigorous set-theoretic formulation of the intuitive geometric idea of real numbers as points on a line (or in other words, lengths of line segments).

It maybe argued that geometrical arguments are not precise enough by the current standards of mathematical rigor, but there is no denying the fact that it is a great pedagogical tool, providing meaning and motivation for great many concepts; and dynamic geometry softwares (such as the free software GeoGebra) has raised this to a whole new level.

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Mathematics behind image processing

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Abstract : We review few of the well known techniques in image processing using mathematical modeling. We mainly concentrate and illustrate some of the results available in literature for image processing using partial differential equations. Finally we present few recent results for texture in painting in images.

1 Introduction

Images can be seen as the best way of communication without boundaries of languages. For example, to say that one can not take a "U" turn at a particular point on the road, instead of explicitly writing it in some language, one image would serve the purpose. This explains the necessity and importance of images. The technological advancements have improved imaging techniques and quality and nature of images. Today digital images have encompassed each and every corner of our life. Family pictures taken using ordinary cameras, medical images taken using various techniques like magnetic resonance imaging (MRI), images taken by satellites using infra red cameras are few types of them. Images of family and friends are taken for pleasure. Images in medical field help in detecting, localizing and analyzing the diseases and can be helpful to decide the course of medical help to be provided to the patient. Satellite images help in prediction of weather, to quantify changes in vegetation, sea level, forest cover and to design remedial measures.

Many a times images captured by cameras can not be used as they are and some transformations need to be done on them to extract useful information from them. Images captured may contain some undesired information called noise or missing data due to loss of information during transmission or because of limitations in the imaging techniques. Some times end user is interested only in particular part of the image or is interested in analyzing and comparing certain parts of similar images. All these problems and requirements need to be tackled using computational assistance. Digital image processing is the use of computer algorithms to perform processing on digital images to extract useful information.

But what does a digital image mean and how it can be represented mathematically? A digital image is a numeric representation of a real image. Idea is to put a regular grid on an image and assign a digital value to each square on a grid say average brightness in that square. Each square on a grid is called a "pixel" which stands for picture element. In general representation of a black and white or gray scale image as it should be called appropriately, each pixel value varies between 0 and 255; 0 for black and 255 for white. Another important characteristic of an image is its size or resolution which is nothing but number of rows and columns in the grid used to digitize the image. Thus from a

Figure 1: Image Rotation by 45 degrees around center of the image

mathematical view point any matrix which has integer values lying between 0 and 255 will represent a gray scale image when displayed on a computer. For colour images, each pixel is assigned with three intensity values each one representing red, green and blue level in the image. These values together would represent a colour or RGB image. Thus gray scale image is two dimensional array whereas to represent colour image we require three such two dimensional arrays.

Suppose one needs to increase brightness of a gray scale image. Simplest operation, keeping in mind that image is a 2D array would be to add a fixed value say b to each of its values. If b is strictly positive, intensity value at each pixel would increase and we will get brightened image. Similarly if one needs to rotate the image we can use matrix rotation operation to achieve this. For example suppose we need to rotate the image by a an angle α counterclockwise around origin. This means a point (x, y) in the original image is mapped onto a point (X, Y) in the resultant image. Where relation between these points is given by:

$$X = x \cos \alpha - y \sin \alpha \tag{1.1}$$

$$Y = x \sin \alpha + y \cos \alpha \tag{1.2}$$

Figure 1 illustrates the numerical implementation of rotation around center of the image by an angle of 45 degrees.

Thus we have seen that using the array representation of an image we can do simple processing to get a new image. But many a times more complicated and involved operations must be done to extract useful information from images. Various image processing problems are broadly classified and explained briefly below.

- **Image enhancement**: In image enhancement, the goal is to accentuate image features for subsequent analysis or for image display. Image defects which could be caused by the digitization process or by faults in the imaging set-up (for example, bad lighting) can be corrected using image enhancement techniques. For example problems like contrast and edge enhancement, pseudo coloring, noise filtering, sharpening and magnifying come under image enhancement.. It is useful in feature extraction, image analysis, and visual information display.

- **Image segmentation**s: Segmentation procedures partition an image into its constituent parts or objects. This partitioning depends on the user requirements and problem can become extremely difficult depending upon image content. A segmentation could be used for object recognition, image compression, image editing, removal or revival of various portion in an image/video.

- **Image restoration**: It comprises removal or minimization of known degradation in an image. This includes deblurring of images degraded by the limitations of a sensor or its environment, noise filtering, and correction of geometric distortion or nonlinearities due to sensors.

- **Image reconstruction**: Image reconstruction from projections is a special class of image restoration problems where a two (or higher) dimensional object is reconstructed from several one dimensional projections.

- **Image in painting**: Image in painting deals with filling in missing information in an image. In painting must be done in a plausible way which is undetectable for human eye.

2 Modeling image by PDE

As we have noted digital image is discrete information of intensity values arranged in an array. But actual image which is perceived by us is continuous variation of intensity values. In an image, boundary of an object would describe sudden changes in intensity values. Gradient of intensity represents variation in intensity values and large gradient values would define the boundaries of objects in the image. Thus real image can be modeled as piecewise continuous function from connected subsets of 2 or 3 dimensional space. This motivates to look at the image as a solution of partial differential equations (pde). We would look at the solution of PDE in weak sense so as to recover the discontinuities.

Let us look at a restoration or noise removal problem. Let us assume that image is represented by u_0. We would like to denoise and obtain the denoised image which is denoted by u. Thus we can write $u_0 = Ru + \eta$ where η represents noise component and R is a linear transformation. We would be able to recover u if we minimize the functional

$$F(u) = \int_\Omega |Ru - u_0|^2 \tag{2.1}$$

By using calculus of variation, this minimization problem can be converted to PDE. Solving this PDE numerically would give us our required u. But in general such minimization problems are ill posed and one can not find a solution which depends continuously on initial data. We therefore regularize the functional which we wish to minimize. Thus instead of solving problem (2.1), we solve another minimization problem given by

$$\inf_u \{\int_\Omega |Ru - u_0|^2 + \lambda \int_\Omega \phi(|\nabla u|)\} \tag{2.2}$$

Where ϕ is a smooth function and $\lambda > 0$ is a weight factor . First term on RHS in 2.2 is a fidelity term and second term is a smoothing term. Above minimization problem gives rise to a well posed pde. Thus we look for best solution u which fits given data, whose gradient is low and noise is removed. The example of restoration using above method is presented in figure 2.

Figure 2: Denoising the image

Figure 3: Denoising the image using diffusion equation

Instead of following above methodology, we may also like to look at the image as a solution of a PDE. Let us take the simplest diffusion equation or heat equation as model for image. Taking given image as a initial value we can then solve

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0 \\ u(0,x) = u_0(x) \end{aligned} \tag{2.3}$$

Here u_0 is given image and we seek u which is a solution of above equation as denoised image. As we know solution of heat equation is given by convolution with Gaussian kernel, the solution u would be smoothened out version of u_0. This does remove noise from original image, but boundaries of different objects are also smoothened out. Numerical implementation of above equation on the noisy image in Figure 2 is illustrated in Figure 3.

We observe from above figure 3 that edges or boundaries of objects are not preserved in this denoising thus more finer modeling needs to be done. One such idea is to modify diffusion equation by weight function. Thus we may solve

$$\begin{aligned} \frac{\partial u}{\partial t} - div(c|\nabla u|) = 0 \\ u(0,x) = u_0(x) \end{aligned} \tag{2.4}$$

where c can be smooth or piecewise smooth function or a constant. This is a celebrated idea of using anisotropic diffusion equation for denoising introduced by Perron and Malik. [3]

Figure 4: In painting of an image

Above example also emphasizes need to find boundaries of object in the image. As in the denoising, we would like o treat them differently. As mentioned earlier boundaries also play important role in segmentation problem as well as in painting problem. Finding gradient of an image is one of the way to find boundaries. A vast literature is available on sementation problems which use various PDE methods. One among them is solving Neumann and Dirichlet boundary conditions problems simultaneously to localize the discontinuities [1]

3 Image In painting

In painting is a process of reconstructing lost or deteriorated parts of the image based on the background information. The term in painting is derived from the ancient art of restoring paintings by professional restorers in museums. Digital image in painting tries to imitate this process and perform the in painting of an image in an undetectable way. Image in painting finds its applications in removal of scratches in an image, repairing damaged areas in images, recovering lost blocks during wireless image transmission, recovering lost information while image zooming and super-resolution , removing undesired objects from an image. Image in painting is applied to remove red-eye, the stamped date from the image or remove unwanted text from the image etc. An example of in painting by removing text on an image is illustrated in figure 4.

3.1 In painting images with Texture

In painting can be done using diffusion equation provided the area to be in painted is small and background information can be smoothly diffused to fill in the missing region. This may not work for images which have large texture variation. Texture is a measure of image coarseness, smoothness and regularity. Images with texture contain regions characterized more by variation in the intensity values than one value of intensity for the given image. Figure 5 shows two examples of images with different kinds of textures.

In painting an image which has texture variation is a challenging problem. Texture in painting is tackled by methods which combine PDE and statistical methods. The idea is to decompose image information in it structure part and texture part. The missing information from stucture image is in painted using usual method whereas image in painting

Figure 5: Images with texture

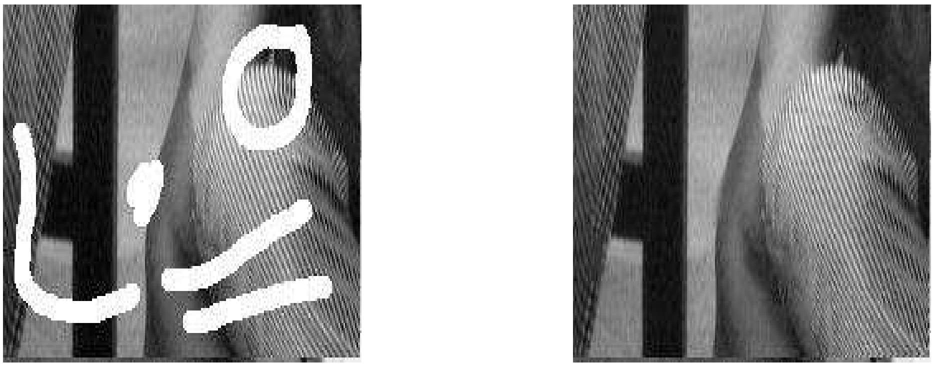

Figure 6: Simultaneous structure and texture In painting

in texture part is done using texture synthesis. These two different in painted images are then superimposed to give a final in painted image. Such an approach is used in [2] and is illustrated in figure 6.

3.2 In painting using Texture Differentiation

We have another approach to tackle this problem. Instead of segmenting image in structure and texture we only use texture differentiation of an image. First we segment image in various textures. then we locate correct texture patch using background information for missing region. This reduces computational time considerably as we are not looking for best match in whole of the image. Moreover, we are assured of getting correct texture as we are looking only in texture segment which matches with background information. Results of this algorithm on Barbara image are showed in figure 7.

4 Conclusion

We have reviewed few of the important image processing problems and have seen how they can be formulated using PDEs. We have shown how this can be efficiently used for image in painting to recover texture in the image. Further work to enhance the results is underway. We would like to conclude that partial differential equations help is resolving many issues in image processing. Problems in various fields like medical imaging, satellite imaging,

Figure 7: Texture In painting using texture differentiation

astronomical imaging can be resolved by applying pde methods. Each one of them has different goals and intricacies hence modeling that is underlying pde/ methodology to solve pde needs to be modified accordingly.

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Phase Space analysis

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A two dimensional linear system is of the form

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}$$

where a, b, c, d are parameters. This can be expressed as

$$\dot{X} = AX \tag{1}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

The movement of $X = (x, y)$ in the xy plane (called phase plane) satisfying (1) is called trajectory.

How the qualitative nature can be analysed, even in the case of systems, where solutions are available. The method which will be explained in this lecture can be applied even for system not having analytic solution.

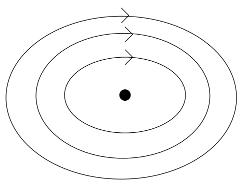
Let us consider, $m\ddot{x} + kx = 0$, the governing equations of simple harmonic oscillator. This can be solved in terms of sines and cosines. Generally, it is impossible to solve non-linear equation in the form of an analytical solution. It is clear that the state of any system can be characterised by x (current position) and $\dot{x}$ (velocity). Rename $\dot{x} = y$ and the above equation can be written to a system of equations such that

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{-k}{m}x = -\omega^2 x \quad \text{with } \omega^2 = \frac{k}{m}. \end{aligned}$$

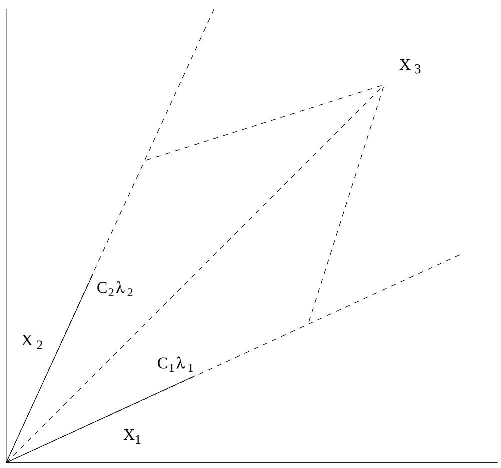
This will assign vector $(\dot{x}, \dot{y}) = (y, -\omega^2 x)$ at each point (x, y), which is called a vector field on the phase plane. To find trajectory at (x_0, y_0), assume an imaginary particle at the point and follow as per the equation. The point (x_0, y_0) is called phase point. It is clear that the phase point at (0,0) is motionless, $\sin(\dot{x}, \dot{y}) = (y, -\omega^2 x) = (0, 0)$ which means that origin is a fixed point. But phase point starting anywhere else circulates around the origin and eventually return to its starting point. Such trajectories are called closed orbits. The set of all trajectories in phase space is called phase portrait. We can also directly found the orbits as follows:

$$\begin{aligned} \frac{dx}{dt} &= y \quad \text{and} \quad \frac{dy}{dt} = -\omega^2 x \\ \therefore \quad \frac{dy}{dx} &= \frac{-\omega^2 x}{y} \\ \implies \quad \omega^2 x dx + y dy &= 0 \\ \therefore \quad \frac{\omega x^2}{2} + \frac{y^2}{2} &= \text{constant} \end{aligned}$$

This equation represents an ellipse.



and hence the fixed point (0,0) corresponds to state equilibrium of the system. The closed orbits are periodic motions.



We claim that

$$X(t) = C_1 e^{\lambda_1 t} X_1 + C_2 e^{\lambda_2 t} X_2$$

is the general solution of the system.

Why it is a general solution

(1) We know that $e^{\lambda_1 t} X_1$ and $e^{\lambda_2 t} X_2$ are solutions of $\dot{X} = AX$ and hence any linear combination say

$$X(t) = C_1 e^{\lambda_1 t} X_1 + C_2 e^{\lambda_2 t} X_2$$

is a solution of $\dot{X} = AX$.

(2) The solution satisfies $X(0) = X$, the initial condition.

(3) It is the only one solution of the system by existence and uniqueness theorem states below:

Existence and Uniqueness Theorem

An initial value problem $\dot{X} = f(x)$, with $X(0) = X_0$ has a unique solution $X(t)$ on some interval of t, say, $(-a, a)$ about $t = 0$ if (1) f is continuous and (2) all partial derivatives $\dfrac{\partial f_i}{\partial x_i}$ are continuous on some open connected set $D \subset \mathbb{R}^n$ [where $X_0 \in D$]. In otherwords, existence and uniqueness are guaranteed for the problem $\dot{X} = f(x)$, if f is continuously differentiable.

NOTE:

The above theorem says that two different trajectories never intersect. If it intersects there would be two solutions starting from the same point and hence violate the uniqueness of the theorem.

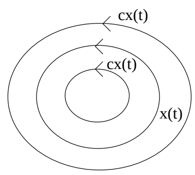
NOTE:

In two-dimensional phase space, suppose there is a closed orbit C in the phase plane, then any trajectory starting inside C is trapped inside C forever.

Limit cycles

In general, linear systems and non-linear systems exhibits closed orbits or trajectory. In linear systems, if $X(t)$ is a solution of the system $\dot{X} = AX$, So is $CX(t)$ for any $C \neq 0$. Hence they are never isolated. Also a slight disturbances to the system will persist for ever. At the same time non-linear systems can have isolated closed trajectory called Limit cycles. Hence neighbouring trajectories will spiral toward or away from the limit cycle depending on the structure of the system.

If all neighbouring trajectories approach the limit cycle, we say that the limit cycle is stable or attracting. Otherwise it is called unstable.

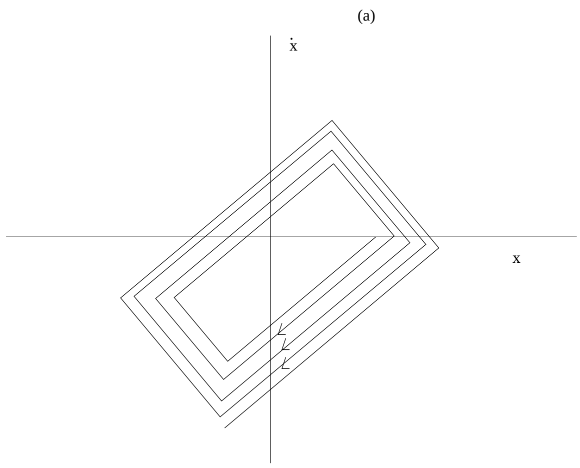


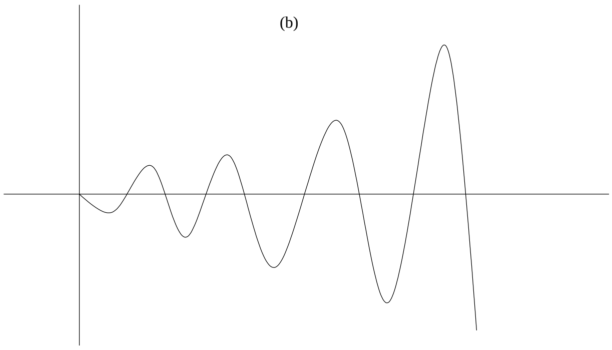
As an example, consider the well-known Van der Pol equation;

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

Historically, the above equation is very important in the nonlinear electrical circuits used in first radios.

If we numerically integrate the equation for $\mu = 1.5$ starting at $(x, \dot{x} = (0.5, 0)$ at $t = 0$ we will graph as follows.





We can prove that the equation has a unique limit cycle for $\mu > 0$. (Later).

Ruling Out Closed Orbits

A system $\dot{X} = F(X)$ is called a gradient system, if it can be expressed in the form of $\dot{X} = -\nabla V(X)$, where $V(X)$ is a continuously differentiable single valued scalar function called potential function.

Theorem 1. Closed orbits are impossible in gradient systems, (which does not mean that closed orbits are always possible in non-gradient systems).

Proof. Consider a system $\dot{x} = f(x,y)$ and $\dot{y} = g(x,y)$, which is a gradient system. $\therefore$ $\exists\ V(x,y)$ such that $\dot{X} = -\nabla V$

$$\begin{aligned}
\implies (\dot{x},\dot{y}) &= -\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right) \implies \left[\frac{\partial v}{\partial x} = -\dot{x} \quad \& \quad \frac{\partial v}{\partial y} = -\dot{y}\right] \\
\implies \frac{\partial \dot{x}}{\partial y} &= \frac{\partial \dot{y}}{\partial x}
\end{aligned}$$

which is the necessary condition for gradient system.

Assume that closed orbit $(x(t), y(t))$ exists for the system. Hence on one completion of a cycle with period T, we have

$$\begin{aligned}
X(0) &= X(T) \quad \text{ie } (x(0),y(0)) = (x(T),y(T)) \\
\therefore\ V(X(0)) &= V(X(T)) = \text{constant} \qquad \text{(say)} \\
\therefore\quad \text{change in } V &= 0 \\
\therefore\quad \nabla V &= 0 \qquad (2)
\end{aligned}$$

Note that, rate of change in V is $\frac{dv}{dt}$.
Hence, change over a period of one cycle is given by,

$$\begin{aligned}
\nabla V &= \int_0^T \frac{dv}{dt}.dt \\
&= \int_0^T \left(\frac{\partial v}{\partial x}.\frac{dx}{dt} + \frac{\partial v}{\partial y}.\frac{dy}{dt}\right) dt \\
&= \int_0^T (-\dot{x}\dot{x} + -\dot{y}\dot{y})dt = -\int_0^T (\dot{x}^2 + \dot{y}^2)dt < 0 \quad (\dot{x},\dot{y}) \neq 0
\end{aligned}$$

which is a contradiction to eqn (2) ($(\dot{x},\dot{y}) = 0 \implies (x,y)$ is a fixed point). □

Definition: Let X^* be a fixed point of the system

$$\dot{X} = F(X)$$

then a continuously differentiable real-value function $V(X)$ is called a Lyapunov function, if

(1) $V(X) > 0 \quad \forall\, X \neq X^*$

(2) $V(X) = 0 \quad \forall\, X = X^*$

(3) $\dot{V}(X) < 0 \quad \forall\, X \neq X^*$

Theorem 2. If there is a Lyapunov function corresponding to the system $\dot{X} = F(X)$, then the fixed points X^* is asymptotically stable. Also no closed orbits there exists.

Example:

Consider $\dot{x} = -x + 4y$ and $\dot{y} = -x - y^3$.

$\implies$ $(0,0)$ is a fixed point.

Fixed points are given by $\dot{x} = 0$ and $\dot{y} = 0$

$$\begin{aligned} \text{Let } V &= x^2 + 4y^2 > 0 \quad \forall\, (x,y) \neq (0,0) \\ V &= 0 \quad (x,y) = 0 \\ \text{Also } \dot{V}(X) &= \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} \\ &= 2x(-x+4y) + 8y(-x-y^3) \\ &= -(2x^2 + 8y^4) < 0 \quad \forall\, (x,y) \neq (0,0) \end{aligned}$$

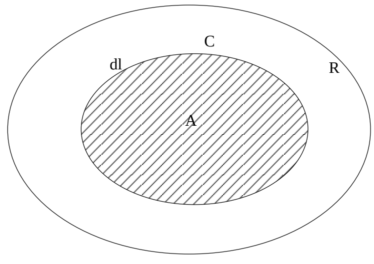
$\therefore$ $\exists$ no closed orbits.

Note: $\exists$ no systematic way to construct Lyapunov functions. Devine inspiration is required for such a construction.

The third method for ruling closed orbits is based on Green's theorem and known as Dulae's Criterion.

Theorem 3. Let $\dot{X} = f(X)$ be a continuously differentiable vector field defined on a simply connected subset R of a plane. If there exist a continuously differentiable real-valued function $g(X)$ such that $\nabla.\left(g(X)\dot{X}\right)$ has one sign through out R, then there exists no closed orbits lying inside R.

Proof. Let C is a closed orbit lying entirely in the region, R. Let A denote the region inside C, then by Green's theorem, we have



$$\int_A\int \nabla.\left(g(X)\dot{X}\right)dA = \oint_C (g(X)\dot{X}).\vec{n}\,dl$$
$$= \oint_C g(X)(\dot{X}.\vec{n})dl$$

where $\vec{n}$ is the outward normal and dl is the element of arc length along C. Since C is a trajectory on R^2, the vector $\dot{X}$ is the tangent vector and $\vec{n}$ is orthogonal to it and hence

$$\dot{X}.\vec{n} = 0 \quad \text{everywhere.}$$
$$\therefore \quad \int_A\int \nabla.(g(X)\dot{X})dA = 0$$

which is a contradiction, since $\nabla.g(X)\dot{X}$ has one sign throughout R.

$\therefore$ $\exists$ no closed orbits lying inside R. □

Example: $\dot{x} = x(2-x-y)$ and $\dot{y} = y(4x-x^2-3)$ has no closed orbits for $x > 0,\ y > 0$.

Take $g(X) = \dfrac{1}{xy}$ is continuous for $x > 0,\ y > 0$.

$$\therefore \nabla.g(X)\dot{X} = \frac{\partial}{\partial x}(g(X)\dot{x}) + \frac{\partial}{\partial y}(g(X)\dot{y})$$
$$= \frac{\partial}{\partial x}\left(\frac{2}{y} - \frac{x}{y} - 1\right) + \frac{\partial}{\partial y}\left(4 - x - \frac{3}{x}\right)$$
$$= \frac{-1}{y} < 0 \quad \text{for} \quad y > 0$$

$\therefore$ By Dulae's criterion, $\exists$ no closed orbits in the first quadrant.

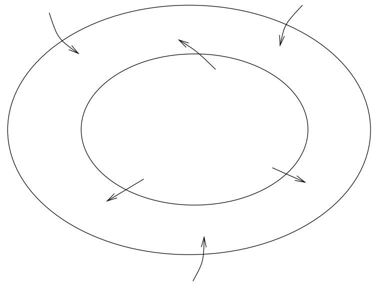
NOTE:

This has also the same drawback of Lyapunov method.

Theorem 4 (Poincare - Bendixson Theorem)**.** Let $\dot{X} = f(X)$ is a continuously differentiable vector field on an open set containing a closed bounded subset R of the plane, where R doesn't contain any fixed points, and $\exists$ a trajectory C, which is confined in R, (means starts in R and stays in R for all future time). Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. However, R contains a closed orbit.

NOTE:

It is one of the very important result in non-linear dynamics, because it ruled the behaviour of irregular nature in two-dimensional systems. Applying the above theorem is not easy generally. The main problem is How can we be sure that a confined trajectory C exists in R. The standard trick is to construct a trapping region R. That is, a closed connected set such that the vector field points inward everywhere on the boundary of R. Then all trajectories in R are confined. If $\exists$ no fixed points, then the theorem ensures that R contains a closed orbit.



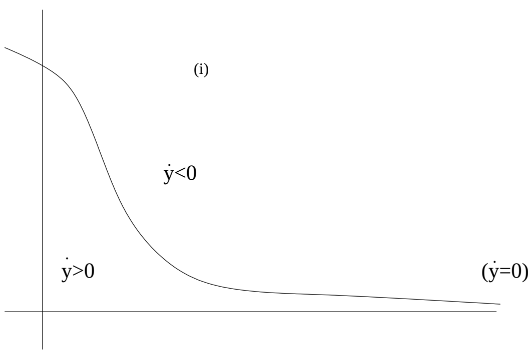
The idea of nullclines

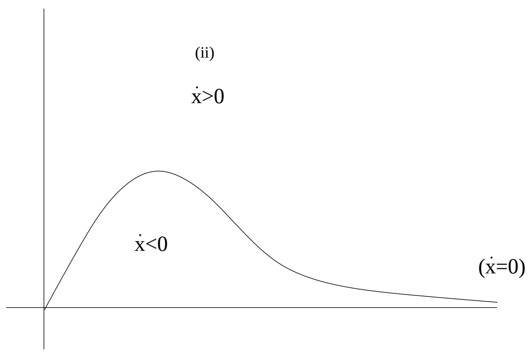
Draw vectors at every point (x_0, y_0) along the direction of $f(x_0, y_0)i + g(x_0, y_0)j$ where $\dot{x} = f(x, y),\ \dot{y} = g(x, y)$ is a dynamical system. The x- nullcline is a set of points in the phase plane with $\dot{x} = 0$. This can be found out by solving $f(x, y) = 0$. Similarly the y-nullcline can be found out by solving $g(x, y) = 0$. For example, consider the system

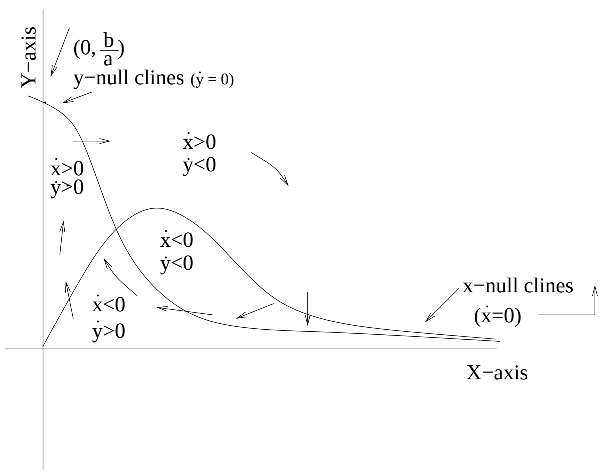
$$\begin{aligned} \dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y, \quad \text{where } a, b > 0 \end{aligned}$$

The nullclines are given by $\dot{x} = \dot{y} = 0$

$\therefore \quad y = \dfrac{x}{a + x^2}$ is the x- nullclines and $y = \dfrac{b}{a + x^2}$ is the y - nullclines. These nullclines can be sketched as below:







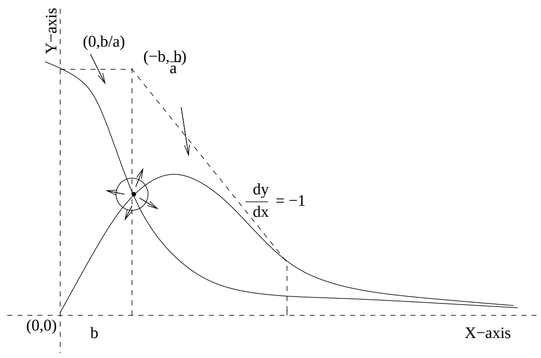
Note that direction of flow will be vertical on $\dot{x} = 0$ and horizontal on $\dot{y} = 0$. The direction can be determined by the signs of $\dot{x}$ and $\dot{y}$. If x is sufficiently large the x^2 will dominate

on $\dot{x}$ and $\dot{y}$ and hence $\dot{x} = x^2y \quad \text{and} \quad \dot{y} = -x^2y$
Hence, $\dfrac{dy}{dx} = \dfrac{\dot{x}}{\dot{y}} = -1$ along trajectories.

$$\text{Now,} \quad \dot{x} + \dot{y} = b - x < 0 \quad \text{if } b < x \Longrightarrow \dot{x} - (-\dot{y}) < 0$$

for $b < x \Longrightarrow \quad -\dot{y} > \dot{x} \quad$ for $b < x$ and

$$\text{Hence } -\dot{y} > \dot{x} \quad \text{for, } x > b \text{ and } -\dot{y} < \dot{x} \quad \text{for } x < b$$
$$\text{Hence } \frac{dy}{dx} = \frac{\dot{x}}{y} > -1 \quad \text{for } x < b \text{ and } \frac{dy}{dx} = \frac{\dot{x}}{\dot{y}} < -1 \quad \text{for } x > b$$



Hence $\dfrac{dy}{dx}$ is more negative than -1 and hence vectors are more steeper than the line from $\left(b, \dfrac{b}{a}\right)$ with slope $\dfrac{dy}{dx} = -1$. This implies that the vector fields point inward on the diagonal line. Thus the region become a trapping region. Even now we cannot conclude that there exist a closed orbit, since there exist a fixed point inside the region. If it is a repeller, we can conclude that $\exists$ a closed orbit by considering a punctured origin. The repeller drives all neighbouring trajectories into the region. Now we have to prove that the fixed point is a repeller. The fixed points are given by

$$-x + ay + x^2y = 0 \quad \text{and} \quad b - ay - x^2y = 0 \quad \Longrightarrow \quad b - x = 0 \quad \Longrightarrow x = b$$

$$\begin{aligned}
\therefore \quad -b + ay + b^2y &= 0 \\
y &= \frac{b}{a+b^2} \\
\therefore \quad X^* &= \left(b, \frac{b}{a+b^2}\right)
\end{aligned}$$

Now

$$\begin{aligned}
J &= \begin{bmatrix} -1 + 2xy & a + x^2 \\ \\ -2xy & -(a+x^2) \end{bmatrix} \\
&= \begin{bmatrix} -1 + \dfrac{2b^2}{a+b^2} & a + b^2 \\ \\ -\dfrac{2b^2}{a+b^2} & -(a+b^2) \end{bmatrix}
\end{aligned}$$

RESULT: Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix, its eigenvalues are given by $\begin{vmatrix} \lambda - a & b \\ c & \lambda - d \end{vmatrix} = 0$

$$
\begin{aligned}
\text{That is} \quad (\lambda - a)(\lambda - d) - bc &= 0 \quad \text{or} \quad \lambda^2 - (a+d)\lambda + ad - bc = 0 \\
\therefore \quad \lambda &= \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2} \\
&= \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2 - 4(ad-bc)}{4}} \\
&= \frac{\text{Trace}}{2} \pm \sqrt{\frac{Trace^2 - 4 \;\; Determinant}{4}} \\
&= \frac{\tau}{2} \pm \sqrt{\frac{\tau^2 - 4\Delta}{4}}
\end{aligned}
$$

$\therefore$ λ has +ve real root if Trace > 0 and -ve real root if Trace < 0

$\Longrightarrow$ Unstable (Trace > 0) and Stable (Trace < 0) provided that Determinant > 0.

Hence for the above matrix,

$$
\begin{aligned}
\text{Trace, } \tau &= -1 + \frac{2b^2}{a + b^2} - (a+b)^2 \\
&= \frac{-a - b^2 + 2b^2 - (a+b)^2}{a + b^2} \\
&= \frac{-a - b^2 + 2b^2 - a^2 - 2ab^2 - b^4}{a + b^2} \\
&= \frac{-b^4 + 2ab^2 - b^2 + a^2 + a}{a + b^2} \quad \text{and} \\
\text{Determinant, } \Delta &= -\left(-1 + \frac{2b^2}{a + b^2}\right)(a + b^2) + \frac{2b^2}{a + b^2}(a + b^2) \\
&= (a + b^2) - 2b^2 + 2b^2 = a + b^2 > 0
\end{aligned}
$$

$$
\text{Hence, } \tau = \frac{-b^4 + 2ab^2 - b^2 + a^2 + a}{a + b^2} \begin{array}{ccc} < & & \text{Stable} \\ = & 0 & \text{Centre} \\ > & & \text{Unstable} \end{array}
$$

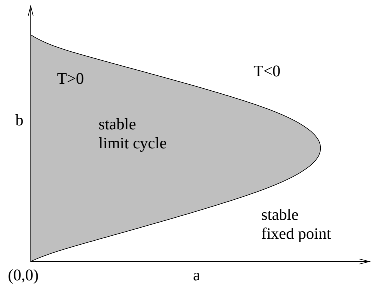
$$
\Longrightarrow \quad b^4 + 2ab^2 - b^2 + a^2 + a \begin{array}{ccc} > & & \text{Stable} \\ = & 0 & \text{Centre} \\ < & & \text{Unstable} \end{array}
$$

$$
\begin{aligned}
\text{Now,} \quad b^4 + 2ab^2 - b^2 + a^2 + a &= 0 \text{ or} \\
p^2 + (2a - 1)p + a^2 + a &= 0 \text{ (where } p = b^2) \\
\Longrightarrow \quad p &= \frac{1 - 2a \pm \sqrt{(2a-1)^2 - 4(a^2 + a)}}{2} \\
&= \frac{1}{2}\left(1 - 2a \pm \sqrt{1 - 8a}\right) \\
b^2 &= \frac{1}{2}\left(1 - 2a \pm \sqrt{1 - 8a}\right) \\
\text{Hence, } \tau &= -\frac{b^4 + (2a-1)b^2 + (a + a^2)}{a + b^2}
\end{aligned}
$$

Hence the fixed point is unstable for $\tau > 0$, and stable for $\tau < 0$. The dividing line $\tau = 0$ occurs when

$$b^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1 - 8a}\right)$$

This defines a curve in (a, b) space, as shown in Figure.



The parameters on the origin corresponding to $\tau > 0$ guaranteed the existence of a closed orbit and $\tau < 0$ guaranteed non-existence of a closed orbit, since stable fixed points exist in the region, where $\tau < 0$.

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Conditional expectation and its role in theory of estimation

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Abstract : In this talk, I will discuss some important properties of conditional expectation and will explain how it occurs as mean square estimate.

Conditional Distribution

Let X and Y be two continuous random variables with joint probability density function (pdf) f. Then the conditional distribution function (df) of X, given $Y = y$ is defined as $\lim_{\epsilon \to 0^+} P\{X \le x | y - \epsilon < Y \le y + \epsilon\}$ and is denoted by $F(x|y)$.

Define the conditional density function of X, given $Y = y$, denoted by $f(x|y)$ as a non-negative function satisfying $F(x|y) = \int_{-\infty}^{x} f(x|y)\ dt$ for all $x \in \mathbb{R}$.

At every point (x, y) where f is continuous and the marginal pdf $f_Y(y) > 0$ and is continuous, we have

$$\begin{aligned} F(x|y) &= \lim_{\epsilon \to 0^+} \frac{P\{X \le x,\ Y \in (y-\epsilon, y+\epsilon]\}}{P\{Y \in (y-\epsilon, y+\epsilon]\}} \\ &= \lim_{\epsilon \to 0^+} \frac{\displaystyle\int_{-\infty}^{x} \int_{y-\epsilon}^{y+\epsilon} f(u,v)\ dv\ du}{\displaystyle\int_{y-\epsilon}^{y+\epsilon} f_Y(v)\ dv} \end{aligned}$$

Dividing the numerator and the denominator by 2ϵ and passing to the limit as $\epsilon \to 0^+$, we have

$$F(x|y) = \frac{\displaystyle\int_{-\infty}^{x} f(u,y)\ du}{f_Y(y)} = \int_{-\infty}^{x} \frac{f(u,y)}{f_Y(y)}\ du$$

It follows that there exists a conditional pdf of X, given $Y = y$, that is expressed by

$$f(x|y) = \frac{f(x,y)}{f_Y(y)},\ f_Y(y) > 0.$$

Conditional expectation

If h is a Borel-measurable function and X and Y are continuous random variables, then the conditional expectation of $h(X)$, given Y, written as $E(h(X)/y)$ is defined by

$$E\left(h(X)|y\right) = \int_{-\infty}^{\infty} h(x)\ f(x|y)\ dx \quad \text{if} \quad f_Y(y) > 0$$

From this definition it is clear that $E(h(X)/y)$ assumes different values for different values of y.

Hence, $E(h(X)|Y)$ itself is a random variable. The following result shows that the average value of this random variable is not different from the average value that $h(X)$ assumes.

Result

$$E\left(E(h(X)|Y)\right) = E\left(h(X)\right)$$

Proof

$$\begin{aligned}
E\left(E(h(X)/Y)\right) &= \int_y E(h(x)/y)\ f_Y(y)\ dy \\
&= \int_y \left[\int_x h(x)\ f(x/y)\ dx\right]\ f_Y(y)\ dy \\
&= \int_y \left[\int_x h(x)\ f(x,y)\ dx\right]\ dy \\
&= \int_x h(x) \left[\int_y f(x,y)\ dx\right]\ dy \\
&= \int_x h(x)\ f_X(x)\ dx = E(h(X))
\end{aligned}$$

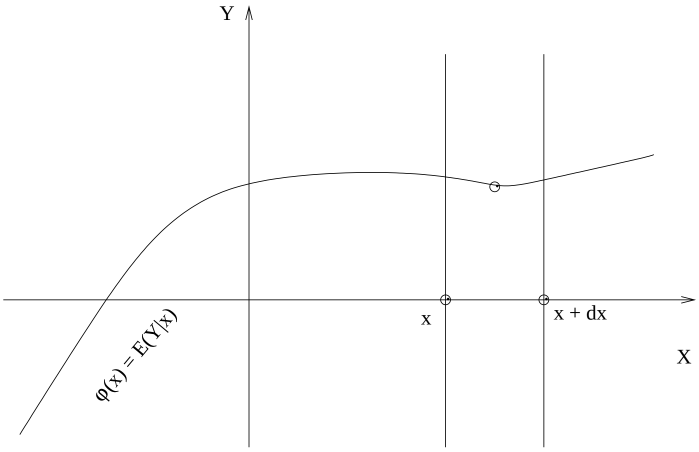
Hence the result.

Note 1:

For a given x, $E(Y|x)$ is the center of gravity of the masses in the vertical strip $(x, x+dx)$. The locus of these points, as x varies from $-\infty$ to ∞, is the function

$$\psi(x) = \int_{-\infty}^{\infty} y\ f(y|x)\ dy = E(Y|x),$$

known as the regression line.



Note 2:

In general, $\psi(x) = E(Y|x)$ is not a straight line however, if the random variables X and Y are jointly normal, then $\psi(x) = \mu_Y + r\dfrac{\sigma_Y}{\sigma_X}(x - \mu_X)$, a straight line.

Note that if X and Y are jointly normal with zero mean, then

$$
\begin{aligned}
f(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2r\ xy}{\sigma_X\ \sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right\} \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} \exp\left\{-\frac{\left(y - \dfrac{r\sigma_Y x}{\sigma_X}\right)^2}{2\sigma_Y^2(1-r^2)} - \frac{x^2}{2\sigma_X^2}\right\}
\end{aligned}
$$

so that $f(y|x) = \dfrac{1}{\sigma_Y\sqrt{2\pi(1-r^2)}} \ \exp\left(\dfrac{-\left(y - \dfrac{r\sigma_Y x}{\sigma_X}\right)^2}{2\sigma_Y^2(1-r^2)}\right).$

Then, it can be shown that

$$E(Y/x) = \mu_Y + r\ \sigma_Y\left(\frac{x - \mu_X}{\sigma_X}\right).$$

The L^P - spaces

If $X : \Omega \to \mathbb{R}^n$ is a random variable and $p \in [1, \infty)$ is a constant, we define the L^p - norm of X, denoted by $||X||_p$ by

$$
\begin{aligned}
\|X\|_p &= \left[\int_\Omega |X(\omega)|^p\ dP(w)\right]^{1/p} \quad \text{or simply we write} \\
\|X\|_p &= \left(\int_\Omega |X|^p\ dP\right)^{1/p}
\end{aligned}
$$

The corresponding L^p - spaces are defined by

$$L^p(P) = L^p(\Omega) = \{X : \Omega \to \mathbb{R}^n;\ \|X\|_p < \infty\}.$$

L^p - spaces are Banach spaces, ie, complete normed linear spaces. In particular, $L^2(p)$ is even a Hilbert space, ie, a complete inner product space with inner product $\langle X, Y \rangle = E(XY), \ X, Y \in L^2(p)$ so that $\|X\|_2^2 = E(X^2)$. Now, let us assume that we want to estimate the random variable Y by means of the information we have with the random variable X. In particular, suppose that the linear function aX is used to estimate Y. We choose the constant a in such a way that the mean square error estimate is minimum. ie, $e = \|Y - aX\|_2^2 = E\left((Y - aX)^2\right)$ is minimum.

The necessary condition for which is

$$\frac{\partial e}{\partial a} = 0 \implies E\left((Y - aX)X\right) = 0$$

which means that $Y - aX$ is orthogonal to X.
Conversely, assume that $Y - aX$ is orthogonal to X.
Then,

$$\begin{aligned} E\left((Y - \bar{a}X)^2\right) &= E\left(((Y - aX) + (a - \bar{a})X)^2\right) \\ &= E\left((Y - aX)^2\right) + (a - \bar{a})^2 \ E(X^2) \end{aligned}$$

Therefore, $E\left((Y - \bar{a}X)^2\right) \geq E\left((Y - aX)^2\right)$ for any a.

This shows that $E\left((Y - aX)^2\right)$ is minimum if and only if $Y - aX$ is orthogonal to X.

ie, if and only if $a = \dfrac{E(XY)}{E(X^2)}$.

Non-linear mean square estimation

Suppose we wish to estimate Y not by a linear function but by a non-linear function $g(X)$ of the random variable X. Our problem now is to find the function $g(X)$ such that the mean square error

$$\begin{aligned} e &= \|Y - g(X)\|_2^2 = E\left((Y - g(X))^2\right) \\ &= \int_x \int_y (y - g(x))^2 \ f(x, y) \ dx \ dy \end{aligned}$$

is minimum.

$$\text{Now,} \quad e = \int_x f_X(x) \left(\int_y (y - g(x))^2 \ f(y|x) \ dy \right) \ dx$$

Since the integrands are positive, e is minimum if the inner integral is minimum for every x. But, as per our previous discussion, the inner integral is minimum if and only if $g(X) = E(Y/X)$. Thus the optimum $g(X)$ is the regression line $\psi(X) = E(Y/X)$.

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Brownian Excursions

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Abstract : The study of the behaviour of Brownian motion around a point was initiated by Paul Levy, who introduced the notion of the 'local time' at a point. Building on Levy's work, Ito's excursion theory lays out the tools for calculating probabilities related to the excursions of Brownian motion above and below a point. In this talk, we look at the case of Brownian excursions into an interval, from the boundary.

1 Preliminaries

We have a probability space (Ω, F, P). A random variable is a function
$T := [0, \infty)$ is the time parameter set.
A collection of random variables $\{X_t : t \in T\}$ is called a stochastic process.
Notation: $\{X \in A\} := \{\omega : X(\omega) \in A\}$.
A statement S holds almost surely (a.s.), if

$$\text{P } \{\omega : S \text{ is true for } \omega\} = 1.$$

2 Two points of view

For each $\omega \in \Omega$ the collection of real numbers $(X_t(\omega))$ can be viewed as a function from $T \to \mathbb{R}$ called a (random) trajectory of the process (X_t) corresponding to $\omega \in \Omega$
Thus, a stochastic process is both a collection of random variables $(X_t : t \in T)$ as well as a collection of random functions on T viz. the trajectories $\{t \to X_t(\omega) : \omega \in \Omega\}$.
We will consider only the case when all the trajectories are realised as continuous functions on T. i.e. we are dealing with a collection of random continuous functions on $T = [0, \infty)$.

3 Functionals associated with a continuous function

Let $h : T \to \mathbb{R}$ be a fixed continuous function, $h(0) = 0$. Let $Z(h) := \{t : h(t) = 0\}$. This is a closed set . Its complement $Z(h)^c$ is a union of open intervals.
$Z(h)^c = \bigcup_i (\alpha_i, \beta_i)$

Example 3.1. (a) $h(t) := sin(t)$

(b) $h(t) := t.sin(\frac{1}{t})$

Let $a < b$ be two real numbers and (a, b) the corresponding open interval in the real line. Let $h(.)$ be a continuous function on T as above. For each $t > 0$, we define $C_t := C_t(h, (a, b)) :=$ number of crossings of (a, b) by $h(.)$ during $[0, t]$. Note that as ba, C_t and that in any finite time interval a continuous function can have only a finite number of crossings. When we replace $h(.)$ by the (random) trajectory of a stochastic process (X_t) viz. we take $h(.) = X.(\omega)$ we get random functionals of the trajectories viz.

$$C_t(\omega) := C_t(X.(\omega), (a, b))$$

Similarly

$$Z(\omega) := Z(X.(\omega)) := \{t : X_t(\omega) = 0\}$$

and

$$Z(\omega)^c = \bigcup_i (\alpha_i(\omega), \beta_i(\omega))$$

4 Excursions into an interval

Let $a < b.W : [0, \infty) \to \mathbb{R}$, continuous. Let

$$\begin{aligned} Z_{a,b} &:= \{t : W_t \leq a or W_t \geq b\} \\ Z_{a,b}^c &:= \{t : W_t \in (a, b)\} \\ &:= \bigcup_i (\gamma_i, \delta_i). \end{aligned}$$

Definition 4.1. A stochastic process $(W_t)_{t \geq 0}$ is a 1-dimensional standard Brownian motion iff

(a) $W_t - W_s \sim N(0, t - s), 0 \leq s < t$

(b) It has independent increments i.e. for $0 < t_1 < t_2 < \cdots < t_n$. $W_{t1}, W_{t2} - W_{t1}, \cdots W_{tn} - W_{tn-1}$ are independent random variables.

(c) $W_0 \equiv 0$.

(d) $t \to W_t(\omega)$ is continuous for every $\omega \in \Omega$.

5 The Zeroes of Brownian motion

Basic result : almost surely, $Z = \{t : W_t = 0\}$ is an uncountable, closed, perfect set of zero Lebesgue measure.

$$Z^c = \bigcup_i (\alpha_i, \beta_i)$$

Note that during an excursion interval,$(\alpha_i, \beta_i), W_t \neq 0$. Hence either $W_t > 0$ for every $t \in (\alpha_i, \beta_i)$ or $W_t < 0$ for every $t \in (\alpha_i, \beta_i)$. For $W_{\alpha_i} = W_{\beta_i} = 0$.

6 Problem

Note that for two excursion intervals (α_i, β_i) we cannot say $\alpha_i < \alpha_j$ and $\beta_i < \beta_j$ if $i < j$.

Equivalently, the problem is to obtain a good description of the excursions of the Brownian path $s \to W_s$ during the time $[0, t]$?

Itos solution: We can do this in the interval $[0, \tau_t]$ for certain random times - called stopping times - τ_t .

7 Stopping times

A random time is a map $\sigma : \Omega \to [0, \infty)$. A stopping time σ with respect to (W_t) is a random time with the property that $(\sigma \leq t) \subset \Omega$ is determined by $\{W_u : u \leq t\}$.

Example 7.1.

$$\sigma = inf\{u : W_u > 1\}$$
$$\tau = sup\{u : W_u > 1\}$$

Note that $W_\sigma(\omega) := W_{\sigma(\omega)}(\omega) = W_t(\omega)$ where $t = \sigma(\omega)$ is a random variable.

8 Adapted Functionals

A functional $T_t(.), t \geq 0$ defined on continuous functions is said to be adapted to the process (X_t) if the value of the functional $T_t(X.(\omega))$ on the random trajectory $s \to X_s(\omega), s \geq 0$ depends only on the part of the trajectory during $[0, t]$ viz. $X_s(\omega), 0 \leq s \leq t$.

Example 8.1. $T_t(h) :=$ sup $h(s)$ is not an adapted functional. $t - 1 \wedge 0 \leq s \leq t + 1$ Note that the number of crossings C_t defined earlier is an adapted functional.

Theorem 8.1. (K.B.Athreya and B.Rajeev, 2013, Sankhya) Let $\{C_t, t \geq 0\}$ be the number of crossings of (a, b) by the Brownian motion (W_s) during $[0, t]$, and X have the standard normal distribution. Then as $t \to \infty$ we have

$$\frac{C}{\sqrt{t}} \to^d \frac{|X|}{2(b-a)}$$

Definition 8.1. A continuous adapted process $(L_t)_t \geq 0$ is called a local time for (W_t) at the point zero iff

(a) $\forall \omega \in \Omega, t \to L_t(\omega)$ is non decreasing.

(b) It increases only on Z i.e.

$$\int_0^t I_Z(s) dL_s = L_t$$

almost surely or equivalently

$$\int_0^t I_{Z^c}(s)dL_s = 0$$

almost surely.

9 Levys construction of local time

Let $\epsilon > 0$ and define $C^\epsilon(t) :=$ Number of crossings of $(0, \epsilon)$ by the path $s \to W_s$ during the time interval $[0, t]$. Note that , in earlier notation $C^\epsilon(t) = C_t(W., (0, \epsilon))$.

Theorem 9.1. almost surely, the following hold for each $t > 0$:

$$\lim_{\epsilon \to 0} C^\epsilon(t) = \infty$$

and

$$\lim_{\epsilon \to 0} \epsilon C^\epsilon(t) = L_t$$

10 Tracking Excursions through Local time

Recall $Z^c = \bigcup_i (\alpha_i, \beta_i)$. Note that $L_{\beta_i} - L_{\alpha_i} = 0$, a.s. Let

$$\tau_t := inf\{s > 0 : L_s > t\}$$

Facts

(a) Almost surely, $t \to \tau_t$ is a non-decreasing right continuous function.

(b) $L_{\tau_t} = t$.

(c) When $\Delta\tau_t \neq 0, (\tau_t -, \tau_t) = (\alpha_i, \beta_i)$ for some i.

(d) For each $t > 0, \tau_t$ (and consequently $\tau_t -$) are stopping times with respect to (W_t).

11 Excursions of (W_t)

Let $t > 0$. Suppose $\Delta\tau_t neq 0$. Then the excursion of (W_t) in the interval $(\tau_t -, \tau_t)$ is given by the function

$$e_t(s) := W_{\tau_t - + s}, 0 \leq s \leq \tau_t - \tau_t -$$

. Let

$$U := \{\omega : [0, \infty) \to \mathbb{R}, \omega(0) = 0, \omega(R) = 0, \omega(t) \neq 0, 0 < t < R, \omega(\cdot) continuous\}$$
$$=: U^+ \cup U^-$$
$$U^+ := \{\omega \in U : \omega(t) > 0\}.$$

Note that if $\Delta\tau_t \neq 0, e_t \in U$. Let $\ulcorner \subset U$ be a subset of excursions. We define random variable.

$$N_t^\ulcorner := \sharp\{s \leq t : \Delta_{\tau_s} \neq 0, e_s(\cdot) \in \ulcorner\}$$
$$= \text{number of excursions during [0, t] that lie in the set} \ulcorner.$$

For many sets $\ulcorner, N_t^\ulcorner \equiv \infty$.

Example 11.1. if $\ulcorner := \{\omega \in U : sup|\omega(s)| < 1\}$ then $N_t^{\ulcorner} \equiv \infty$ because there are infinitely many points $s \in [0,t]$ with $\Delta\tau_s = 0$, and $sup_u|e_s(u)| = sup_u|W_{\tau_s+u}| < 1$. On the other hand $N_t^{\ulcorner} < \infty$ almost surely. Note that the number of events N_t during $[0,t]$ as defined above actually correspond to the number of excursions of (W_s) during $[0,\tau_t]$.

12 K.Itos description of Brownian excursions

Theorem 12.1. (a) there exist , $\ulcorner \in U, n \geq 1$ such that $U = {}_n\bigcup_{n\geq 1}\ulcorner_n$ and $N_t^{\ulcorner_n} < \infty$ for all $t \geq 0$, almost surely for each $n \geq 1$.

(b) Suppose $N_t^{\ulcorner} < \infty$ almost surely, for all $t \geq 0$. Then $(N_t^{\ulcorner})_{t\geq 0}$ is a Poisson process with parameter $n(\ulcorner) := \frac{1}{t}EN_t^{\ulcorner}$. In particular, for $0 < t_1 < t_2 < \cdots < t_n, N_{t_1}^{\ulcorner}, N_{t_2}^{\ulcorner} - N_{t_1}^{\ulcorner}, \cdots, N_{t_n}^{\ulcorner} - N_{n_1}^{\ulcorner}$ are independent Poisson random variables.

13 Markov Property

The Brownian motion (W_t) is a Markov process: if $0 < s < t + s$

$$P(W_{t+s} \in A|W_u, u \leq s) = P(W_{t+s} \in A|W_s).$$

It is a strong Markov process: if σ is a finite random (stopping) time and $t > 0$

$$P(W_{t+\sigma} \in A|W_u, u \leq \sigma) = P(W_{t+\sigma} \in A|W_\sigma)$$

14 Independent Increments and Markov Property

It is known that the property of Independent Increments implies the (ordinary) Markov property.

If in addition, the paths are continuous, as in Brownian motion, then we can show for a finite stopping time σ

$(W_{t+\sigma} - W_\sigma)_{t\geq 0}$ is independent of $\{W_t : t \leq \sigma\}$

In particular (W_t) is a strong Markov process.

15 Sketch of Proof:

The events $N_{t_n}^{\ulcorner} - N_t^{\ulcorner}$ are determined by the Brownian motion (W_t) during $(\tau_{t_{n1}}, \tau_{t_n}]$.

The independence of increments of $(N_t^{\ulcorner})$ follows from the independence of the increments of Brownian motion (W_t) during the intervals $(0, \tau_{t_1}], (\tau_{t_1}, \tau_{t_2}] \cdots, (\tau_{t_{n1}}, \tau_{t_n}]$. Also we note that a counting process with independent increments has to be a Poisson process.

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Controllability and Observability of Linear Systems

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Consider the n-dimensional control system described by the vector differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in (t_0, \infty) \tag{1}$$

$$x(t_0) = x_0$$

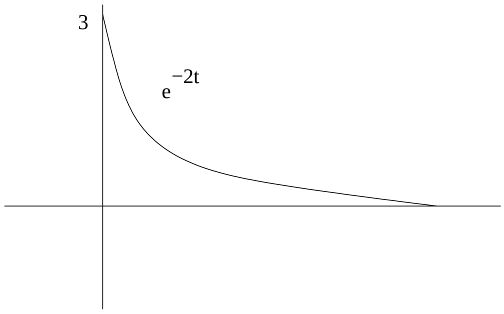
where, $A(t) = (a_{ij}(t))_{n\times n}$ is an $n \times n$ matrix with entries are continuous functions of t defined on $I = [t_0, t_1]$, $B(t) = (b_{ij}(t))_{n\times m}$ is an $n \times m$ matrix with entries are continuous function of t on I. The state $x(t)$ is an n-vector, control $u(t)$ is an m-vector. We first deal with controllability of one dimensional system which described by a scalar differential equation.

What is a control system ?

Consider a 1-dimensional system

$$\frac{dx}{dt} = -2x, \quad x(0) = 3$$

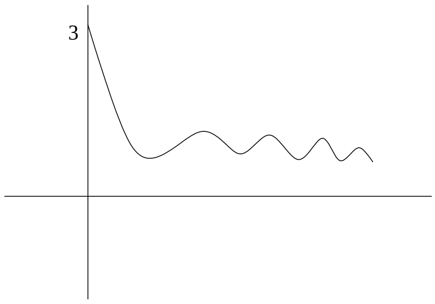
The solution of the system is $x(t) = 3e^{-2t}$ and its graph is shown in the following figure



If we add a nonhomogeneous term $sin(t)$ called the forcing term or control term to it then the system is given by

$$\frac{dx}{dt} = -2x + sin(t)$$

$$x(0) = 3$$



Observe that the solution or the trajectory of the system is changed. That is, the evolution of the system is changed by adding the new forcing term to the system. Thus the system with a forcing term is called a control system.

Controllability Problem

The controllability problem is to check the existence of a forcing term or control function $u(t)$ such that the corresponding solution of the system will pass through a desired point $x(t_1) = x_1$.
We now show that the scalar control system

$$\dot{x} = ax + bu$$

$$x(t_0) = x_0$$

is controllable. We produce a control function $u(t)$ such that the corresponding solution starting with $x(t_0) = x_0$ also satisfies $x(t_1) = x_1$. Choose a differentiable function $z(t)$ satisfying $z(t_0) = x_0$ and $z(t_1) = x_1$. For example, by the method of linear interpolation, $z - x_0 = \frac{x_1 - x_0}{t_1 - t_0}(t - t_0)$. Thus the function

$$z(t) = x_0 + \frac{(x_1 - x_0)}{t_1 - t_0}(t - t_0)$$

satisfies

$$z(t_0) = x_0, z(t_1) = x_1$$

A Steering Control using $z(t)$: The form of the control system

$$\dot{x} = ax + bu$$

motivates a control of the form

$$u = \frac{1}{b}[\dot{x} - ax]$$

Thus we define a control using the funtion z by

$$u = \frac{1}{b}[\dot{z} - az]$$

$$\dot{x} = ax + b[\frac{1}{b}[\dot{z} - az]]$$

$$\dot{x} - \dot{z} = a(x - z)$$

$$\frac{d}{dt}(x-z) = a(x-z)$$
$$x(t_0) - z(t_0) = 0$$

Let $y = x - z$

$$\frac{dy}{dt} = ay$$
$$y(t_0) = 0.$$

The unique solution of the system is $y(t) = x(t) - z(t) = 0$. That is, $x(t) = z(t)$ is the solution of the controlled system satisfying the required end condition $x(t_0) = x_0$ and $x(t_1) = x_1$. Thus the control function

$$u(t) = \frac{1}{b}[\dot{z}(t) - az(t)] \text{ is a steering control.}$$

Remark : Here we have not only controllability but the control steers the system along the given trajectory z. This is a strong notion of controllability known as trajectory controllability. Trajectory controllability is possible for a time-dependent scalar system $\dot{x} = a(t)x + b(t)u \ : \ b(t) \neq 0 \quad \forall \quad t \in [t_0, t_1]$ In this case the steering control is

$$u = \frac{1}{b(t)}[\dot{z} - a(t)z]$$

n-dimensional system with $m = n$

Consider an n-dimensional system $x = Ax + Bu$, where $A$and $B$are $n \times n$ matrices and B is invertible matrix. Now consider a control function as in the case of scalar systen, given by

$$u = B^{-1}[\dot{z} - Az]$$

where, $z(t)$ is a n-vector valued and differentiable function satisfying $z(t_0) = x_0$ and $z(t_1) = x_1$. Using this control we have

$$\dot{x} = Ax + BB^{-1}[\dot{z} - Az]$$
$$\dot{x} - \dot{z} = A(x-z)$$
$$\dot{x}(t_0) - z(t_0) = 0$$
$$\Longrightarrow x(t) = z(t)$$

Remark: If $BB^{-1} = I$, that is, if B has right inverse then also the system is trajectory controllable. **When $m < n$**:
When $m < n$ we consider the system

$$\dot{x} \quad = \quad A(t)x + B(t)u \tag{1}$$

$$\begin{aligned} x(t_0) &= x_0 \\ A(t) &= (a_{ij}(t))_{n\times n}, \\ B(t) &= (b_{ij}(t))_{n\times m} \end{aligned}$$

Definition(Controllability) : The system (2) is controllable on an interval $[t_0, t_1]$ if $\forall \quad x_0, x_1 \in \mathbb{R}^n, \quad \exists$ controllable function $u \in L^2([t_0, t_1] : \mathbb{R}^m)$ such that the corresponding solution of (2) satisfying $x(t_0) = x_0$ also satisfies $x(t_1) = x_1$: Since x_0 and x_1 are arbitrary this notion is also known as exact controllability or complete controllability.

Subspace Controllability : Let $D \subset \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$ and if the system is controllable for all $x_0, x_1 \in D$ then we say that the system is controllable to the subspace D.

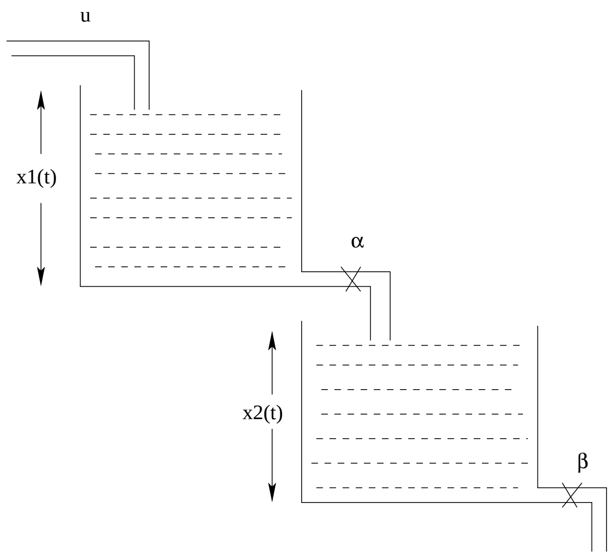
Approximate Controllability: If D is dense in state space then the system is approximately controllable. But in $\mathbb{R}^n$, $\mathbb{R}^n$ is the only dense subspace of $\mathbb{R}^n$. Thus approximate controllability is equivalent to complete controllability in $\mathbb{R}^n$. For the subspace D we have

$$D \subseteq \mathbb{R}^n \text{ and } \quad \bar{D} = \mathbb{R}^n \quad \text{implies} \quad D = \mathbb{R}^n$$

Null Controllability : If every non - zero state $x_0 \in \mathbb{R}^n$ can be steered to the null state $0 \in \mathbb{R}^n$ by a steering control then the system is said to be null controllable.

We now see examples of controllable and uncontrollable systems.

Example: Tank Problem :



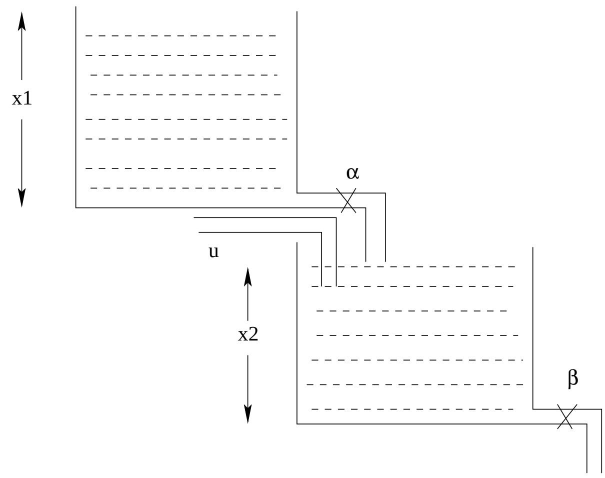
Let $x_1(t)$ be the water level in Tank 1 and $x_2(t)$ be the water level in Tank 2. Let α be the rate of outflow from Tank 1 and β be rate of outflow from Tank 2. Let u be the supply of water to the system. The system can be modelled into the following differential equations:

$$\frac{dx_1}{dt} = -\alpha x_1 + u$$
$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2$$

Model - 1 :

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Model - 2 :



$$\frac{dx_1}{dt} = -\alpha x_1$$
$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2 + u$$
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Obviously the second tank model is not controllable because supply can not change the water level in Tank 1. We will see later that the Model 1 is controllable whereas the model 2 is not controllable.
Controllability analysis can be made in many real life problems like :

(i) Rocket launching Problem, Satellite control and control of aircraft

(ii) Biological System : Sugar Level in blood

(iii) Defence: Missiles & Anti-missiles problems.

(iv) Economy- regulating inflation rate

(v) Eology: Predator - Prey system

Solution of the Controlled System using Transition Matrix :
Consider the n-dimensional linear control system:

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0$$

Let $\Phi(t_,t_0)$ be the transition matrix of the homogeneous system $\dot{x} = A(t)x$. The solution of the control system is given by (using variation of parameter method)

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t_,\tau)B(\tau)u(\tau)d\tau$$

The system is controllable iff for arbitrary initial and final states x_0, x_1 there exists a control function u such that

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

We first show that for linear systems complete controllability and null-controllability are equivalent.

Theorem :The linear system (1) is completely controllable iff it null-controllable.

Proof : It is obvious that complete controllability implies null-controllability.We now show that null-controllability implies complete controllability. Suppose that x_0 is to be steered to x_1.

Suppose that the system is null-controllable and let $w_0 = x_0 - \Phi(t_0, t_1)x_1$. Thus there exists a control u such that

$$\begin{aligned} 0 &= \Phi(t_1, t_0)w_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\ &= \Phi(t_1, t_0)[x_0 - \Phi(t_0, t_1)x_1] + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\ &= \Phi(t_1, t_0)x_0 - x_1 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\ x_1 &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\ &= x(t_1) \end{aligned}$$

$\Longrightarrow \quad u$ steers x_0 to x_1 during $[t_0, t_1]$

Conditions for Controllability :

The system (1) is controllable iff $\exists \quad u \in L^2(I, \mathbb{R}^m)$ such that

$$x_1 = \Phi(t_0, t_1)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

$$\text{ie,} x_1 - \Phi(t_0, t_1)x_0 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Define an operator $C : L^2(I, \mathbb{R}^m) \to \mathbb{R}^n$ by

$$Cu = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Obviously, C is a bounded linear operator and Range of C is a subspace of $\mathbb{R}^n$. Since x_0, x_1 are arbitrary, the system is controllable iff C is onto.

Range(C) is called the Reachabie set of the system.

Theorem : The following statements are equivalent:

(a) The linear system (1) is completely controllable.

(b) C is onto

(c) C^* is 1-1

(d) CC^* is 1-1

In the above result, the operator C^* is the adjoint of the operator C. We now obtain the explicit form of C^*.

Adjoint Operator :The operator $C : L^2(I,\mathbb{R}^m) \to \mathbb{R}^n$ defines its adjoint $C^* : \mathbb{R}^n \to L^2(I,\mathbb{R}^m)$ in the following way:

$$\begin{aligned}
< Cu, v >_{\mathbb{R}^n} &= < \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau, v >_{\mathbb{R}^n} d\tau \\
&= \int_{t_0}^{t_1} < \Phi(t_1,\tau)B(\tau)u(\tau)d\tau, v >_{\mathbb{R}^n} d\tau \\
&= \int_{t_0}^{t_1} < u(\tau), B^*(\tau)\Phi^*(t_1,\tau)v >_{\mathbb{R}^n} d\tau \\
&= < u, B^*()\Phi^*(t_1,\tau)v >_{L^2(I,\mathbb{R}^m)} \\
&= < u, C^*v >_{L^2(I,\mathbb{R}^m)} \\
(C^*v)(t) &= B^*(t)\Phi^*(t_1,t)v \\
\text{Using C* we get CC* in the form } CC^* &= \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\phi^*(t_1,\tau)d\tau
\end{aligned}$$

Observe that $CC^* : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded linear operator. Thus, CC^* is an n by n matrix.
Thus we have from the previous theorem that the system (1) is controllable $\Longleftrightarrow$ C is onto $\Longleftrightarrow$ CC^* is 1-1
$\Longleftrightarrow$ CC^* is an invertible matrix.
The matrix CC^* is known as the Controllability Grammian for the linear system and is given by
Controllability Grammian

$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\Phi^*(t_1,\tau)d\tau$$

By using inverse of the controllability Grammian we now define a steering control as given in the following theorem.
Theorem :The linear control system is controllable iff $W(t_0,t_1)$ is invertible and the steering control that move x_0 to x_1 is given by
$u(t) = B^*(t)\Phi^*(t_1,t)W^{-1}(t_0,t_1)[x_1 - \Phi(t_0,t_1)x_0]$
Proof : Controllability part is already proved earlier. We now show that the steering control defined above actually does the tranfer of states. The controlled state is given by

$$\begin{aligned}
x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau \\
x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)B^*(\tau)\Phi^*(t_1,\tau)W^{-1}(t_0,t_1)[x_1 - \Phi(t_0,t_1)x_0]d\tau \\
x(t_1) &= \Phi(t_1,t_0)x_0 + W(t_0,t_1)W^{-1}(t_0,t_1)[x_1 - \Phi(t_0,t_1)x_0] \\
x(t_1) &= x_1
\end{aligned}$$

Remark :Among all controls steering x_0 to x_1, the control defined above is having minimum L^2- norm (energy). We will prove this fact later.
Define a matrix Q given by

$$Q = [B|AB|....A^{n-1}B]$$

It can be shown that Range of $W(t_0, t_1)$ = Range of Q

Controllability of the linear system and the rank of Q are related by the following Kalman's Rank Test.

Theorem (Kalman's Rank Condition) : If the matrices A and B are time - independent then linear system (1) is controllable iff

$$\text{Rank}(B|AB|...|A^{n-1}B) = n$$

Proof : Suppose that the system is controllable.

Thus the operator $C : L^2(I, \mathbb{R}^m) \to \mathbb{R}^n$ defined by

$$Cu = \int_{t_o}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

is onto.We now prove that

$$\mathbb{R}^n = \text{Range(C)} \subset \text{Range(Q)}.$$

Let $x \in \mathbb{R}^n$ then $\exists u \in L^2(I, \mathbb{R}^n)$ such that

$$\int_{t_o}^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau = x$$

Expand $e^{A(t_1-\tau)}$ by Cayley - Hamilton's Theorem.

$$\int_{t_o}^{t_1} [P_0(0) + P_1A + P_2A^2 + ... + P_{n-1}A^{n-1}]Bu(\tau)d\tau] = x$$

$$\Longrightarrow x \in \text{Range}[B|AB|A^2B|..........|A^{n-1}B]$$

Conversely, Suppose that condition holds but system is not controllable. ie,Rank of $W(t_0, t_0) \neq n$

$$\Longrightarrow \exists v \neq 0 \in \mathbb{R}^n \text{such that} W(t_0, t_1)v = 0$$

$$\Longrightarrow v^T W(t_0, t_1)v = 0$$

$$\int_{t_o}^{t_1} v^T \Phi(t_1, \tau)BB^*\Phi^*(t_1, \tau)vd\tau = 0$$

$$\Longrightarrow \int_{t_o}^{t_1} ||B^*\Phi^*(t_1, \tau)v||^2 d\tau = 0$$

$$\Longrightarrow B^*\Phi^*(t_1, t)v = 0 \quad t \in [t_0, t_1$$

$$\Longrightarrow v^T\Phi(t_1, t)B = 0 \quad t \in [t_0, t_1]$$

$$v^T e^{A(t_1-t)}B = 0 \quad t \in [t_0, t_1]$$

Let $t = t_1$, $v^T B = 0$

Differentiating $v^T e^{A(t_1-t)}B = 0$ with respect to t and putting $t = t_1$

$$-v^T AB = 0............v^T A^{n-1}B = 0$$

$$\Longrightarrow (v \perp \text{Range}[B|AB|....|A^{n-1}B])$$

Hence Rank of $Q \neq n$

Rank condition is violated and thus we get a contradiction and thus the system is controllable.

Examples : Tank Problem: Model I.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$Q : [B : AB] = \begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix}$$

Rank $Q = 2 \quad \Longrightarrow$ System is controllable.

Model - 2 :

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix};$$

rank$(Q) = 1 \not< 2$

$\Longrightarrow$ System is not controllable.

Computation of Steering Control :

$$\begin{aligned} Cu &= w \\ CC^*v &= w \end{aligned}$$

where $u = C^*v$. The system is controllable iff

$$\begin{aligned} & C \text{ is onto.} \\ & \Longleftrightarrow C^* \text{is} 1-1. \\ & \Longleftrightarrow CC^* \text{is} 1-1. \\ & \Longleftrightarrow CC^* \text{is invertible.} \end{aligned}$$

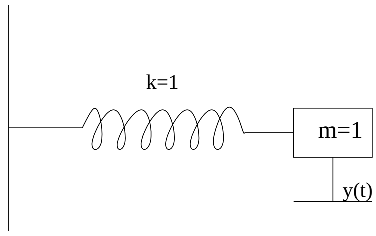
If CC^* is invertible then

$$\begin{aligned} v &= (CC^*)^{-1}w \\ u &= C^*(CC^*)^{-1}w \end{aligned}$$

is the steering control.

Controllability Example :

Spring Mass System : Consider a spring mass system having unit mass and with spring constant 1. By Newton's law of motion we have the following differential equation.



$$y'' + y = 0$$

$$\text{Let} \quad x_1 = y$$

$$x_2 = y'$$

$$\dot{x_1} = x_2$$

$$\dot{x_2} = y'' = -y = -x_1$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Transition Matrix by Laplace Transform Method :

We know that $e^{At} = L^{-1}\{(sI - A)^{-1}\}$

$$(sI - A) = \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 + 1}\begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}^T = \begin{pmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{pmatrix}$$

$$L^{-1}\{sI - A\}^{-1} = L^{-1}\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \sin t \end{pmatrix}$$

Another Way - Matrix Expansion:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + ...$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -I$$

$$A^3 = -A$$

$$A^4 = I$$

$$A^5 = A$$

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} + \begin{bmatrix} \frac{-t^2}{2!} & 0 \\ 0 & \frac{-t^2}{2!} \end{bmatrix} + \begin{bmatrix} 0 & \frac{-t^3}{3!} \\ \frac{-t^3}{3!} & 0 \end{bmatrix} + \begin{bmatrix} \frac{-t^4}{4!} & 0 \\ 0 & \frac{-t^4}{4!} \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + ... & t - \frac{t^3}{3!} + \frac{t^5}{5!} - ... \\ -t + \frac{t^3}{3!} - ... & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - ... \end{bmatrix} \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \end{aligned}$$

Let the initial state and the desired final states be given by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ Transition Matrix is given by,

$$\Phi(T,t) = \begin{pmatrix} \cos(T-t) & \sin(T-t) \\ -\sin(T-t) & \cos(T-t) \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Controllability Grammian is given by,

$$\begin{aligned} W(0,T) &= \int_0^T \begin{pmatrix} \sin(T-t) \\ \cos(T-t) \end{pmatrix} \begin{pmatrix} \sin(T-t) & \cos(T-t) \end{pmatrix} dt \\ &= \begin{pmatrix} \frac{1}{2}(T - \sin 2T) & \frac{1}{4}(1-\cos 2T) \\ \frac{1}{4}(1-\cos 2T & \frac{1}{2}(T + \frac{1}{2}\sin 2T) \end{pmatrix} \\ W^{-1}(0,T) &= \frac{4}{t^2 - \frac{1}{2}(1-\cos 2T)} \begin{pmatrix} T + \frac{1}{2}\sin 2T & \frac{1}{4}(\cos 2T - 1) \\ \frac{1}{4}(\cos 2T - 1) & \frac{1}{2}(T - \frac{1}{2}\sin 2T) \end{pmatrix} \end{aligned}$$

The steering control is

$$u(t) = \frac{4}{T^2 - 1/2(1-\cos 2T)}(0,1) \begin{pmatrix} \cos(T-t) - \sin(T-t) & \\ \sin(T-t) & \cos(T-t) \end{pmatrix} W^{-1}(0,T) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{T^2 - \frac{1}{2}(1-\cos 2T)} \{[(T - \frac{1}{2})\cos(T-t) + \sin(T-t)] + \frac{1}{2}[\cos(T+t) - \sin(T+t)]\}$$

Minimum norm control :We now prove that the steering control defined in the above discussion is actually an optimal control.

Theorem: The control function defined by $u_0 = B^*(t)\Phi^*(t_1,t)W^{-1}(t_0,t_1)x_1$ is a minimum norm control among all other controls steering the system from state x_0 to state x_1. That is, $||u_0|| \leq ||u||$ for all other steering controllers u in $L^2(I,\mathbb{R})$

Proof:

Let $u = u_0 + u - u_0$ and hence we have

$$\begin{aligned} ||u||^2 &= ||u_0 + (u-u_0)||^2 \\ &= < u_0 + (u-u_0), u_0 + (u-u_0) > \\ &= < u_0, u_0 > + < u_0, u - u_0 > + < u - u_0, u_0 > + < u-u_0, u-u_0 > \\ &= ||u_0||^2 + ||u-u_0||^2 + 2\text{Re} < u_0, \quad u - u_0 >_{L^2} \end{aligned}$$

Now,

$$\begin{aligned}
< u_0, u - u_0 >_{L^2} &= \int_{t_0}^{t_1} < u_0(t), u(t) - u_0(t) >_{\mathbb{R}^m} dt \\
&= \int_{t_0}^{t_1} < B^*(t)\Phi^*(t_1,t)W^{-1}(t_0,t_1)x_1, u(t) - u_0(t) > dt \\
&= < W^{-1}(t_0,t_1)x_1, \int_{t_0}^{t_1} \Phi(t_1,t)B(t)[u(t) - u_0(t)]dt > \\
&= < W^{-1}(t_0,t_1)x_1, \quad x_1 - x_1 > \\
&= 0
\end{aligned}$$

Since both u and u_0 are steering controllers.

Thus

$$\begin{aligned}
&||u||^2 = ||u_0||^2 + ||u - u_0||^2 \\
\text{or} \quad &||u||^2 - ||u_0||^2 = ||u - u_0||^2 \geq 0 \\
&||u||^2 \geq ||u_0||^2
\end{aligned}$$

for all steering controllers u.

Adjoint Equation : An equation having solution x in some inner product space is said to be adjoint of an equation with solution p in the same inner product space if $< x(t), p(t) >$ = constant.

Theorem : The adjoint equation associated with $\dot{x} = A(t)x$ is $\dot{p}(t) = -A^*(t)p$

Proof :

$$\begin{aligned}
\frac{d}{dt} < x(t), p(t) > &= < \dot{x}(t), p(t) > + < x(t), \dot{p}(t) > \\
&= < A(t)x, p(t) > + < x(t), -A^*(t)p(t) > \\
&= < x(t), A^*(t)p(t) > + < x(t), -A^*(t)p(t) > \\
&= < x(t), 0 >= 0
\end{aligned}$$

$\therefore \quad < x(t), p(t) > =$ constant.

Theorem : If $\Phi(t, t_0)$ is the transition matrix of $\dot{x}(t) = A(t)x$ then $\Phi^*(t_0, t)$ is the transition matrix of its adjoint system $\dot{p} = -A^*(t)p$.

Proof :

$$\begin{aligned}
I &= \Phi^{-1}(t,t_0)\Phi(t,t_0) \\
0 = \frac{d}{dt}I &= \frac{d}{dt}[\Phi^{-1}(t,t_0)\Phi(t,t_0)] \\
&= \frac{d}{dt}[\Phi^{-1}(t,t_0)]\Phi(t,t_0) + \Phi^{-1}(t,t_0)\dot{\Phi}(t,t_0) \\
&= \dot{\Phi}(t_0,t)\Phi(t,t_0) + \Phi(t_0,t)A(t)\Phi(t,t_0) \\
0 &= [\dot{\Phi}(t_0,t) + \Phi(t_0,t)A(t)]\Phi(t,t_0) \\
\Longrightarrow \dot{\Phi}(t_0,t) &= -\Phi(t_0,t)A(t) \\
\dot{\Phi}^*(t_0,t) &= -A^*(t)\Phi^*(t_0,t) \\
\Longrightarrow \Phi^*(t_0,t) & \quad \text{is the transition matrix of the adjoint system.}
\end{aligned}$$

Remark : The system is self adjoint if $A(t) = -A^*(t)$ and in this case

$$\begin{aligned} \Phi(t,t_0) &= \Phi^*(t_0,t) \\ &= \Phi^{-1}(t,t_0) \\ \Phi(t,t_0)\Phi^*(t,t_0) &= I \end{aligned}$$

Observability

Problem of finding the state vector knowing only the output y over some interval of time$[t_0, t_1]$.

Consider the input free system

$$\dot{x}(t) = A(t)x(t) \tag{2}$$

with the observation equation

$$y(t) = C(t)x(t),$$

where $C(t) = (c_{ij}(t))_{m\times n}$ matrix having entries as continuous functions of t.

Let $\Phi(t,t_0)$ be the transition matrix. The solution is $x(t) = \Phi(t,t_0)x_0$

Thus

$$y(t) = C(t)\Phi(t,t_0)x_0 \quad t_0 \le t \le t_1$$

Definition : System (3) is said to be observable over a time period $[t_0, t_1]$ if it is possible to determine uniquely the initial state $x(t_0) = x_0$ from the knowledge of the output $y(t)$ over $[t_0, t_1]$.

The complete state of the system is known if initial state x_0 is known.

Define a linear operator

$$L : \mathbb{R}^n \to L^2([t_0,t_1];\mathbb{R}^m)\text{by}$$

$$(Lx_0)(t) = C(t)\Phi(t,t_0)x_0$$

Thus,

$$y(t) = (Lx_0)(t) \quad t \in [t_0, t_1]$$

The system is observable iff L is invertible.

Theorem :The following statements are equivalent.

(a) The linear system $\dot{x}(t) = A(t)x(t), \quad y(t) = C(t)x(t)$ is observable.

(b) The operator L is 1-1.

(c) The adjoint operator L^* is onto.

(d) The operator L^*L is onto.

Remark : $L^*L : \mathbb{R}^n \to \mathbb{R}^n$ is an $n \times n$ matrix called Observability Grammian.

Finding $L^* : L^2 \to \mathbb{R}^n$:

$$\begin{aligned} < (Lx_0)(.), w(.) >_{L^2(I,\mathbb{R}^n)} &= \int_{t_o}^{t_1} < C(t)\Phi(t,t_0)x_0, w(t) >_{\mathbb{R}^n} dt \\ &= \int_{t_o}^{t_1} < x_0, \Phi^*(t,t_0)C^*(t)w(t) >_{\mathbb{R}^n} dt \\ &= < x_0, \int_{t_o}^{t_1} \Phi^*(t,t_0)C^*(t)w(t)dt >_{\mathbb{R}^n} \\ &= < x_0, L^*w(.) >_{\mathbb{R}^n} \end{aligned}$$

Thus,

$$L^*w = \int_{t_o}^{t_1} \Phi^*(t,t_0)C^*(t)w(t)dt$$

Observability Grammian The observability Grammian is given by

$$M(t_0,t_1) = L^*L = \int_{t_o}^{t_1} \Phi^*(t,t_0)C^*(t)C(t)\Phi(t,t_0)dt$$

The linear system is observable if and only if the observability Grammian is invertible.

Kalman's Rank Condition for Time Invariant System

If A and C are time-independent matrices,then we have the following Rank Condition for Observability.

Theorem : The linear system $\dot{x}(t) = Ax(t),\ y(t) = Cx(t)$ is observable iff the rank of the following Observability matrix O

$$O = \begin{pmatrix} C \\ CA \\ CA^2 \\ ... \\ CA^{n-1} \end{pmatrix}$$

is n.

Proof : The observation $y(t)$ and its time derivatives are given by,

$$\begin{aligned} y(t) &= Ce^{A(t)}x(0) \\ y^1(t) &= CAe^{At}x(0) \\ y^2(t) &= CA^2e^{At}x(0) \\ .... &\quad .... \\ y^{n-1}(t) &= CA^{n-1}e^{At}x(0) \end{aligned}$$

At $t = 0$, we have the following relation.

$$\begin{aligned} y(0) &= Cx(0) \\ y^1(0) &= CAx(0) \\ y^2(0) &= CA^2x(0) \\ .... &\quad .... \\ y^{n-1}(0) &= CA^{n-1}x(0) \end{aligned}$$

The initial condition $x(0)$ can be obtained from the equation.

$$\begin{pmatrix} C \\ CA \\ CA^2 \\ ... \\ CA^{n-1} \end{pmatrix} x(0) = \begin{pmatrix} y^0(0) \\ y^1(0) \\ y^2(0) \\ ... \\ y^{n-1}(0) \end{pmatrix}$$

The initial state $x(0)$ can be determined if the observability matrix on the left hand side has full rank n.
Hence the system is observable if the Kalman's Rank Condition holds true. Converse can be proved easily(exercise).
Reconstruction of initial state x_0 : We have

$$\begin{aligned} y &= Lx_0 \\ L^*y &= L^*Lx_0 \\ x_0 &= (L^*L)^{-1}L^*y \\ x_0 &= [M(t_0,t_1)]^{-1}\int_{t_o}^{t_1} \Phi^*(\tau,t_0)C^*(\tau)y(\tau)d\tau \end{aligned}$$

Duality Theorem :
The linear system

$$\dot{x} = A(t) + B(t)u \tag{3}$$

is controllable iff the adjoint system

$$\left.\begin{aligned} \dot{x} &= -A^*(t) \\ y &= B^*(t)x \end{aligned}\right\} \tag{4}$$

is observable.
Proof : If $\Phi(t,t_0)$ is the transition matrix generated by $A(t)$ then $\Phi^*(t_0,t)$ is the transition matrix generated by $-A^*(t)$.
The system (5) is observable iff the observability Grammian

$$\begin{aligned} M(t_0,t_1) &= \int_{t_o}^{t_1} [\Phi^*(t_0,t)]^*(B^*)(t)^*B^*(t)\Phi^*(t_0,t)dt \quad \text{is non-singular} \\ &\Longleftrightarrow \int_{t_o}^{t_1} \Phi(t_0,t)B(t)B^*(t)\Phi^*(t_0,t)dt \quad \text{is non-singular} \\ &\Longleftrightarrow \int_{t_o}^{t_1} \Phi(t_1,t_0)\Phi(t_0,t)B(t)B^*(t)\Phi^*(t_1,t_0)\Phi^*(t_0,t)dt \quad \text{is non-singular} \\ &\Longleftrightarrow \int_{t_o}^{t_1} \Phi(t_1,t)B(t)B^*(t)\Phi^*(t_1,t)dt \quad \text{is non-singular} \\ &\Longleftrightarrow W(t_0,t_1) \quad \text{is non-singular} \\ &\Longleftrightarrow \text{The system (4) is controllable.} \end{aligned}$$

Example :

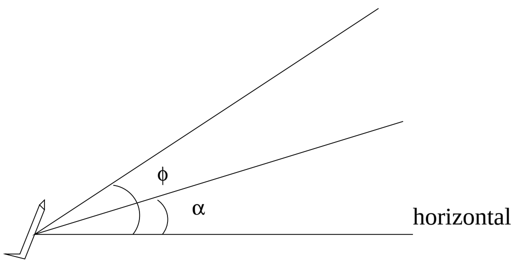
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$\dot{x} = Ax$$

$y(t) = [1, 0, 1]x(t)$. That is, $y = Cx$

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -5 & 16 \\ 1 & -4 & 11 \end{bmatrix}$$

has rank 3.
$\Longrightarrow (A, C)$ is observable.

Airplane Model(linear Model) :



Let us define the following variables: $\phi(t)$:pitch angle $\equiv$ body of the plane inclined to an angle ϕ with the horizontal.
$\alpha(t)$:Flight Path Angle: The path of the flight is along a straight line and it is at an angle α with the horizontal.
$h(t)$: Altitude of the plane at time t.
c: Plane flies at a constant non-zero ground speed c.
w: Natural Oscillation frequency of the pitch angle.
a, b: the constants.
$u(t)$: The control input u is applied to the aircraft by the elevators at the tail of the flight.
$\alpha > 0$ for ascending $\alpha < 0$ for descending.
Now the mathematical model of the system for small ϕ and α is given by

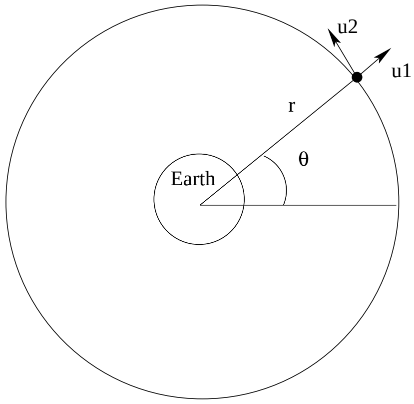
$$\begin{aligned} \dot{\alpha} &= a(\phi - \alpha) \\ \ddot{\phi} &= -w^2(\phi - \alpha - bu) \\ \dot{h} &= c\alpha \end{aligned}$$

Consider the variables:
Let $x_1 = \alpha$,$x_2 = \phi$, $x_3 = \dot{\phi}$, $x_4 = a$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{pmatrix} = \begin{pmatrix} -a & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ w^2 & -w^2 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w^2b \\ 0 \end{pmatrix} u$$

Show that the system is controllable.

Satellite Problem :



$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2 \end{pmatrix}$$

$u_1(t)$ - radial thrusters

u_2 - tangential thrusters

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

Show that the system is observable.

Only radial distance measurements are available: :

$$y_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x(t) = C_1 x(t)$$

$$\begin{bmatrix} C_1 \\ C_1A \\ C_1A^2 \\ C_1A^3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & -w^2 & 0 & 0 \end{pmatrix}$$

has rank 3.

Thus the system is not observable only with radistance measurements.

Only measurements of angle are available :

$$y_2 = [0, 0, 1, 0]x(t)$$

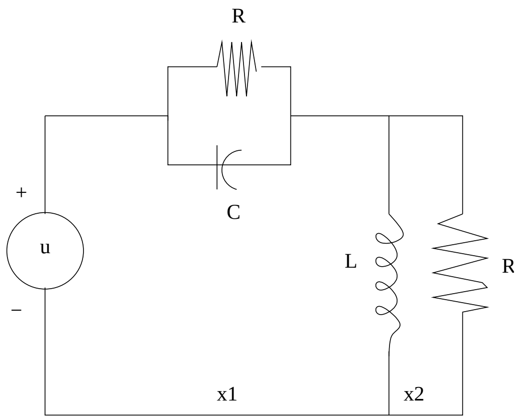
$$= C_2 x(t)$$

$$\text{rank}\begin{bmatrix} C_2 \\ C_2A \\ C_2A^2 \\ C_2A^3 \end{bmatrix} = 4$$

This implies that even with the measurement of angle alone the system is observable.

Electrical Circuit Example :

Consider the following circuit:



The state space representation is given by,

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{pmatrix} u(t)$$

Observation equation is given by

$$y(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u(t)$$

$$Q = [B|AB] = \begin{bmatrix} \frac{1}{RC} & \frac{-2}{R^2C^2} + \frac{1}{LC} \\ \frac{1}{L} & \frac{-1}{RLC} \end{bmatrix}$$

The system is uncontrollable if $det = \frac{1}{R^2LC^2} - \frac{1}{L^2C} = 0$ iff $R = \sqrt{\frac{L}{C}}$

Observation Matrix is

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ \frac{2}{RC} & \frac{-1}{C} \end{bmatrix}$$

It has full rank implies the observability of the system.

Observability Example : Consider the spring mass system considered earlier.

Let the observability equation be given by

$$y(t) = [0, 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$C = [0, 1]$$

$$\text{Observability Matrix} \quad O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rank is 2 $\Longrightarrow$ System is observable.

Computation of initial state x_0

Let $[t_0, t_1] = [-\pi, 0]$

$$\Phi(t, -\pi) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{pmatrix}$$

$$
\begin{aligned}
C\Phi(t,-\pi) &= [0,1]\begin{bmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{bmatrix} \\
&= [-\cos t - \sin t] \\
W(0,\pi) &= \int_{-\pi}^{0} \begin{bmatrix} -\cos t \\ -\sin t \end{bmatrix} \begin{bmatrix} -\cos t & -\sin t \end{bmatrix} dt \\
&= \int_{-\pi}^{0} \begin{bmatrix} \cos^2 t & \sin t \cos t \\ \cos t \sin t & \sin^2 t \end{bmatrix} dt \\
&= \frac{\pi}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Now using the reconstruction formula

$$
\begin{aligned}
\begin{pmatrix} x_1(-\pi) \\ x_2(-\pi) \end{pmatrix} &= \frac{2}{\pi} \int_{-\pi}^{0} \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix} \begin{pmatrix} \frac{1}{2}\cos t & \frac{1}{2}\sin t \end{pmatrix} dt \\
&= \frac{1}{\pi} \int_{-\pi}^{0} \begin{pmatrix} -\cos^2 t & -\cos t \sin t \\ -\sin t \cos t & \sin^2 t \end{pmatrix} dt \\
&= \frac{1}{\pi} \begin{pmatrix} -\frac{\pi}{2} \\ -\frac{\pi}{2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \\
&= x_0 \in \mathbb{R}^2
\end{aligned}
$$

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Pedagogical content knowledge (PCK) development in statistics teaching: what content knowledge does mathematics teachers have and demonstrate during classroom practice ?

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Abstract : This paper presents the findings on mathematics teachers' pedagogical content knowledge (PCK) in statistics teaching. Six mathematics teachers were initially selected for the study based on their school's performance over two years in the senior certificate examination in mathematics, and the four top scorers in the a conceptual knowledge exercise (CKE) in statistics were finally selected for this study . The study adopted a qualitative research method. The data on the teachers' PCK were collected through lesson observations, questionnaires, interviews, video recordings, teachers' written reports, and document analyses. The results of the study show that the teachers possess topic specific subject matter content knowledge and use of procedural and conceptual knowledge to teach statistics in school mathematics. The implications for mathematics education programmes are discussed.

1 Introduction

In an attempt to improve learners' achievement in mathematics and sciences, several researchers use the terms 'subject matter knowledge' and 'subject matter content' to describe the kind of knowledge that teachers need for teaching (Shulman, 1986; Ma, 1999; Vistro-Yu, 2003; Jong, 2003; Jong, Van Driel and Verloop, 2005; Halim and Meerah, 2002; Rollnick, Bennett, Rhemtula, Dharsey and Ndlovu 2008). In terms of mathematics teaching, Plotz (2007) refers to subject matter content knowledge as 'mathematical content knowledge'. With regard to PCK development in statistics teaching, it is necessary to define what each of the concepts means so that they can be used to define the PCK constructs used in statistics teaching. Plotz (2007) argues that mathematical content knowledge is largely acquired by studying mathematics in school, and this may be described as 'in-school acquired knowledge'. Van Driel, Verloop and De Vos (1998), Jong (2003), and Jong et al. (2005), describe subject matter knowledge as knowledge obtained through formal training at universities and colleges which may be regarded as disciplinary education. Subject matter knowledge, one can therefore conclude, is acquired through formal training in a subject area.

Ball and Bass (2000 argue that the subject matter knowledge needed by teachers is found not only in the topics of to be learned but also in the practice of teaching itself. In other words, knowing the content of a subject is not enough to qualify a teacher to teach; what makes a teacher capable of teaching is how well he or she facilitates learning. According to these authors, little is known about the way in which 'knowing' a specific topic in a list of topics affects a teacher's capabilities, and if one expects to identify the subject matter content knowledge needed for teaching from the curriculum without focusing on practice as well, not much will be gained (Ball and Bass, 2000; Plotz, 2007). Plotz's (2007) study also reveals that mathematical content knowledge and pedagogical knowledge are both required for effective teaching and can enhance development of PCK. He further stress that teachers' prior knowledge is also needed for effective content knowledge transformation and understanding since prior knowledge aids teachers in the design of problem solving activities during classroom practice.

Capraro, Capraro, Parker, Kulm and Raulerson (2005) researched the role of mathematics content knowledge in developing pre-service teachers' PCK using performance in a previous mathematics course, a pre- and post-test assessment instrument, success in the state-level teacher certification examination, and journals. Their study outlined the connection between mathematics content knowledge and pedagogical knowledge in developing PCK. A total of 193 undergraduate students in integrated method block courses were involved in the research project and the findings indicated that teachers' previous mathematical abilities are valuable predictors of students' success in teacher certificate examinations. In addition, mathematically competent pre-service teachers exhibited progressively more PCK since they had been exposed to mathematical pedagogy comprising subject matter content and teaching practice during their mathematics method courses. To be pedagogically effective in teaching a topic, it is necessary to have comprehensive understanding of it.

However, the South African mathematics (Grades 10–12) teaching force is made up mainly of practitioners who have three-year teaching diplomas obtained from the old (pre-1994) colleges of education (Rollnick et al., 2008). According to these authors less than 40% of these teachers hold a junior degree on the subjects they teach and mathematics content only measures up to that of first year at a university. In this study the key question is, given that the teachers show competence or understanding of these concepts in mathematics, irrespective of their training, how does this influence their teaching and therefore their PCK for teaching statistics in school mathematics?

2 Conceptual framework

Research reports by Manouchehri (1976) indicated that subject matter content knowledge consists of an explanatory framework and the rules of evidence within a discipline. According to Jong (2003), subject matter content knowledge of prospective mathematics teachers is acquired primarily during disciplinary education (Jong, 2003). This knowledge consists of substantive content knowledge and syntactic content knowledge (SnBarnes, 2007). Substantive content knowledge refers "to the concepts, principles, laws, and models in a particular content area of a discipline (SnBarnes, 2007)." Syntactic content knowledge is the "set of ways in which truth or falsehood, validity or invalidity are established" (Schwab, 1978, cited in Shulman, 1986). In practice, teachers should not only be able to define the acceptable truths in a domain, but also to explain, in theory and in practice, why these truths are worth knowing and how they relate to other propositions in and outside the discipline.

Both types of subject matter knowledge (substantive and syntactic) are needed for teachers' development of PCK to create adequate understanding of the nature of the subject matter and how it should be taught (Jong, 2003). It is therefore assumed that mathematics teachers with good PCK have both types of subject matter content knowledge and are able to apply this knowledge in making the topic understandable to learners. This assumption is given empirical support by Wu (2005), who indicated that teachers with good PCK have a firm command of subject matter knowledge and are able to design instructional materials that allow learners to grasp what they teach. Muijs and Reynolds (2000) call them effective teachers.

Other scholars, such as Carpenter, Fennema, Petterson and Carey (1988), Even (1993), Manouchehri (1997), Van Driel et al. (1998), Halim and Meerah (2002), Tsangaridou (2002), Viri (2003) and Hill (2008) have studied the influence of subject matter knowledge on the PCK of pre-service, novice, and expert teachers. These studies reveal that teachers' content knowledge goes a long way towards determining the level of PCK. Subject matter content knowledge is a key components of PCK that was assessed in this study.

Methodology

The methodology for the study consisted of two phases. In the first phase, six identified mathematics teachers undertook a written exercise to assess their conceptual knowledge of statistics. The results of this exercise were used to select the four best-performing teachers for the second phase of the study.

The second phase consisted of a concept mapping exercise (CME), lesson observations, interviews, teachers' written reports, and document analyses to produce rich detailed descriptions of participating teachers' PCK in the context of teaching data-handling concepts at school level. The CME was used to indirectly assess content knowledge and the teachers' conceptions of the nature of school statistics and how it is to be taught. The qualitative data obtained were analysed to determine individual teachers' content knowledge, related pedagogical knowledge, and how they developed their PCK in statistics teaching. The analysis was based on iterative coding and categorisation of responses and observations in order to identify themes, patterns, and gaps in school statistics teaching. Commonalities and differences, if any, in the PCK profiles of the four participating teachers were determined and analysed.

The validity of the CKE was conducted by giving exercises to the teachers to ascertain whether CKE could be used to assess their knowledge of school statistics and to select participants for the study. A concept map was given to the same mathematics teachers to determine whether the CME would allow them to list topics according to Grades 10, 11 and 12 and arrange them in logical order, such that one topic formed the basal knowledge of the next for each of the grades. Second, they were required to decide whether the memorandum was appropriate for answering the CME. The interview, questionnaire, and teacher written reports were validated by mathematics education experts using a set of criteria to establish whether these instruments contained appropriate information to determine teachers' mathematics educational background for developing subject matter content knowledge in statistics teaching (Ijeh, 2013).

The reliability of the CKE was established through the Kuder-Richardson split half procedure (KR-20, KR-21). The reliability index was 0.81. The CME and memorandum were given to four school mathematics teachers who did not participate in the study and who were physically located outside the study site to avoid contamination. There were consistencies in the responses of the mathematics teachers in the anticipated answers of the CME. The reliability of the teacher interviews, questionnaires, and written reports was determined by school mathematics teachers who were not involved in the study to determine the extent to which the instruments were likely to yield consistent responses (Cresswell, 2008) in terms of assessing the mathematics teachers' educational background that may have enabled them to develop their topic-specific PCK in statistics teaching.

3 Result and Discussion

Teacher A was observed teaching histogram construction and box-and-whisker plots in a step-wise fashion using the recommended mathematics textbooks and work schedule. For example, when he was asked, *"What learning and teaching support materials do you use in teaching statistics?"* he responded, *"I use classroom mathematics textbooks recommended by the Department of Basic Education and the work schedule."*

He started the lesson by asking the learners to name the components of measures of central tendency such as modes, medians, and means of ungrouped data to determine their prior knowledge of histogram construction. The *learners responded: The components of measures of central tendency are mode, median and mean.*

The components of measures of central tendency having been identified, the teacher and learners prepared a frequency table from the raw data. Using this table, the histogram was constructed by first drawing its horizontal and vertical axes (see Figure 1). The axes were labelled with data values on the horizontal axis, and frequencies on the vertical axis. A scale was chosen by the teacher, who stated that the highest and lowest values of the frequencies and data values, as well as the dimensions of the graph paper provided, had been considered. Next, the bars of the histogram were drawn by joining the line of best fit (see Figure 1). Teacher A's lesson showed that he had adopted a rule-oriented procedural approach to teaching histogram construction.

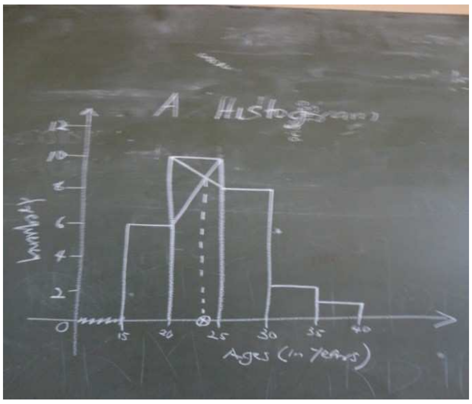

Figure 1 : A histogram constructed by the teacher and learners during lesson

In teaching the construction of box-and-whisker plots, he gave further evidence of using procedural knowledge, focusing primarily on rules and algorithms, rather than conceptual knowledge. The procedural approach requires simply plugging the data into the appropriate formulae, and then working out the correct values of the quartiles for the box-and-whisker plots (see Figure 2). For example , using the formula; $Q_1 = \frac{(n+1)^{th}}{4}$ to calculate the position of Q_1; $Q_1 = 20$; $Q_2 = 23$ and $Q_3 = 27$,, all the values were obtained from the ogive that the learners had been working on. The most challenging aspect for this teacher was knowing how to move from an algorithmic stage to a conceptually meaningful one as far as the students' learning was concerned.

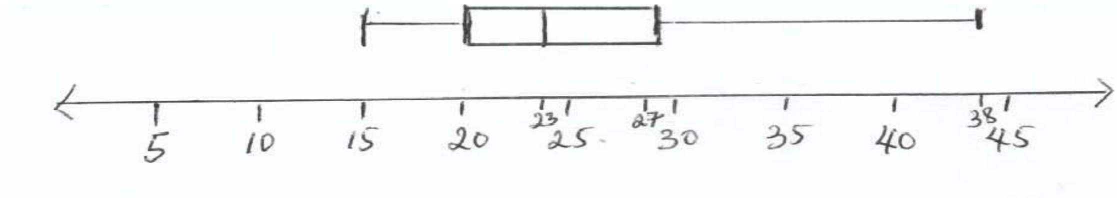

Figure 2 : A box-and-whisker plot constructed with the values $Q_1 = 20$; $Q_2 = 23$ and $Q_3 = 27$.

However, he used a conceptual teaching approach during the lesson and demonstrated the mathematical connections and relationships between ogives and box-and-whisker plots by describing how quartiles were obtained from the ogive and used in the construction of the box-and-whisker plot (see Figure 2), the relationships between the ogive and box-and-whisker plot, and the calculation of the first, second, and third quartiles. A description of the number line on which the box-and-whisker was drawn, with its mathematical connections, were also elucidated during his lesson (see Figure 2). A conceptual-based instructional approach endeavours to provide the reasons that make algorithms and formulae work (Peal, 2010). The emphasis is placed on the learners' understanding of the relationships and connections between important statistical concepts such as the use of quartiles to construct the box-and whisker plots on a number line (see Figure 2). Overall, Teacher A implemented more of a rule-oriented procedural knowledge approach in teaching histogram and box-and-whisker plot construction than a conceptual one. To summarise, he used both knowledge approaches except that one was dominant.

Through non-verbal cues of nodding their heads, the learners indicated that they grasped the lesson on histogram construction through the use of conceptual knowledge better than when Teacher A adopted a rule-oriented approach. This observation was supported by the fact the learners were able to recall and apply the procedures posed by him. For example, the learners calculated the percentage of learners in the age group of (26-40) years as 37% using the frequency table and histogram that was constructed (Figure 1).

Teacher A's preference for the use of procedural knowledge in teaching histograms was confirmed in the learners' workbooks (document analysis) and during the interview. It was discovered that the learners had written down the teacher's rules or steps on how to construct histograms and box-and-whisker plots, as well as the diagrams of histogram and box-and-whisker plots. Teacher A might have adopted the use of procedural knowledge because the construction of histograms which demands that specific procedural rules must be followed, is consistent with a conceptual understanding of the term. In studies conducted by Flockton, Crooks and Gilmore (2004) and Leinhardt et al (1990) on graphing, they stress that the construction of graphs requires a sequence of drawing the axes, choosing the scale, labelling the axes, plotting the points, and joining the lines of best fit. The order of the steps, in the case of Teacher A, demonstrated the required knowledge and skills for histogram construction.

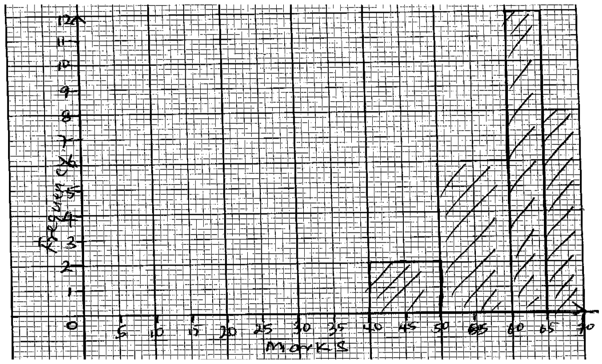

Figure 3 : An example of an incomplete classwork exercise on histogram construction due to incorrect scaling of the data axis.

As observed, the learners experienced learning difficulties, particularly labelling the data axis with incorrect scales, which could mean that he possibly presented his lesson in a limited way, that is, solely procedurally, without providing the reasons underlying the procedures and clarifying the relationship between concepts (a conceptual knowledge approach) in histogram construction (see Figure 3).

During class work, the learners tried to draw a histogram, which could not be accommodated on the graph paper provided because they scaled the data axis incorrectly (see Figure 3). It may be said that Teacher A's PCK in terms of subject matter content knowledge presentation did not always reveal the required variety of ways of presenting the data handling topics to his learners for ease of access. In some instances, he demonstrated the use of both procedural and conceptual knowledge in teaching histograms and box-and-whisker plots, but he predominantly used a set of algorithms to demonstrate graph construction. In the main lesson on histogram and box-and-whisker plots, he displayed factual knowledge, procedural proficiency, and conceptual understanding of the data handling topics that were taught.

4 Teacher B

Teacher B planned and taught his statistics lessons on bar graphs and ogives from the recommended mathematics textbooks and work schedules. He used a predominantly rule-driven formal procedural approach to statistical graphs (see Figure 4). As observed, in starting his lessons he tried to identify learners' prior knowledge of the new topic. For instance, he introduced bar graph construction and interpretation with a pre-activity that assessed learners' understanding of the way in which to prepare a frequency table. For example he asked,*prepare a frequency table of the following scores: 2, 3, 4, 5, 5, 6, 4, 7, 5, 6.* His use of pre-activities as diagnostic strategies to identify learners' pre-existing knowledge was also attested to in his responses to the teacher interview, questionnaire, and written reports.

Teacher B taught graphical constructions of bar graphs and ogives according to the learning outcomes of data handling as stated in the mathematics curriculum (DoBE, 2010).

These outcomes require that learners should be able to use appropriate measures of central tendency and spread to collect, organise, analyse, and interpre *prepare a frequency table of the following scores: 2, 3, 4, 5, 5, 6, 4, 7, 5, 6.*t data, in order to establish statistical and probability models for solving related problems (DoBE, 2011). Teacher B followed precisely the order in which the learning outcomes were stated in teaching his learners how to construct bar graphs and ogives. In practice this meant, as observed in his lesson, drawing the axes, choosing the scale, labelling the axes, plotting the points, and joining the line of best fit - in that order (see Figure 4). Teacher B demonstrated his PCK for drawing bar graphs in line with the sequence described.

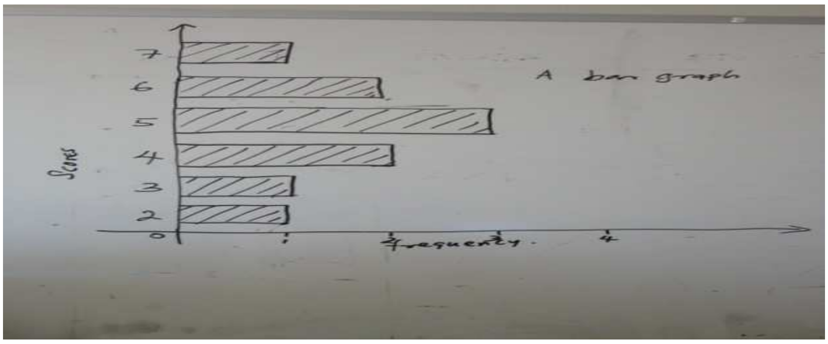

Figure 4 : Bar graph of the scores of learners in test on how to construct, analyze, and interpret a bar graph.

Flockton *et al* (2004) confirm that for a person to understand a graph, he or she should be able to use the construction skills of drawing the axes, labelling the axes, plotting the points, and joining the line of best fit to construct a graph.

Teacher B's assumed PCK on bar graphs and ogive constructions could be characterised as procedural in terms of his lesson planning and teaching approach. Teacher B's predominant use of a formal procedural approach was also triangulated in the analysis of his learners' workbooks (document analysis). The learners drew the bar graph and wrote down the teacher's steps on how to construct bar graphs and ogives. Teacher B might have been influenced to adopt a formal procedural approach because of the learning outcomes of data handling as laid down in the Curriculum and Assessment Policy Statement (CAPS) (DoBE, 2012). Besides, the construction of bar graphs and ogives demands specific procedural rules (Flockton *et al*, 2004 and Leinhardt *et al,* 1990).

Having said that, when the teacher merely taught them the rules for constructing bar graphs, some learners experienced misconceptions, confusing bar graphs with histograms, and histograms with ogives. Teacher B can be said to have presented his lesson in a limited way with insufficient explanations of how to choose the scales of grouped data (consisting of histogram, frequency polygon, ogive, and scatter plot) that are used to analyse and interpret large data. Further, Teacher B seems not to have the flexibility to present the topics to the learners in different ways because his lessons were presented solely according to a procedural knowledge approach.

A detailed description of the construction of bar graphs and ogives using a conceptual knowledge approach would have been ideal in presenting the lesson and would have avoided possible misconceptions and learning difficulties that the learners encountered in the lesson. Conceptual knowledge involves understanding mathematical ideas and procedures and includes basic arithmetic facts (Engelbrecht, Harding & Potgieter, 2005). It is rich in relationships among important mathematical concepts such as calculating the quartile positions and locating the quartile itself on the ogive, class intervals and boundaries, frequencies, and cumulative frequencies of an ogive. But Teacher B's teaching of bar graphs and ogives was dominated by a procedural knowledge approach which involves following a rule or procedure without a detailed explanation of the relationships and mathematical connections between the concepts being learned, such as calculating a quartile position and locating it in an ogive. Thus, the teacher is probably unable to present his lesson in a variety of ways to ensure better comprehension and understanding.

Baker *et al* (2001) and Bornstein (2011) note that a teacher who is unable to present mathematics content to learners in a variety of ways tends to expose them to learning difficulties, such as constructing a histogram instead of an ogive because of the use of an incorrect scale for labelling the data axis as was observed during the lesson on ogives. A combined approach of both procedural and conceptual knowledge would have helped to deepen the learners' understanding and would have avoided the misconceptions and learning difficulties that the learners had developed during the lesson, as suggested by Engelbrecht, Harding & Potgieter (2005).

5 Teacher C

Teacher C also displayed evidence of a procedural rather than a conceptual knowledge approach in his lessons on the construction of ogives and scatter. Repetition. Schneider and Stern (2010) view conceptual knowledge as mastery of the core concepts and principles and their interrelations in the mathematics domain; knowledge rich in relationships. On the other hand, procedural knowledge can be viewed as consisting of rules and procedures for solving mathematics problems. Procedural knowledge in mathematics allows learners to solve problems quickly and efficiently because it is to some extent automated through drill work and practice.

Teacher C demonstrated the requisite knowledge of and skills for constructing ogives and scatter plots in a step-by-step manner. For example, in his teaching on ogives, he moved from algorithmic to a conceptually meaningful stage. He began his lesson on ogives by trying to identifying the learners' prior knowledge of the concept of ogives through oral questioning, and the accuracy of the homework on histograms that had previously been taught (see Figure 5). Subsequently, using a cumulative frequency table prepared by the learners, an ogive was constructed by first drawing its horizontal and vertical axes. The data values were labelled on the horizontal axis (the upper class boundaries), and the cumulative frequencies on the vertical axis (see Figure 5). A scale was chosen by the teacher, who indicated that he had chosen it by considering the highest and lowest values of the frequency and data values. The points were plotted and the line of best fit was joined to produce the ogive (see Figure 5).

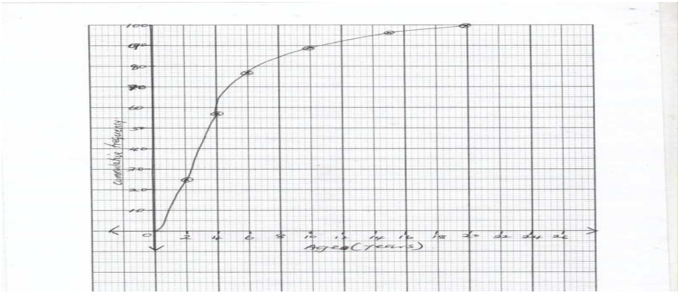

Figure 5 : An Ogive of age distribution of sample of 100 cars owners park in a car park.

This process of constructing an ogive from grouped data depicted a rule-oriented procedural approach. His procedural knowledge in teaching ogives (which was understandable to his learners) is believed to have been developed as a result of five years' mathematics teaching experience, and using the recommended lesson plan and work schedule of the Department of Education (DoE, 2010). The same procedural approach was used to teach scatter plots. Some of the factors that may have contributed to Teacher C teaching scatter plots in a step-wise manner, following a particular order or sequence, could be attributed to the way in which the learning outcome of data handling is stated in the mathematics curriculum (DoBE, 2010). The document indicates that competency in graphing requires that the learner is able to construct, analyse, and interpret statistical and probability models to solve related problem. The construction of graphs, as stated, entails scaling, drawing axes, labelling the axes, plotting points, and joining the line of best fit (Flockton *et al*, 2004; Leinhardt et al, 1990). Teacher C followed this sequence for teaching scatter plots. In the lesson he gave a full explanation of how to construct a scatter plot before demonstrating how to analyse and interpret it. The learners did their classwork in groups. They were presented with exercises on scatter plots, and were requested to analyse and interpret the plots to determine whether there was a correlation between the variables X and Y.

Teacher C's preferred procedural approach to teaching the topic was confirmed in the learners' workbooks, portfolios, teacher interview, and written reports. Owing to the limited use of conceptual rather than procedural knowledge – namely knowledge of the core concepts and principles and their interrelations in teaching ogive and scatter plots, some learners displayed misconceptions and learning difficulties in their analysis and interpretation of scatter plots. For example, a negatively correlated linear scatter plot was interpreted by the learners as having no correlation because of an outlier that lay far from the line of best fit (see Figure 6).

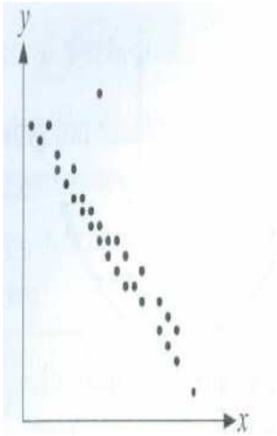

Figure 6 : A negatively correlated scatter plots.

This misconception could be attributed to the rule-oriented approach that had been adopted to describe the construction of scatter plots which did not allow for sufficient explanation of the interrelationships among the data values, frequencies, lines of best fit, and outliers (Ijeh, 2013). The learning difficulty of interpreting a negatively correlated scatter plot as having no correlation owing to outliers may further indicate that in teaching construction of scatter plots the teacher did not explain an outlier, line of best fit, type and nature of correlation, and how the presence of an outlier affects the correlation of the X and Y variables of the scatter plot.

What can be gleaned from the discussion so far is that teachers need to possess deep conceptual understanding of the mathematics concepts that they are teaching and must be able to illustrate why mathematical algorithms work and how these algorithms can be used to solve problems in real-life situations (Nicholson & Darnton, 2005). The learning difficulties experienced by the learners were subsequently addressed by Teacher C during post-activity discussions (instructional strategy); a strategy that was frequently used by him during his lessons on ogives and scatter plots.

An important task of any teacher is to attempt to transform the content to be taught in such a way as to make it comprehensible to learners (Mohr & Townsend, 2002). Teacher C also displayed evidence of a conceptual approach by providing the reasons that make the algorithm and formula work, and by explaining the relationships between important statistical concepts as well as the mathematical connection between them during the lessons on ogives. It was significant that more learners seemed to possess a better grasp of the topic and were able to construct and interpret ogives by means of this approach rather than a procedural one. In the particular lessons observed, Teacher C explained the mathematical connections and relationships between quartile positions and the quartiles and how quartiles can be used to interpret ogives. In doing so, Teacher C could be regarded as having displayed adequate subject matter content knowledge of school statistics.

6 Teacher D

In Teacher D's observed lessons, it was noted that he had planned and taught his lessons on bar graphs and histograms using the Department of Basic Education's mathematics work schedule, and the recommended textbooks as sources of information. This was confirmed in the teacher questionnaire and interview. For instance when he was asked, *"What learning and teaching support materials do you use in teaching statistics?"* Teacher D responded, *"I use classroom mathematics by Laridon et al, 2006 and the work schedule."*

During his teaching of bar graph and histogram construction, he used more of a procedural approach to teaching bar graphs and histograms than a conceptual one. For example, Teacher D taught the lesson on bar graphs in a step-by-step manner, beginning with pre-activities to identify the learners' prior knowledge of bar graph construction, followed by the preparation of a frequency table compiled by the learners using a familiar daily life example (see Figure 7). In this case, a frequency table was prepared of the number of cars in a car park according to their make. Next, with the help of the frequency table, a bar graph was constructed by drawing its horizontal and vertical axes and labelling them appropriately. A scale was chosen by the teacher with the explanation that this was done by considering the highest and lowest values of the frequencies and the companies that manufactured the cars. Next, the points were plotted and the line of best fit was joined to produce the bar graph (see Figure 7). The teacher's specific strategy for teaching bar graph construction followed a rule-oriented procedural approach using procedural knowledge.

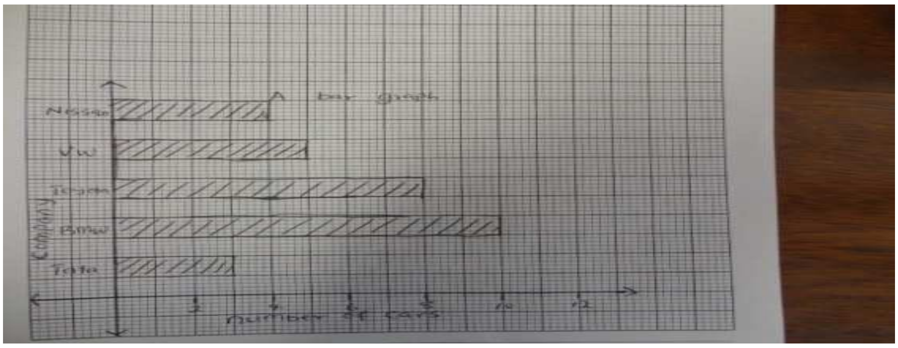

Figure 7 : Bar graph showing the numbers of makes of cars in a car park.

Engelbrecht *et al* (2006) describe the procedural knowledge approach as "following a rule or procedures flexibly, accurately, efficiently and appropriately in completing a given task". For example, in constructing a statistical graph, a procedural knowledge approach requires a series of actions such as drawing the axes, choosing the scale, labelling the axes, plotting the points, and joining the line of best fit. But what may sometimes be challenging is knowing how to move from the procedural stage to a conceptually meaningful one in terms of student learning.

As with the other teachers, Teacher D's procedural knowledge may have been developed over his 15 years of teaching mathematics in high school, using the recommended lesson plan and work schedule for statistics (DoBE, 2010). It could be suggested that although Teacher D possesses adequate ways of presenting bar graph construction to his learners, his PCK may be limited in the sense that he presented his lesson procedurally, an approach that was not always responsive to the learners' needs. Consequently, some of the learners constructed the classwork task without leaving spaces between the bars of the graph. The inability to consider the consistency of spaces between the bars of a graph during lesson presentation resulted in learning difficulties during classroom practice.

According to Shulman (1987), representation involves a teacher thinking through key ideas and identifying alternative ways of presenting them to learners. It is a stage in which suitable examples, demonstrations and explanations are used to build a bridge between the teacher's comprehension of the subject matter and what is required for the learners (Ibeawuchi, 2010). Multiple forms of representations are highly desirable if one is to be successful in the teaching process (Rollnick *et al*, 2008). Teacher D, in certain graphing topics, displayed evidence of an alternative conceptual knowledge approach in teaching histograms. Engelbrecht *et al* (2005) describe a conceptual knowledge approach as "involving an understanding of mathematical ideas and procedures consisting of the knowledge of basic arithmetic facts". It is knowledge rich in relationships and understanding of important statistical concepts in bar graph and histogram constructions. In the lesson observed, Teacher D explained in detail the meaning of a histogram. According to him, **"**a histogram is a graphical representation, showing a visual impression of the distribution of grouped data. It consists of tabular **frequencies** shown as adjacent rectangular bars, erected over discrete intervals, with an area equal to the frequency of the observations in the interval. Unlike the bar graph, a histogram is used to represent a large set of data (e.g. a population census) visually, but with no spaces between the bars". His conceptual approach (presumably PCK) to teaching the construction of a histogram enhanced conceptual understanding of the topic as the learners seemed to be satisfied with Teacher D's conceptual explanation of how to construct a histogram after they had experienced misconceptions and learning difficulties in labelling the data axis. They displayed non-verbal cues of nodding their heads in agreement with the teacher's explanation.

From the lessons observed with Teacher D, he used a procedural rather than a conceptual knowledge approach. His preferred use of this approach was confirmed in the document analysis of the learner workbooks and written reports. The learners had completed the diagrams on bar graphs and histograms efficiently, with indications of the procedures that had been adopted in constructing these statistical graphs. Star (2002) argues that it is important for practising teachers to possess both kinds of knowledge in order to impart teaching to the learners in a meaningful way. The use of both a rule-oriented procedural and a conceptual knowledge approach reveals that teachers are looking for ways of making the teaching of bar graphs and histograms comprehensible and accessible to their learners. Moreover, the construction of graphs demands that a particular order of actions should be followed, consistent with conceptual understanding. Teacher D can therefore be said to possess and demonstrate the required knowledge of bar graph and histogram construction.

7 Conclusion and Recommendation

The four participating teachers taught statistical graphs predominantly using procedural knowledge and less frequently conceptual knowledge. The use of procedural knowledge was to some extent dictated by the nature of the topic which requires learners to be able to collect, organise, analyse, and interpret statistical and probability models to solve related problems (DoBE, 2010). A second factor that leads to the use of procedural knowledge is the way in which statistical graphs should be constructed which involves drawing axes, choosing scales, labelling axes, plotting points, and joining the lines of best fit. Other ways in which the teachers demonstrated the subject matter content knowledge they possess, included the frequent use of mathematics textbooks and CAPS documents. They develop additional subject matter content through using the above-mentioned resources and attending content knowledge workshops. This study has provided a critical analysis of the PCK (subject matter content knowledge) that the selected mathematics teachers demonstrated during the teaching of statistics to enhance continuous improvement in the development of teacher education programmes for in-service and pre-service mathematics teachers.

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Methods of Least Squares and its Geometry

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One of the best known applications of Linear Algebra is method of fitting a function to exponential data called the method of least squares.

Consider an experiment involving two measurable variables x and y where y is approximately at least a linear function of x. Assume that we have some observed values of the variables x and y as $(a_1, b_1), \ldots, (a_n, b_n)$ in the xy-plane. If they are really in a linear relation and the data were free from error they will lie on a straight line whose equation can be written. But in practise, it is not the case often. So we need to find a straight line which 'best fits' the given data. Least squares arise in seeking the best fitting line.

Consider the relation $y = cx + d$. Condition for the line to pass through the points are

$$ca_1 + d = b_1, \ldots, ca_n + d = b_n$$

Most of the cases, these equations will be inconsistent. So we look for reals c and d which come close to satisfying this linear system in the sense that the total error is minimized. A good measure of total error is

$$(ca_1 + d - b_1)^2 + \cdots + (ca_n + d - b_n)^2$$

This is the sum of squares of the vertical deviations of the line from the data points. Squares are taken to avoid negative signs. So the line fitting problem is just a particular instance of a general problem about inconsistent linear systems.

Suppose we have a linear system of m equations in n unknowns $x_1, x_2, \ldots, x_n$; $AX = B$. Since the system is inconsistent, the problem is to find a vector X which minimizes the length of the vector $AX - B$ or a better deal, its square $E = \|AX - B\|^2$.

A vector X which minimizes E is called a least squares of solution of the linear system $AX = B$. A least squares solution is an actual solution if the system is consistent.

Normal System

Consider $AX = B, E = \|AX - B\|^2$. To minimize E.

Put $A = (a_{ij})_{i,j=1}^{m,n}$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. The i^{th} entry of $AX - B$ is $\left(\sum\limits_{j=1}^{n} a_{ij}x_j\right) - b_i$. Therefore

$$E = \|AX - B\|^2 = \sum_{i=1}^{m} \left(\sum a_{ij}x_j - b_i\right)^2$$

which is a quadratic function of $x_1, x_2, \ldots, x_n$. Now we use several variable calculus to get the absolute minima. First we find the critical points of the function E.

$$\frac{\partial E}{\partial x_k} = \sum_{i=1}^{m} 2 \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) a_{ik} = 0$$

$$\Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j \cdot a_{ik} = \sum_{i=1}^{m} a_{ik} b_i, \quad k = 1, 2, \ldots, n$$

This is a new system of linear equations in $x_1, \ldots, x_n$ whose matrix form is

$$\left(A^T A\right) X = A^T B$$

It is called the normal system of the linear system $AX = B$. Solutions of normal system are the critical points of E. Surely E does have an absolute minimum because it is a continuous function with non-negative values. Since E is unbounded when $|x_j|$ is large its absolute minimum must occur at critical points. Therefore

Theorem. Every least square solution of $AX = B$ is a solution of the normal system $(A^T A)X = A^T B$.

Question

(a) What if the normal system is inconsistent ? (Not possible if so no progress).

(b) Even if the normal system is consistent, will all solutions be least squares solutions ?

To resolve this we look at some Linear Algebra results.

Consider an $m \times n$ real matrix A. [We can consider it as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.]

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

Let S be the vector space obtained as the span of column vectors of A, it is called the column space of A. Note that it is a subspace of $\mathbb{R}^m$. Nullspace of A is $N(A) = \{X \in \mathbb{R}^n\}$.

Theorem. Let A be a real matrix. Then

(i) $N(A) = (\text{column space of } A^T)^\perp$

(ii) $N(A^T) = (\text{column space of } A)^\perp$

(iii) column space of $A = \left(N(A^T)\right)^\perp$

(iv) column space of $A^T = (N(A))^\perp$

Here if $V \subset \mathbb{R}^n$ then $V^\perp = \{u \in \mathbb{R}^n \mid \langle u, v \rangle = 0 \text{ for all } v \in V\}$. Moreover, $V \oplus V^\perp = \mathbb{R}^n$.

Note :

If $A : \mathbb{R}^n \to \mathbb{R}^m$ then $A^T : \mathbb{R}^m \to \mathbb{R}^n$ and column space of A^T is a subspace of $\mathbb{R}^n$ and null space of A^T is a subspace of $\mathbb{R}^m$.

Let A be having all its columns linearly independent. So they form a basis for the column space S of A. Projection of a vector X on the column space S is Y where $X - Y \in S^\perp$. That is $(X - Y) \perp S \Rightarrow (X - Y) \cdot$ column vectors of $A = 0$. So find a vector Y such that $(X - Y) \cdot$ column vectors of $A = 0$. Then Y is a projection of X on S.

Theorem. Let A be a real $m \times n$ matrix. Then $A^T A$ is a symmetric $n \times n$ matrix whose null space equals the null space of A and whose column space equals the column space of A^T.

Proof. We have $(A^T A)^T = A^T A$. So $A^T A$ is symmetric. Let $S =$ column space of A. Then $N(A^T) = S^\perp$.

$$\begin{aligned} X \in (A^T A) &\Leftrightarrow (A^T A)X = 0 \\ &\Leftrightarrow A^T(AX) = 0 \\ &\Leftrightarrow AX = N(A^T) \end{aligned}$$

That is $AX \in S^\perp$. But $AX \in S$. Since S is the span of columns of A, $S \cap S^\perp = \{0\}$, hence $AX = 0 \Rightarrow X \in N(A)$. If $X \in N(A)$ then $AX = 0 \Rightarrow A^T AX = 0 \Rightarrow X \in N(A^T A)$. So $N(A^T A) = N(A)$. Column space of $A^T A = (N(A^T A))^\perp = (N(A))^\perp =$ column space of A^T. □

Theorem (Fundamental Theorem about the MLS)**.** Let $AX = B$ be a linear system of m equations in n unknowns.

(a) The normal system $(A^T A)X = A^T B$ is always consistent and its solutions are exactly the least squares solutions of $AX = B$.

(b) If $\operatorname{rank} A = n$, then $A^T A$ is invertible and there is a unique least square solution of the normal system, namely
$$X = (A^T A)^{-1} A^T B$$

Proof. We have column space of $A^T A =$ column space of A^T. Therefore, column space of $[A^T A | A^T B] =$ column space of $A^T A$. Since the extra columns of $A^T B$ is a linear combination of the columns of A^T and thus belong to the column space of $A^T A$. So the rank of the cofficient matrix and the rank of the augmented matrix of the normal system are the same. This implies that the normal system is consistent.

Now we show that every solution of the normal system is a least square solution of $AX = B$. Suppose X_1, X_2 are two solutions of the normal system. Then $A^T(X_1 - X_2) = A^T B - A^T B = 0$. So $Y = X_1 - X_2 \in N(A^T A) = N(A) \Rightarrow AY = 0$. Now $AX_1 - B = A(Y + X_2) - B = AX_2 - B \Rightarrow E = \|AX - B\|^2$ has same value for $X = X_1$ and $X = X_2$. Thus all solutions of the normal system give the same error vlaue for E. Since every least square solution of $AX = B$ is a solution of the normal system, it follows that the solutions of the normal system constitute the set of all least squares solution, as claimed.

Let $\operatorname{rank} A = n$. Then $\operatorname{rank} A^T A = n$. Therefore $A^T A$ is invertible and so $X = (A^T A)^{-1} A^T B$ is the unique solution. □

Geometry of Least Squares Process

Let $AX = B$; A is an $m \times n$ matrix and let S be the column space of A. Least squares solution are the solutions of the normal system $A^T AX = A^T B$. In otherwords

$$A^T(B - AX) = 0$$
$$\Rightarrow (B - AX) \in N(A^T) = S^\perp$$

Therefore X is a least square solution of $AX = B$ if and only if $(B - AX) \in S^\perp$. Now $B = (B - AX) + AX$ and $AX \in S$. That is B can be uniquely written as the sum of its projections on S and $S^\perp$ since $(B - AX) \in S^\perp$ and AX is the projection of B on S. Thus

Theorem. Let $AX = B$ be an arbitrary linear system and let S be the column space of A. Then the column vector X is a least square solution of the system $AX = B$ if and only if AX is the projection of B on S.

Optimal Least Squares Solutions

If rank $A < n$ there will be infinitely many least square solutions. Now to see how one can choose one solution that is in certain sense optimal.

Natural way is to select one least square solution with minimal length. So optimal least square solution of $AX = B$ is a least square solution X whose length $\|X\|$ is as small as possible.

Method to find optimal least square solution

Let $U = N(A) =$ (column space of $A^T)^\perp$. Suppose X is a least square solution of $AX = B$. Then there exists a unique expression $X = X_0 + X_1$ where $X_0 \in U$ and $X_1 \in U^\perp$. Then

$$AX = AX_0 + AX_1 = AX_0 \quad (\text{since } x_0 \in N(A))$$

Thus $AX - B = AX_1 - B$. So X_1 is also a least square solution of $AX = B$. Now

$$\begin{aligned} \|X\|^2 &= \|X_0 + X_1\|^2 \\ &= (X_0 + X_1)^T (X_0 + X_1) \\ &= X_0^T X_0 + X_1^T X_1 \\ &= \|X_0\|^2 + \|X_1\|^2 \end{aligned}$$

Therefore

$$\|X\|^2 = \|X_0\|^2 + \|X_1\|^2 \geq \|X_1\|^2$$

So if X is an optimal solution, then $\|X\| = \|X_1\|$ so that $\|X_0\| = 0$. Thus $X = X_1 \in U^\perp$. So optimal least square solution must belong to $U^\perp =$ column space of A^T. Now we show that there is a unique least square solution in $U^\perp$.

Let $X, \hat{X}$ be two solutions in $U^\perp$. Then $AX, A\hat{X}$ both equal to projection of B on the column spaces of A. We have

$$A(X - \hat{X}) = 0 \Rightarrow X - \hat{X} \in N(A)$$

But

$$X, \hat{X} \in U^{\perp} \Rightarrow X - \hat{X} \in U \cap U^{\perp} = \{0\} \Rightarrow X = \hat{X}$$

Hence X is the unique optimal least square solution belonging to $U^{\perp}$. Thus we have

Theorem. A linear system $AX = B$ has a unique optimal least square solution namely the unique vector X in the column space of A^T such that AX is the projection of B on the column space of A.

Instructional design and the genetic decomposition of a concept

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Abstract : This paper reports on the APOS (Actions, Processes, Objects and Schema) approach to difficulties experienced by first year engineering students at a University of Technology in constructing the concept of the factor theorem. A proposed initial genetic decomposition (IGD), describing the mental constructs which students are supposed to make in order to understand the chain rule was suggested. Instructional treatment followed the activities, classroom discussions and exercises (ACE) model proposed by Dubinsky (1991). This paper in particular presents the discussions on interviews with group representatives seeking clarity on responses to four exercises on differentiation of trigonometric functions done collaboratively in class on the understanding of the chain rule. In a class of 78, students worked collaboratively in 12 groups of about six participants each. The interviews in this study were conducted with selected individuals from the different groups for clarity and explanations on written responses. This was done to get feedback on how the students perceived the chain rule and to fulfil the verification purpose where the group response was clarified. These interviews were semi-structured and questions were open-ended. They followed a guide designed to elicit the students' understanding of the chain rule based on the tasks given. Analysis of results revealed to a greater extent a process understanding of the chain rule concept and when using the Triad (intra-, inter- and trans-) mechanism to explain the interview discussions, it was revealed that most students operated on inter- stage. Differentiation as a process was complete with most groups but they struggled with basic algebraic manipulations, understanding of composition and decomposition of functions.

Keywords : APOS, genetic decomposition, trigonometric functions, chain rule, calculus and Triad mechanism

1 Introduction

The main issue in this paper is how students conceptualise mathematical learning in the context of calculus with specific reference to the chain rule. The paper focuses on how students use the chain rule in finding derivatives of composite functions (including trigonometric ones). The research was based on the APOS (Action-Process-Objects-Schema) approach in exploring conceptual understanding displayed by first year, University of Technology students in learning the chain rule in calculus. Dubinsky & McDonald (2001) suggested that APOS theory as a tool can be used objectively to explain students' difficulties with a broad range of mathematical concepts and recommended ways in which students can learn these concepts. They further argued that this theory can point us towards pedagogical strategies that lead to marked improvement in (1) student learning of complex or abstract mathematical concepts, and (2) students' use of these concepts to prove theorems, provide examples, and solve problems.

A schema for a certain mathematical concept is an individual's collection of actions, processes, objects and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept (Dubinsky & Mc Donald, 2001). Asiala et al, (2004) asserted that an individual's schema is the totality of knowledge which for her is connected consciously or unconsciously to a particular mathematical topic, for example an individual may have a function schema, derivative schema, chain rule schema.

The chain rule is used to find the derivatives of composite functions. Kaplan (1984) referred to it as a function of functions. A composite function is a function that is composed of two or more functions. For the two functions f and g, the composite function or the composition of f and g, is defined by

$$(f \circ g)(x) = f(g(x))$$

The function $g(x)$ is substituted for x into the function $f(x)$. For example, the function $h(x) = (3x - 9)^4$ could be considered as a composition of the functions, $f(x) = x^4$ and$g(x) = 3x - 9$. However, it could also be written as a composition of $f(x) = (3x)^4$and $g(x) = x - 3$. Often, a function can be written as a composition of several, different combinations of functions. One must be careful to consider the domain of the respective functions.

The chain rule allows us to find the derivative of composite functions. The chain rule states that if f and g are differentiable functions and $F(x) = f(g(x))$, then F is differentiable and the derivative of F is given by $F'(x) = f'(g(x))g'(x) = (fog)'(x)$

In Leibniz notation, if $y = f(u)$, $u = g(x)$ and y and u are differentiable functions, then$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. Kaplan (1984) chose to call this rule, the composition rule since the function to be differentiated is a composition of other functions. The same applies when a function is a product we use the product rule to get its derivative. The first year syllabus deals with a combination of a maximum of five functions that can be used in the composition. We can have more than one composition in a problem. The students should now be able to decompose the given function into its elementary pieces one step at a time. Kaplan then proposed the following table of derivatives with all possible compositions of functions. All of the formulas in the table were derived from the general chain rule with *f(x)* as one of the main functions, $x^n; e^x; \ln x; \sin x; \cos x$ and an arbitrary function *g(x)*.

The chain rule is of important use to other areas of calculus. These include: (1) Finding the marginal Physical Productivity Function of the workers in Business economics $(\frac{dP}{dx})$ for $P = 10(3x + 2)^3 - 10$, (2) Revenue changing when given a revenue function like $R(x) = 25(x+2)^2 + 20x - 5$,(3) Higher order differentiation used to calculate demand, cost and profit in business and (4) Calculations of rates on physical body relationships including body weight and surface area, cell growth, blood flow and other physical quantities. It is important for the students at this stage to know which formula to use and how to use it without computing the derivatives of the component functions. They must be able to identify whether a constant times a function, sum of functions, product, quotient, composition or piecewise functions are given in the problem. The implication here is that they should be well versed with function algebra.

The chain rule is included in several studies in mathematics education literature. Some of them are about teaching of the chain rule (Lutzer, 2003; Mathews, 1989; Thoo 1995; Uygur & Ozdas, 2007) while others are on understanding the rule. Uygur (2010) who studied the cognitive development of applying the chain rule through the three worlds of mathematics suggested that the instructional way of presenting the chain rule changed focus to encourage students to obtain the chain rule with some life-related problem situations. In contrast, verifying the chain rule by using either or both graphing software or graphics calculator and an algebraic approach was considered for developing teaching and learning strategies of the chain rule in the mathematics teaching program of South Australia (SACE Board of South Australia, 2009). Uygur (2010) further noted that as much as there was an absence of studies on structural development of the chain rule, there was also a need for a study on students' applying the chain rule to second order derivatives and to two-variable composite functions. It was noted also, that the prerequisite knowledge of composite function is another significant notion for applying the chain rule by raising awareness of the relation among various cases. Uygur inferred that variable notion is another significant prerequisite knowledge in the embodied world of the cognitive development of the chain rule. Novotna and Hoch (2008) had indicated the importance of structural knowledge in applying the chain rule in the cognitive development of mathematical concepts. Students' application of the chain rule was analyzed within Tall's (2007) framework containing three levels of understanding which considered symbolic development. Their study addressed the structural development of the chain rule. On the contrary this study focused on the discussion of the types of structures constructed by students when learning the chain rule with the view to clarifying their understanding: (i) of the composition of function and (ii) of the derivative.

2 Theoretical Framework

The main mechanism for an individual to obtain new mathematical meaning is for him/her to construct mental representations of direct experiences relevant to that concept. A structured set of mental constructs which might describe how the concept can develop in the mind of an individual is called the **genetic decomposition** of that particular concept. The initial genetic decomposition (IGD) of the concept of the chain rule suggested below guided the researcher's teaching instruction in class and the construction of the interview and discussion tasks. APOS ascertains that to understand a mathematical concept begins with manipulating previously constructed mental or physical objects in the learner's mind to form **actions**; actions would then be interiorised to form **processes** which are then encapsulated to form **objects** (Dubinsky, 1991). These objects could be de-encapsulated back to the processes from which they are formed, which would be finally organized in **schemas**. Understanding the chain rule was explored in relation to the schema relevant to it. For an elaboration of these concepts refer to Maharaj (2010, p43).

This study has therefore adopted the APOS approach (Dubinky, 1991a), based on intuitive appeal as there has been little empirical research done documenting its impact on students' conceptual understanding of the chain rule in the African continent. Also, APOS has been used in research focusing on understanding of various mathematical concepts, (Pascual, 2004; Sfard, 1991; Tall, 1994; Dubinsky, 1991a; De Vries, 2001; Gray & Tall, 2002; Clark et al, 1997). This study was conducted according to a specific framework for research and curriculum development in advanced mathematics education, which guided the systematic enquiry of how students acquire mathematical knowledge and what instructional interventions contribute to student learning. The framework consists of three components: theoretical analysis, instructional treatment, and observations and assessment of student learning as proposed by Asiala et al (2004) and illustrated in Figure1.

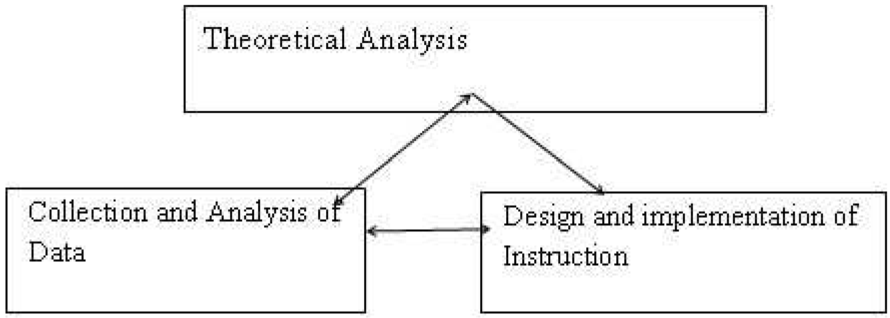


Figure1: Theoretical Framework for Research

Theoretical analysis includes the initial genetic decomposition of the concept which specifies the mental constructs which a learner is expected to have in order to understand the chain rule. The initial genetic decomposition of the chain rule was assumed as:

For a student to have his or her function schema

(a) He/she had developed a process or object conception of a function and

(b) Has developed a process or object conception of a composition of functions.

For a derivative schema,

(a) He/she had developed a process conception of differentiation

(b) The student then uses the previously constructed schemas of functions, composition of functions and derivative to define the chain rule. In this process the student recognized a given function as the composition of two functions, took their derivatives separately and the multiplied them.

(c) The student recognized and applied the chain rule to specific situations. The initial genetic decomposition is modelled in Figures 2 and 3 in the following section and is adopted from Jojo (2013).

$$f^{'}(x) = 2\tan x \sec^2 x f^{'}(x) = 2\tan x \sec^2 x f(x) = \tan^2 x f(x) = \tan^2 x$$

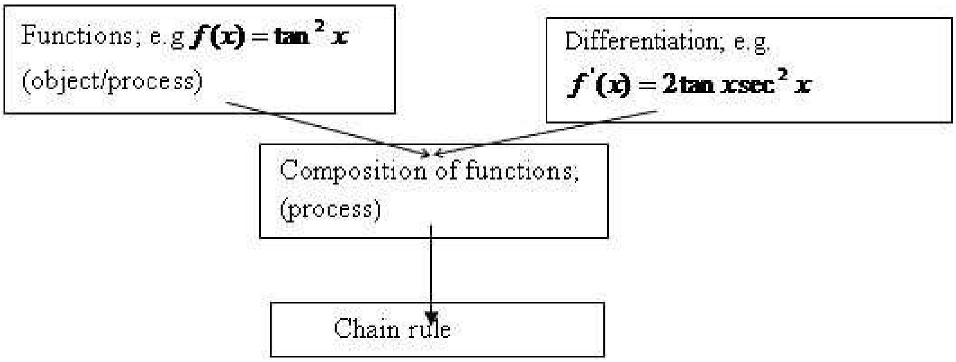


Figure 2: Initial genetic decomposition of the chain rule

3 Instructional Treatment

The design and implementation of instructional treatment was based on the initial genetic decomposition (IGD). The pedagogical approach included three sequential lessons intended to increase the students' understanding of the chain rule to the object or schema stages of APOS. These included revision of function notations, composition of functions and differentiation of trigonometric using product and quotient rules. The chain rule was defined for the purposes that the students should not only know it but be able to remember it, use it and apply it to various problems. Scaffolding was used to assist students to attain a higher level of understanding by encouraging creative and divergent thinking (Brush & Saye, 2001, Mcscosker & Diezman 2009). Anghileri (2006) asserts that students actively construct meaning as they engage significantly within established mathematical practices. These tools in a mathematics classroom could include diagrams, pictures, technology, mathematics formulas and hints for an effective solution process.

The principles of effective mathematics teaching drawn from educational theories of Piaget illustrated that learning required interaction to develop: (1) a deep conceptual understanding, (2) positive relationships and (3) a classroom community. This social interaction leads to gradual, incremental changes in thought and behaviour of learners and through which interaction with other learners, allows them to examine, clarify and change their conceptual understanding. This study sought to explore how actions, processes and objects of the chain rule schema could be coordinated as mental structures to enhance the learning of the concept and access it in situations where it needs to be applied.

Students were provided with activities in class that were designed to induce them to make the suitable mental constructions as suggested by the initial genetic decomposition. There were key instruments designed to assess the content knowledge and pedagogical content knowledge of the student in presenting the mathematics content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK) regarding the concept of the chain rule. This study also considered the factors influencing the development of the lecturer's knowledge and how the knowledge is related to lecturer performance and student achievement. The researcher was the lecturer in this study. She had always experienced problems in teaching this concept.

The chain rule is the underlying concept in many applications of calculus: implicit differentiation, solving related rate of change problems, applying it in the fundamental theorem of calculus and solving differential equations. Research (Hassani, 1998) into the nature of students' understanding of the concepts underlying the calculus showed significant gaps between their conceptual understanding of the major ideas of calculus and their ability to perform procedures based on these ideas. The chain rule states that if $g(x)$ is a function differentiable at c and f is a function differentiable at$g(c)$, then, the composite function fog given by $(fog)(x) = f(g(x))$is differentiable at c and that$(fog)'(c) = f'(g(c)).g'(c)$. Cottrill (1999) asserted that: (1) conventional wisdom holds that students' conception of the chain rule (as with other rules) is that of symbol manipulation, (2) the conception of the chain rule appeared to be a straight-forward manipulation of symbols which could easily be applied in problem situations and (3) concluded that an application based on symbol manipulation carries a heavy requirement for the function to be given by an expression, fostering students' tendencies toward instrumental understanding, where they are unable to apply the chain rule.

The research aimed *to find out whether students can construct an underlying structure of the chain rule in dealing with composition or decomposition of functions*. This focus was accomplished by: Determining the students' actual engagement with tasks in groups and how these tasks link with the expected outcomes highlighted in the initial genetic decomposition.

Ernest (1991) asserts that Mathematics Education understood in its simplest and most concrete sense concerns the activity or practice of teaching mathematics. He further asserts that learning is inseparable from teaching. This process involves the exercise of the mind and intellect in thought, enquiry, and reasoning. Similarly, the *interpretive research paradigm* seeks to explore real human and social situations and uncover the meanings, understandings and interpretations of the actors involved. It was therefore evident that in exploring how students conceptualized the understanding of the chain rule, APOS could be used objectively to (1) explain students' difficulties with the chain rule and (2) suggest ways that students can learn the chain rule. More specifically APOS could lead us towards pedagogical strategies that in turn lead to marked improvement in (1) student learning of the chain rule and (2) students' use of this concept to solve problems in calculus.

Literature

Hiebert & Carpenter (1992) asserted that learning mathematics with understanding involves making connections among ideas, and that those connections are considered to facilitate the transfer of prior-knowledge to novel situations. With regard to the psychological approach to learning, the constructivist idea is that understanding is a continuing activity of individuals organizing their own knowledge structures, a dynamic process rather than an acquisition of categories of knowing (Confrey, 1994; Gagnon & Collay, 2001; Piaget, 1948/1973; Pirie & Kieren, 1994). According to Bransford, Brown & Cocking, (2000), a mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. They further asserted that the degree of understanding is determined by the number and the strength of the connections made with previously acquired mathematics. Thus a mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections.

It is therefore assumed that well-connected and conceptually grounded ideas enable their holders to both remember them and see them as part of a larger whole within which each part shares reciprocal relationships with other parts, (Pirie & Kieren, 1994).
Wiggins, (1993) defined understanding as something different that emerges when we are required to reflect upon achievement, in verifying or criticizing, re-thinking and re-learning what we know. Wiggins & McTighe (1998) further identified several inter-related aspects of understanding including (1) explanation, (2) interpretation, (3) contextual applications, (4) perspective, (5) empathy and (6) self-knowledge. Not all of these apply to each learning situation and they are not hierarchical or mutually exclusive. Students who can explain their ideas justify understanding by making connections and inferences. Those who apply knowledge demonstrate their ability to use what they have learnt in complex situations. Lastly, those who show self-knowledge recognize the limits of their understanding.

4 Data collection and analysis

Worksheets with four exercises on the use of the chain rule were issued to 12 groups of about six students each. There was space provided below each task in the worksheet for students' responses. This was done to reinforce the learning that took place in the three sequential lesson components. The aim was to provide students with opportunities to make applications of the chain rule they learnt and prepare them for the mathematics in which chain rule would be applied. The students worked collaboratively. The activities were designed to foster the students' development of mental structures called for in the initial genetic decomposition. The genetic decomposition assumed the actions, processes, and objects that play a role in the construction of a mental schema for dealing with the chain rule.
Whilst working in groups students discussed their results and listened to explanations given by fellow students. The students worked collaboratively on mathematics tasks designed to help them use the mental structures that they had built during previous lessons. In some cases, students worked on a task as a group, whilst in other cases they worked as individuals and then compared notes, and then negotiated a group solution to the problem. They then reported their results in the class. During this process, the emphasis was on: (1) discussions, (2) reflection explanations by the researcher where appropriate, (3) completion of the tasks by the students, and (4) understanding the use and application of the chain rule.
As the researcher moved from group to group, she noticed that some students used a lead pencil to record their responses on the worksheet. They were trying to avoid mistakes and allow correction of an incorrect response without spoiling the worksheet. In some groups, after transcriptions of agreed responses, all the members of the group satisfied themselves that the submitted response was appropriate. They argued from time to time of the positions where brackets should be inserted. Even after submissions of completed worksheets, other students continued convincing and teaching the inquisitive students on how the chain rule works.
It was so interesting to watch the students refering back to their notes in their books before attempting the questions. Asked about this Zazi, (one group member) answered*: I remember a problem that you did for us, it looked like this one. So I want to compare*

and then differentiate this one. Although Zazi is operating in the action stage, he needed to gain experience constructing actions similar and corresponding to differentiating using the chain rule. The experience of differentiation using the chain rule was built upon in subsequent activities like those in the worksheet, where he was asked to reconstruct familiar actions as general manipulations.

The researcher noticed that students in some groups would first copy a task in the worksheet onto their books. They would then work on it as individuals after which they compared their answers. Students argued and agreed upon certain responses. Individuals justified how they arrived at their responses. This way they taught each other and gave verbal descriptions of actions taken in their own words. They then repeated the actions many times with different tasks in their books and in the worksheet. Thus the worksheet helped the students interiorise the actions.

It was also noticed that most students in different groups were operating in the Intra-stage of the Triad. They had a collection of rules of differentiation with no recognition of relationships between them. Those students were helped by others who reflected on using the chain rule by applying actions to dynamic processes. The latter group had created an object of the chain rule. At the same time they applied actions on differentiation and as such the process of differentiating using the chain rule was encapsulated to form an object. The worksheets were analyzed for meaning which is one of the mechanisms necessary for understsnding a concept. These included detecting (1) the connections made by students to other concepts, (2) calculations made using the chain rule, (3) the chain rule technique used, and (4) mental images on which the chain rule is based.

All the groups applied the chain rule to the first task $y = \tan^2(3x + e^{\sqrt{x^2+1}})$ correctly using the *straight* form technique although only two out of twelve groups presented a solution with brackets, when they differentiated the composite function inside the brackets in the given task. One of the groups who left out the bracket then went on to detach the derivative 3 of $3x$ from the + sign. This 3 now multiplied the first two functions (see Extract).

Differentiate:

$$
\begin{aligned}
y &= \tan^2\left(3x + e^{\sqrt{x^2+1}}\right) \\
\frac{dy}{dx} &= 2\tan\left(3x + e^{\sqrt{x^2+1}}\right)\sec^2\left(3x + e^{\sqrt{x^2+1}}\right)\cdot 3 + e^{\sqrt{x^2+1}}\cdot\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x \\
&= 2\tan\left(3x + e^{\sqrt{x^2+1}}\right)\sec^2\left(3x + e^{\sqrt{x^2+1}}\right)\cdot 3 + e^{\sqrt{x^2+1}}\cdot\frac{1}{2\sqrt{x^2+1}}\cdot 2x \\
&= 6\tan\left(3x + e^{\sqrt{x^2+1}}\right)\sec^2\left(3x + e^{\sqrt{x^2+1}}\right) + e^{\sqrt{x^2+1}}\frac{2x}{2\sqrt{x^2+1}} \\
&= 6\tan\left(3x + e^{\sqrt{x^2+1}}\right)\sec^2\left(3x + e^{\sqrt{x^2+1}}\right) + e^{\sqrt{x^2+1}}\frac{x}{\sqrt{x^2+1}}
\end{aligned}
$$

Extract 1: One group's presentation of task 1

This mistake was not detected by any of the other members of the same group. Those students struggled with the connection of previously learnt algebraic skills like use of brackets where appropriate and manipulation of algebraic terms in a function. The calculations presented after differentiating using the chain rule successfully were therefore not correct for seven out of twelve responses received. The mental images constructed by the seven groups in using the chain rule were incomplete. Although the actions were interiorized into

processes, the processes were not encapsulated to objects. This could partly be attributed to previous knowledge of algebraic skills which were just actions and never interiorized. According to the Triad students in the said groups saw the chain rule as a procedure of differentiation which could not be connected or related to other processes applied to functions. Thus most students operated in the Intra- stage regarding task 1. This concurs with what Lakof & Nunez (1997) asserted that mathematics begins with direct human experience and ends there for some people. According to APOS, we observed that some students could only go as far as the action stage.
The second problem $y = (cos^2x + e^{\ sinx})^{\ 2}$ was presented correctly by nine out of twelve groups. Only one group avoided the use of the chain rule by squaring the given function and then differentiating. This was a brilliant idea but still required them to apply the chain rule on the individual terms, cos^4x, $2cos^2x\ e^{\ sinx}$ and e^{2sinx}. They then used *straight* form technique to differentiate. Those students were connecting the given function to a square of a binomial. Thus a part of understanding the concept of the chain rule is a mental process involving sorting out the given function, dealing with its composition, and connecting the two to find the derivative. They indicated a process construction of mental images since they transformed the given function to a trinomial which was operated on by repeating the actions of differentiation. Their work has been captured in Extract 2.

$$y = \cos^4 x + 2\cos^2 x \mathrm{e}^{\sin x} + \mathrm{e}^{2\sin x}$$
$$y' = 4\cos^3 x \sin x - 4\cos x \mathrm{e}^{\sin x} \sin x + 2\cos^2 x \mathrm{e}^{\sin x} \cos x + \cos x \mathrm{e}^{2\sin x}$$

Extract 2: Chain rule application after squaring a binomial

The third task required students to differentiate implicitly using the chain rule. Five groups out of twelve groups introduced natural logarithms on both sides of the equation before differentiating. They explained that they connected the relationships of exponentials in the right hand side function with logarithms which would get rid of the exponent. In this way they ended up with simple expressions on both sides and thus allowed them to use the *straight* form technique of chain rule differentiation (see Extract 3).
Differentiate implicitly

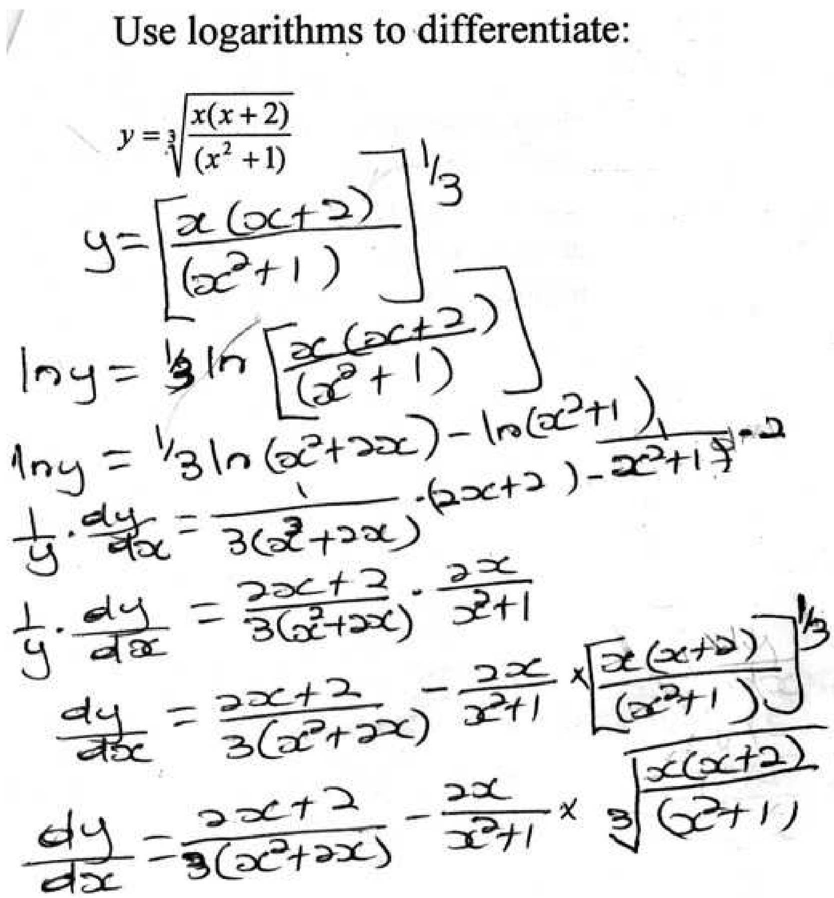
$$\begin{aligned}
\sin(x+y) &= \mathrm{e}^{2y+x^2} \\
\ln\sin(x+y) &= \ln\mathrm{e}^{2y+x^2} \\
\ln\sin(x+y) &= 2y + x^2 \\
\Rightarrow \frac{\cos x + y}{\sin(x+y)}\left[1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right] &= 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2 \\
\Rightarrow \cot(x+y)\left[1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right] &= 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2 \\
\Rightarrow \cot(x+y) + \cot(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2 \\
\Rightarrow (\cot(x+y) - 2y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 2 - \cot(x+y) \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2 - \cot(x+y)}{\cot(x+y) - 2y}
\end{aligned}$$

Extract 3: Differentiation using natural logarithms

Their calculations indicated a full understanding of the use of the chain rule. They operated in the Trans- stage of the triad since they could reflect on relationships between various objects from previous stages. They displayed coherence of understanding of differentiation rules and composition of functions. Three of the five groups presented responses of full construction of mental images of the chain rule and a connection between understanding of algebraic manipulations of the derivative and function composition. The other seven groups applied the chain rule directly using the *straight* form technique and then processed the resulting function to get the derivative. Two of the responses indicated a transition from an operational to a structural mode of thinking since they brought the concept of the chain rule into existence and used it with caution, and preferred it over other methods of differentiation.

$$y = \sqrt[3]{\frac{x(x+2)}{(x^2+1)}}$$

The last task involved differentiating by applying the chain rule.
Generally, one of two strategies was employed by students. The first form technique called for a specific connection between application of natural logarithms and differentiation. Only two groups displayed a coherent collection of the logarithmic rules and differentiation. Those groups were operating in the Trans- stage since they reflected on the explicit structure of the chain rule and were also able to operate on the mental constructions which made up their collection. Those students presented responses showing internal processes for manipulating logarithmic objects. Their schema enabled them to understand, organize, deal with and make sense out of application of the product rule, quotient, logarithmic rules and the chain rule. The other three groups could not apply logarithmic rules correctly and as such could not process the differentiation of the given task. This is illustrated in 4 where students resolved the surd form of the function correctly and took natural logarithms both sides of the equation. The interpretation of logarithms was then incorrect since a bracket was left out in step three of the response. Thus the function differentiated was not the originally given one. Even in their process of differentiation some brackets were still left out when they should have been there. Also the derivative of the last term, $-ln(x^2+1)$ in step four was recorded as $\frac{1}{x^2+1}2$ instead of $\frac{1}{x^2+1}2x$. In the next step the subtraction sign has been left out and then restored back again in the following one. The students in this group's actions indicated that they knew which steps to follow when differentiating. Their mental manipulations did not react to external cues of basic algebraic manipulations and as such transformation was not complete and their actions were not interiorized. Those students did not recognize the relationships between application of natural logarithms and algebraic manipulations resulting in multiplications when they were due and subtractions where appropriate. They perceived differentiation as a separate entities and even the rules applied were not remembered correctly.



Extract 4: Incorrect application of chain rule in differentiation

The other group employed the *straight* form technique after converting the surd form to its exponential form. However, they did not then utilize the product and quotient rules appropriately. Their actions were not interiorized with regards to logarithms and this had an impact on applying the chain rule in the given task. Their mental images could not be related to the string of symbols forming the expression, since they could not interpret both the symbols and or manipulations. Since calculations reflect the active part of mental constructions, the rules for these students were not perceived as entities on which actions could be made. Dubinsky (2010) asserts that in such cases the difficulty does not depend on the nature of the formal expressions, but rather in the loss of the connections between the expressions and the situation instructions.

5 Conclusion

The students' responses discussed above indicate that the instructional pedagogy should accommodate presentation of tasks that evoke rigorous deductive reasoning enabling the students to write and reflect on how they construct various mental images. A wide range of interactions between students themselves and between students and the researcher were discussed.

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A study on analytic solution procedures for solving Differential Equations

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Most of the problems occurring in real life, when mathematically modelled, turn out to be nonlinear differential equations. Since we are familiar with solving linear equations, we linearize the equation and get a solution! Analytic solutions to nonlinear differential equations are important since they give insight to the physical nature of the problem. In this lecture, we shall discuss some of the analytic solution procedures for solving differential equations. We concentrate on the following methods:

(a) Taylor series method

(b) Picard's method

(c) Adomian Decomposition Method (Shooting Type)

(d) Laplace-Adomian Decomposition Method (Shooting Type)

There are some more methods which are available in the literature like Homotopy analysis Method (Introduced by Liao S. J. in 1992), Homotopy Perturbation Method (introduced by J.H. He in 1998) and Variational Iteration Method (introduced by J.H. He in 1999).

Taylor series method

Given the first order differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$ the method calculates the higher order derivative from the differential equation and they are evaluated at the point x_0 using the initial condition and plugged in into the Taylor series of expansion of y about the point x_0.

Let us consider the problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x - y,\ y(0) = -1.$$

The exact solution is

$$y(x) = -3\mathrm{e}^{-x} - 2x + 2$$

We will develop a relationship between y and x by expanding y about x using Taylor series and finding the coefficients.

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + y''(x_0)\frac{1}{2!}(x - x_0)^2 + y'''(x_0)(x - x_0)^3\frac{1}{3!} + \cdots$$

Let $x - x_0 = h$. Then

$$y(x) = y(x_0) + y'(x_0)h + y''(x_0)\frac{1}{2!}(h)^2 + y'''(x_0)(h)^3\frac{1}{3!} + \cdots$$

The first term in the R.H.S of this equation is known from the initial condition

$$y(x_0) = y(0) = -1.$$

The second, third, fourth terms are obtained by successive differentiation of our equation:

$$\begin{aligned}
y'(x_0) &= y'(0) = -2 \times 0 - (-1) = 1 \\
y''(x_0) &= -2 - y'(x_0) \\
y''(x_0) &= y''(0) = -2 - 1 = -3 \\
y'''(x_0) &= -y''(x_0) \\
y'''(x_0) &= y'''(0) = -(-3) = 3
\end{aligned}$$

Now we write our series solution for y as

$$\begin{aligned}
y(h) &= y(0) + y'(0)h + \frac{y''(0)}{2!}h^2 + y'''(0)\frac{h^3}{3!} + \cdots \\
&= -1 + 1.0h - 1.5h^2 + 0.5h^3 - 0.125h^4 + \text{ error}
\end{aligned}$$

The main disadvantage of Taylor-series method it becomes awkward if the derivates become complicated.

Picard's method

Given the first order differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$ the method calculates the successive approximations y_n of the solution as follows:

$$y_n(x) = \ y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,\mathrm{d}t$$

with $y_0(x) = y_0$.

Consider the problem $y' = y^2,\ y(0) = 1$.
Here $x_0 = 0,\ y_0 = 1$ and $f(x, y) = y^2$. Hence the initial approximation to the solution is $y_0(x) = 1$. The successive approximations are calculated using the Picard's machine

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,\mathrm{d}t,\ n \geq 0.$$

A simple calculation yields,

$$\begin{gathered}
y_1(x) = 1 + x, \\
y_2(x) = 1 + x + x^2 + \frac{x^3}{3} \\
y_3(x) = 1 + x + x^2 + x^3 + \frac{2}{3}\frac{x^4}{4} + \frac{1}{3}x^5 + \frac{x^6}{9} + \frac{x^7}{63}
\end{gathered}$$

and one can easily see that the above sequence converges to $1 + x + x^2 + \cdots + x^n + \cdots$.

Adomian's decomposition method

This method consists of

(a) Splitting the given operator equation into linear and nonlinear parts;

(b) Operating by the inverse of the linear operator on both sides;

(c) Decomposing the unknown function into a sum, whose components are to be determined;

(d) Identifying the terms arising out of source terms and initial and/or boundary conditions as the initial term of the sum and

(e) Obtaining the successive terms of the sum in terms of the initial term using Adomian polynomials.

Consider the same problem discussed above. The nonlinear term $f(y) = y^2$ can be decomposed into Adomian polynomials given by

$$A_0 = y_0^2, \qquad A_1 = 2y_0y_1,$$
$$A_2 = 2y_0y_2 + y_1^2, \qquad A_3 = 2y_0y_3 + 3y_1y_2, \ldots$$

The successive terms of the solution are obtained as

$$y_0 = 1, \qquad y_1 = x$$
$$y_2 = x^2, \qquad y_3 = x^3, \ldots$$

and the solution is obtained as

$$y = 1 + x + x^2 + \cdots + x^n + \cdots$$

Laplace-Adomian Decomposition Method

The Shooting type Laplace-Adomian Decomposition Algorithm consists of

(a) Converting the given integro-differential equation into an ordinary differential equation and applying Laplace Transforms or we can take Laplace transform directly to differential equation or integral equation;

(b) If the initial conditions are not given, we take the conditions as parameters and convert the nonlinear term in terms of Adomian polynomials;

(c) Equating like powers and applying inverse Laplace transform to obtain successive approximations and

(d) Summing up the approximations to get the closed form solution.

Numerical Experiments

Problem 1. Consider the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = u + u^3$$

with boundary conditions $u(0) = 1,\ u'(1) = 0,\ 0 < x < 1.$
Taking Laplace transform and simplifying, we get

$$L(u(x)) = \frac{s}{s^2-1} + \frac{a}{s^2-1} + \frac{1}{s^2-1}L(u^3(x)).$$

The nonlinear term $f(u) = u^3$ can be decomposed into Adomian polynomials given by

$$A_0 = u_0^3,\ A_1 = 3u_0^2u_1,\ A_2 = 3u_0^2u_2 + 3u_0u_1^2,\ A_3 = 3u_0^2u_3 + 6u_0u_1u_2 + u_1^3, \ldots$$

$$\begin{aligned}
L(u_0(x)) &= \frac{s}{s^2-1} + \frac{a}{s^2-1}\\
L(u_1(x)) &= \frac{1}{s^2-1}L(A_0)\\
L(u_2(x)) &= \frac{1}{s^2-1}L(A_1)\\
L(u_3(x)) &= \frac{1}{s^2-1}L(A_2)
\end{aligned}$$

By taking the Laplace inverse, we obtain

$$u_0(x) = a\sinh x + \cosh x$$

The next approximations are obtained as

$$\begin{aligned}
u_1(x) = &-\frac{1}{64}\mathrm{e}^{-3x}(a-1)^3 + \frac{1}{64}\mathrm{e}^{3x}(a+1)^3 + \frac{1}{32}\left(12ax - 12a^3x - 3a^2 - 1\right)\cosh x\\
&+\frac{3}{32}\left(4x - 4a^2x + 3a^3 - 7a\right)\sinh x
\end{aligned}$$

and so on. Defining the sum up to the $n-1$ terms as $S_n(x)$ and using the second condition $u'(1) = 0$, we get approximations to $u'(0) = a$, which in turn should be substituted in $S_n(x)$ to get the approximate analytic solution. The values of successive approximations to $u'(0)$ are obtained as shown in the table.

n	a_n
1	-0.761594156
2	-1.021864369
3	-1.054972945
4	-1.064092629

The bounds obtained by Arthurs and Arthurs are -1.049 and -1.077.

The next table presents the results obtained from Laplace-Adomian's method and Shooting type Adomian's decomposition method.

x	STLADA S_3	STLADA S_4	STADM S_6	STADM S_7
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.2	0.82421498	0.82235015	0.82374727	0.82242841
0.4	0.70465913	0.70076649	0.70371973	0.70092610
0.6	0.62739970	0.62143415	0.62613357	0.62161233
0.8	0.58433808	0.57667882	0.58301196	0.57661480
1.0	0.57067025	0.56230240	0.56938739	0.56183884

Problem 2. Now consider the differential equation $\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \mathrm{e}^u$ with boundary conditions $u(0) = 0,\ u(1) = 0,\ 0 < x < 1.$

The nonlinear term $f(u) = e^u$ can be decomposed into Adomian polynomials given by

$$A_0 = \mathrm{e}^{u_0},\ A_1 = \mathrm{e}^{u_0}u_1,\ A_2 = \mathrm{e}^{u_0}u_2 + \frac{1}{2}u_1^2\mathrm{e}^{u_0},\ A_3 = \mathrm{e}^{u_0}\left(u_3 + \frac{1}{6}u_1^3 + u_1u_2\right), \ldots$$

$$\begin{aligned}
L(u_0(x)) &= \frac{a}{s^2} \\
L(u_1(x)) &= \frac{1}{s^2}L(A_0) \\
L(u_2(x)) &= \frac{1}{s^2}L(A_1) \\
L(u_3(x)) &= \frac{1}{s^2}L(A_2) \\
u_0(x) &= ax \\
u_1(x) &= \frac{\mathrm{e}^{ax} - 1}{a^2} - \frac{x}{a} \\
u_2(x) &= \frac{1}{4a^4}\left(\mathrm{e}^{2ax} - 4\,(ax-1)\,\mathrm{e}^{ax} - 5\right) - \frac{x}{2a^3}
\end{aligned}$$

Defining the sum up to the $n-1$ terms as $S_n(x)$ and using the second condition $u\,(1) = 0$, we get approximations to $u'(0) = a$, which in turn should be substituted in $S_n(x)$ to get the approximate analytic solution. The values of successive approximations to $u'(0)$ are obtained as shown in the table.

n	a_n
1	−0.4347754841
2	−0.4604486353
3	−0.4632279437
4	−0.4635776752

The next table presents the comparison with shooting type Adomian's decomposition method and the shooting type Laplace-Adomian's decomposition algorithm.

x	STLADA S_4	STADM S_5	Exact	Error
0.0	0.00000000	0.00000000	0.0000000	0.00000000
0.1	−0.04143043	−0.04142968	−0.041436	1.3442417×10^{-4}
0.2	−0.07325653	−0.07325643	−0.073268	1.5654856×10^{-4}
0.3	−0.09578451	−0.09578178	−0.095800	1.617223×10^{-4}
0.4	−0.10921515	−0.10921339	−0.109240	2.2746246×10^{-4}
0.5	−011367514	−0.11367290	−0.113700	2.1867194×10^{-4}

This problem has the unique explicit solution

$$u(x) = -\ln 2 + \ln\left\{c\sec\left[c\left(\frac{x-0.5}{2}\right)\right]\right\}$$

where c is the root of the $\sqrt{2} = c\ \sec(\frac{c}{4})$ lying between 0 and $\frac{\pi}{2}$, namely $c = 1.3360557$.

Problem 3. Next consider the differential equation $\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -1 - a^2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2$ with the boundary conditions $u(0) = 0,\ u(1) = 0,\ 0 < x < 1$.
The nonlinear term $f(u) = (u'(x))^2$ can be decomposed into Adomian polynomials given by

$$A_0 = (u_0')^2,\ A_1 = 2u_0'u_1',\ A_2 = (u_1')^2 + 2u_0'u_2',\ A_3 = 2u_0'u_3' + 2u_1'u_2', \ldots$$

$$\begin{aligned}
L(u_0(x)) &= \frac{b}{s^2} - \frac{1}{s^3}\\
L(u_1(x)) &= -\frac{a^2}{s^2}L(A_0)\\
L(u_2(x)) &= -\frac{a^2}{s^2}L(A_1)\\
L(u_3(x)) &= -\frac{a^2}{s^2}L(A_2)\\
u_0(x) &= bx - \frac{x^2}{2}\\
u_1(x) &= \frac{1}{3}ba^2x^3 - \frac{1}{12}a^2x^4 - \frac{1}{2}b^2a^2x^2\\
u_2(x) &= \frac{2}{15}ba^4x^5 - \frac{1}{3}b^2a^4x^4 + \frac{1}{3}b^3a^4x^3 - \frac{1}{45}a^4x^6
\end{aligned}$$

Defining the sum up to the $n-1$ terms as $S_n(x)$ and using the second condition $u(1) = 0$, we get approximations to $u'(0) = a$, which in turn should be substituted in $S_n(x)$ to get the approximate analytic solution. The values of successive approximations to $u'(0)$ are obtained as shown in the table.

n	b_n
1	0.5000000000
2	0.5223657595
3	0.5213279281
4	0.5214794109
5	0.5214679845

The next table presents the comparison with shooting type Adomian's decomposition method and the shooting type Laplace-Adomian's decomposition algorithm.

x	STLADA S_4	STADM S_5	Exact	Error
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.04657083	0.04657084	0.04657094	2.36198×10^{-6}
0.2	0.08230374	0.08230375	0.08230398	2.87956×10^{-6}
0.3	0.10757266	0.10757269	0.10757301	3.25360×10^{-6}
0.4	0.12263418	0.12263419	0.12263459	3.34326×10^{-6}
0.5	0.12763825	0.12763827	0.12763868	3.36888×10^{-6}

This problem has the unique explicit solution

$$u(x) = \frac{1}{a^2} \log \left[\frac{\cos\left(a\left(x - 0.5\right)\right)}{\cos\left(0.5a\right)} \right]$$

with $a^2 = 0.49$.

Conclusion

A comparison of Taylor series method, Picard's method, Shooting type Adomian's decomposition method and Shooting type Laplace-Adomian's algorithm has been done. We find that the last two methods provide better results comparison to the first two methods. From the tables we can observe the error is very small in all the cases.

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Preparing school students for a problem-solving approach to mathematics

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Abstract : A problem-solving perspective for using mathematics can be established in school children by laying down roots through classroom activity. The phrase 'problem-solving' has multiple meanings, and I am going to focus on bringing a creative mindset and existing knowledge to bear on non-routine problems. In this talk I will illustrate how this might take place at three different stages of school mathematics: elementary, middle and secondary. I shall also show how this approach can be a basic method for meeting new mathematical ideas.

1 Introduction

The phrase 'problem-solving' has many different meanings in mathematics education. Most curricula internationally now contain some kind of commitment to problem-solving as a curriculum aim and some curricula are more explicit than others about what is meant. In this paper I examine various kinds of problem-solving in mathematics and identify the demands they make on the learner, and hence of pedagogy. Each different type requires different kinds of attention in tasks and lessons, because problem-solving skills vary between problem types. A general problem-solving mindset is not necessarily enough, on its own, to bring about successful mathematical learning through problem solving.

2 Background

My own teaching experience has been in government schools for 11 to 18-year-olds. From this experience I claim that it is possible for most students to experience key ideas in the curriculum through working on extended problem situations, often in collaboration with others.

My experience as a teacher is therefore similar to that described by Boaler (1997) in her comparison of the work of similar students from different schools. In one school, like mine, a problem solving curriculum was used and in another a procedural, step-by-step, way of teaching was pursued. The students in the first school did significantly better than students in the second school in the same examinations both on procedural questions and also on deeper questions. One feature of their success was that if they did not know about the content in a particular question they would use problem-solving strategies to move towards an answer; students in the second school would leave the question out if they did not know how to do it. This study has been used to support arguments for a problem-solving approach to teaching and collaborative classroom practice as ways to improve students' achievement, understanding, and interest in mathematics. Similarly, the introduction of a 'reform' curriculum in the United States has prompted many other countries to promote changes towards problem-solving in mathematics. The international PISA tests have also promoted this shift, and the underlying argument is that the economic and technological demands of the 21st-century require students who are skilled in flexible, multistage application of mathematical ideas to nonroutine situations.

The simplistic argument is: if this is what governments want in their workforce, then this is what students must learn at school. I shall lay out the complexities behind this argument. The research on which I base this paper is mainly classic. Recent research in the area has not, in my view, produced substantial new insights but may have recast them in a socio-cultural discourse. Recent work also gives clear accounts of classroom practice. As an alternative, I am going to focus on the mental activity involved in solving problems.

3 Problematising problem solving

There are three significant questions that need to be addressed during the international rush towards 'problem-solving curricula' in mathematics.

(a) What is meant by 'problem-solving'?

(b) What is learnt through 'problem-solving'?

(c) What are the implications for pedagogy?

There are other significant questions that also arise , such as how problem-solving can be assessed and what forms of knowledge mathematics teachers need, but these are outside the scope of this paper.

4 The meaning of 'problem-solving'

Interest in problem-solving in mathematics as a research domain arose initially in the context of older students' capabilities when presented with questions that required the application of conceptual understanding, rather than the direct and obvious application of learnt techniques. For example, the student might be able to perform solutions of simultaneous equations, where these are like the ones they have practised, but might not be able to begin thinking about this question:

> Consider the set of equations
>
> $$ax + y = a^2$$
> $$x + ay = 1$$
>
> for what values of a does this system fail to have solutions, and for what values of a are there infinitely many solutions ?

This question was posed in Schoenfeld's classic study of the use of problem-solving strategies (1982). His research focused on whether a few undergraduate students could be taught to use problem-solving strategies and hence become better at solving the kind of problems that probe conceptual understanding. In an age of digital mathematical tools, the processes of solving simultaneous equations can be done in nano-seconds, but the solution of problems such as this take some conceptual understanding to tackle, even if the manipulating work is done using a computer algebra system. The relationship between the problem and the student's conceptual understanding is dialogic. Whereas some conceptual understanding is needed to get started, it is likely that the student's knowledge is enriched through working on this problem. For example, the student might not previously have considered that the parameters of a system of equations can have their own internal relations. The problem therefore scaffolds an understanding of simultaneous equations that has a higher level of generality than can be achieved by practising techniques in routine examples. An example with younger children of this need for conceptual understanding when applying learnt procedures might be 'Two mystery numbers on a numberline are three units apart. What could they be?' This question approaches subtraction as 'difference' from an unexpected angle if children have only done given subtractions before.

In school mathematics, the problems presented after practising techniques tend to be word versions of the same technique. This kind of problem-solving has been around in textbooks and teaching for several hundred years and usually the student is expected to apply the technique just practised. This requires them to spot the underlying structure in the word problem and relate it to the technique. Some approaches focus on telling students how to match particular words to particular operations, such as 'how many?' as a cue to multiply. This misses the point and can lead to error. When this is the only kind of problem-solving that is required by curriculum, students depend on knowing what to do to solve the problem: there is a clear method and a clear answer, and teachers can feel secure in maintaining progress through typical curriculum topics. As Hiebert et al. say: "Rather than mastering skills and applying them, students should be engaged in resolving problems. ... the history of problem solving in the curriculum has been infused with a distinction between acquiring knowledge and applying it." (1996, p.18)

A more challenging development of this kind of problem-solving is the presentation of collections of word problems for which the 'routine' is not obvious and students have to decide which operations to apply. For young children, we could pose the problem: 'Molly has five more sweets than Jack, who has one less than Mandeep. How can they share their sweets out equally?' For older students: 'Use compasses and a straight edge to construct a quadrilateral with one right angle between two adjacent equal sides, two opposite parallel sides, and a diagonal whose length is 8 cm. How many possibilities are there?' In each of these the student has to work with the meaning of the relations between the elements presented in the problem: manipulate them; try out different combinations, and in the process become familiar with the relations and properties of the situation. Only then can they apply learnt techniques successfully.

Another kind of problem-solving is to approach new ideas in mathematics through problematising existing knowledge. Similar to the historical genetic development of mathematics, a learner encounters new ideas as they arise through questioning older ideas. This is an ideal pathway for developing young mathematicians who can then tackle Olympiad problems. When this is the predominant kind of problem-solving, students can develop a coherent view across mathematics and a questioning stance towards mathematics. New mathematical developments are seen as implications of earlier mathematical ideas. However, curriculum coverage of particular topics is limited because of the time taken to access each new idea. For example, a classic difficulty in solving linear equations is the development of students' understanding of which moves to make and when. One common approach is to introduce a balance beam model as a metaphor for equality between both sides of the equation. This works well for addition and positive numbers. One teacher I observed using this approach presented its use when negatives are involved as a problem for the class to solve. It took them two lessons to move from the concrete model to the abstract idea of 'doing the same things both sides' which enabled them to handle negatives. It can be argued that the teacher could have led them there, through exposition, fairly quickly and the two lessons could have been spent becoming fluent in using the method. The students who spent two lessons devising this 'rule' for themselves had also developed a strong understanding of what an equation is, rather than merely learning a method. The difference between this kind of problem-solving and the simultaneous equations example above is that the knowledge developed about handling negatives in linear equations is an essential component of future learning, whereas the outcome of the parameter problem has no specific value, but the experience of working with relations between parameters has a more general value.

The final kind of problem-solving I consider here is the solution of problems in context. In this approach the problems might be real, such as to plan a holiday, or to optimise profits in some economic endeavour, or the problems might be 'realistic' in the sense that they relate to situations that are easily imagined. The acknowledged experts in this field are the Freudenthal Institute with their well-theorised approach of Realistic Mathematics Education (e.g. Gravemeijer and Doorman, 1999).

There is therefore a collection of meanings for the phrase 'problem-solving': the routine and nonroutine application of techniques; working on mathematical questions that lead to new curriculum topics; working on mathematical questions that develop pervading mathematical modes of enquiry; and applying mathematics to problems in contexts.

5 Pedagogic issues associated with problem-solving

The ability to solve problems is often stated as a curriculum aim without any differentiation between these types of problem-solving. However, the ability to apply a recently learnt technique to a worded problem is very different from the ability to approach an incompletely-defined realistic problem, such as those found in the workplace, or an unfamiliar mathematical problem. I am going to argue that different kinds of mathematical problem-solving capability require different pedagogical approaches.

A further source of confusion in mathematics education literature is whether research into students' problem-solving is focusing on their development of problem-solving capabilities, their adoption of problem-solving heuristics in particular, or the more general learning of mathematics, including meeting new ideas and becoming fluent with techniques.

6 Routine and non-routine application of techniques in word problems

A plethora of research has been undertaken to understand young children's approach to word problems. Verschaffel, De Corte and Lasure (1994) point to a reluctance to use everyday knowledge in such problems, and Hegarty, Mayer and Monk (1995) find a difference between those who use the size of numbers or particular words as cues to decide what to do, and those who construct mental models to help them solve the problem, the latter group being more successful. From a student's perspective, it is not obvious why they should bother to construct a mental model if all they are being asked to do is repeat a recently-learnt method in a worded context. The actual words may not matter as much as the grammar if the child is thinking 'I have been doing subtraction all week so these problems are about subtraction'. There is no reason why doing routine word problems should prepare students for doing non-routine problems because in the latter they have to understand the relations between components of the problem whereas in the former they only have to act in a patterned ways (Brown and Kuchemann, 1976). Students who are used to the classroom regime in which they are expected to calculate answers quickly and accurately are often reluctant to slow down and reflect on the problem to enable them to choose appropriate techniques to apply. Kahnemann (e.g. Kahnemann and Frederick, 2002) explained this tendency by claiming that the mind operates in two simultaneous systems: one system S1 reacts quickly, intuitively, automatically based on past experience, and the other system S2 reacts more slowly, reflectively, and critically. In mathematics we need both systems, and we also need to have control over which system we use and when, so that when we are starting a problem we need to be using the S2 system and deciding, based on the nature of the task, when to use fluent S1 procedures and when to take deliberate S2 decisions. While this theory explains the findings of Hegarty et al. (1995) it does not give pedagogic advice. For most of the school mathematics taught to younger children, the idea of visualising the situation described in the word problem is an obvious and powerful tool for problem-solving. Explicit attention paid to this in class - not leaving it to chance - can make a difference. This is an example where a specific problem-solving strategy can be taught, adopted as a common practice, and becomes useful. There is the missing link however, which is that children have to understand what actions are expressed by addition, subtraction, multiplication and division. If their understanding of addition is limited to enumerating the outcome of combining two sets, and the problem is about increasing a number, they may not recognise that addition is the appropriate operation even if they have an image of the context. This problem is especially important when children are offered nonroutine problems and given no clues about the associated arithmetic. The pedagogic task is therefore not only to give explicit attention to image building, but also to ensure students have multiple ways to understand the underlying concepts.

In the middle school years the mathematics curriculum generally moves in directions that cannot be visualised so helpfully. For example, it is difficult to observe the geometric fact that, from an external point to a circle, the square of the length of the tangent is equal to the product of lengths from the point to the circle along the secant. It is also difficult to imagine the value of the rate of change of an exponential function. The main arithmetical idea in these years is that of proportion and, while visualisation can give students a good idea of what proportionality means, the relationships between different quantities involved are quite hard to perceive. It is very difficult to connect different sizes of the same shape as instantiating a constant ratio in a diagram or situation. From problem-solving perspective the requirement to understand the problem and also to understand the purpose of different operations is the same as that for younger children, but the strategy of 'visualising the situation' is less likely to give a direct method of solution.

7 Working on mathematical questions that lead to new curriculum topics

Here are three questions that can, in an enquiring community, lead students to appreciate some new-for-them ideas in mathematics:

- I am thinking of two numbers and when I add them together I get 14. What could the numbers be?
- The difference between consecutive terms in a linear sequence is constant. Suppose we have a sequence in which the difference between consecutive terms is itself a linear sequence; what could that sequence be?
- Could we express the size of an angle at a given point in terms of the length of the arc of a unit circle that it supports? Express some common angles and the trigonometric ratios in terms of this new measuring unit.

The assumption behind these questions is that students are willing and permitted to explore, perhaps messily, and thus develop a coherent set of connections between mathematical ideas, grounded in their existing knowledge and understanding. However, experiencing a key idea in one context does not necessarily build up a repertoire of mathematical knowledge and skills that can be applied in more complex problems. It does, however, contribute to the development of a problem-solving mindset. It is of central importance here that the teacher, who has scaffolded higher levels of mathematical thinking by posing the questions, also intervenes to ensure that the new mathematics students encounter becomes part of their repertoire in a generalised form. This can happen through careful repeated use of language, through provision of symbolic representations, and through being given further problems in the same structure. Descriptions of pedagogy that supports this enquiring, problem-solving, mindset often base their claims of success on students' capabilities in solving particular problems, or in taking a problem-solving approach to unfamiliar situations. More rarely do we read of how teachers working in these ways help students to develop a repertoire of technical mathematical methods and a robust mental bank of mathematical concepts.

In the first problem above, students who are easily satisfied with one quick answer will not necessarily grasp the multiple perspectives on the additive relationship that could arise from a deep enquiry of such problems in general. In the second problem it could also be possible to stop at one answer, such as a sequence of square numbers, without teacher intervention to encourage seeking more examples and to pose further questions that extend and generalise the idea. The third problem is perhaps a more obvious introduction to a new idea. The point to be made here is that solving any problem can be an isolated experience which has no effect on either mathematical knowledge, or on the development of mathematical problem solving skills, unless it is part of a planned pedagogical development. In these three cases the curriculum value of the examples I have given is obvious in terms of conceptual knowledge.

8 Working on mathematical questions that develop mathematical modes of enquiry

I have chosen to use the phrase 'mathematical modes of enquiry' to avoid generalised 'problem-solving skills'. Mathematical modes of enquiry include general problem-solving skills, but also include posing questions, following 'what if..?' lines of reasoning, and exploring classes of mathematical objects beyond those that were necessary for solution of the problem (Watson and Mason 1998). In other words, students need to develop curiosity about mathematical situations, seeing 'outside' phenomena in terms of mathematics, and being inventive about selecting, creating and applying mathematical tools for solutions.

I have not mentioned so far the notion of 'open-ended' problems versus closed problems. This is deliberate. A closed problem with one answer, achieved one way, requires, as I have said before, transfer of a known method into a situation that the student recognises as a manifestation of a mathematical relationship. Any problem can be opened up using curious questioning. Any problem can be treated as an open question by someone who does not spot a closed pathway through it. Students who have developed curiosity, and have experience of trying out different directions of exploration, such as by trying special cases, calculating two ways, drawing rough diagrams, breaking down a problem into subgoals, and so on, can tackle more questions than those who rely only on taught methods. As with other aspects of problem-solving, these mathematical habits cannot be an add-on to a more procedural approach - they have to imbue the whole mathematical experience. In addition, teachers could set problems for which their main goal is to help develop self-questioning skills, but if these are always as 'add-on' to formal teaching students are unlikely to develop a problem-solving approach to benefit their learning across the whole of mathematics.

9 Applying mathematics to problems in contexts

So far I have drawn on experience to explain the phenomenon illustrated in Boaler's work, and in other places (e.g. Watson and De Geest 2005; Senk and Thompson, 2003) that students who learn mathematics through complex extended tasks appear to be able to do procedural work similarly to those taught procedurally, but have the additional ability to tackle nonroutine and unfamiliar mathematical situations, and apply mathematics to other spheres of activity.

Another approach to problem-solving in mathematics is that of the Freudenthal Institute, who have an extended experience of teaching mathematics through immersing students in realistic contextual problem solving (e.g. Gravemeijer and Doorman 1999). The word 'realistic' here does not mean 'actual'. Instead it means that students are able to imagine the situation and deduce relations among the quantities involved on the basis their physical experiences. Thus, for example, they might be able to imagine a world in which giants are 2.5 times the height of humans, or a machine in which a fluid flows up hill at a particular rate - it does not have to be possible in the material world.

A simplistic version of real-world problem solving in mathematics do would be to assume that somehow, by solving real-life problems, students can learn mathematics. Students who work frequently on this kind of problem can become adept at developing *ad hoc* solution methods that are specific to the problem itself, for example trial and adjustment methods, measurement methods, numerical approximation methods and so on. Once the problem is solved there may not be any perceived need, by the student or teacher, for a universal method or an abstract idea that a mathematician could see behind the design of the problem. The Realistic Mathematics Education approach structures the problems so that a need for more powerful universal methods arises through a process of 'vertical mathematisation' (Treffers, 1987). Students work on a range of problems that are mathematically similar so that similar solution methods are repeated. A need to encapsulate theses repeated methods as formal mathematical ideas emerges and is orchestrated by the teacher and the published materials. For example, problems that involve proportional reasoning give rise to a notation (a ratio table) that makes sense as a way to store data from the problem. The internal structures of the table can then be used as a problem-solving tool in other problems. For example, the table below can be understood through multiplicative relationships from left to right, or between top row and bottom row, or between any two non-adjacent columns, or as expressions of equivalent fractions or ratios.

1	3	6	9	12	...
?	5	10	15	20	...

The use of ratio tables leads me to a conjecture: that a knowledgeable teacher, or a well-designed textbook, can provide ways to format students' efforts to solve problems so that the underlying formal mathematics becomes more obvious through the layout. The traditional long division algorithm is an example of this idea. The algorithm provides a way of organising a method of division that depends on repeated subtraction from a number which is treated according to the place value of its digits. A procedural approach would focus only on the format - the long division method. A problem-solving approach would generate *ad hoc* methods of division which could then be formatted into the long division algorithm. Something similar happens in early calculus. A procedural approach would focus on using formulae to differentiate polynomials. A problem-solving approach would involve exploring, possibly using a digital graph-plotter, the gradients of a function for different x-values and these findings could be formatted to indicate the underlying gradient function. I am not suggesting that all mathematics can or should be learnt empirically, but that a problem-solving curriculum generates empirical evidence and experience which, if it is left at that point, might not contribute to the creation of a mathematical repertoire which would then be available for further use both in mathematics and outside.

10 Problem solving heuristics

The title of this paper focuses on preparing students to be mathematical problem-solvers. One possible approach would be to teach methods of solving problems.

In his classic work 'how to solve it' Polya (1973) gave a general heuristic for mathematical problem-solving:

- understand the problem
- make a plan
- carry out the plan
- look back

Indeed, this is a general heuristic for any kind of problem-solving. On its own, however, it provides very little advice other than as a framework for keeping track of where you are while solving the problem. He then offered six ways of understanding a mathematical problem, and many of these are mathematically-specific:

- What are you asked to find or show?
- Restate the problem in your own words
- Draw a picture or a diagram
- Is there enough information?
- Do you understand all the words used in stating the problem?
- Pose a question

and 15 possible contributions to making a plan:

- Guess and check
- Make a list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use direct reasoning
- Solve an equation
- Look for a pattern
- Draw a picture
- Solve a simpler problem

- Use a model
- Work backward
- Use a formula
- Be creative
- Use your head

It is clear that some of these suggestions are easier to follow than others. For example, if you cannot see how to apply 'direct reasoning' an instruction to do so is of little use. Similarly, if nothing helpful comes to mind, the instruction to 'use your head ' is pointless. It would also be ridiculous to expect students to learn all these suggestions and to try them all when they are stuck with a problem. Polya's work prompted a research debate about the value of teaching specific problem-solving heuristics. Key research in this area is Schoenfeld (e.g. 1979, 1982) and it is worth returning to the debates of the 1980s to think about the desirability of a problem-solving curriculum.

On the one hand, Schoenfeld was getting mixed results when teaching particular problem-solving strategies to small groups of advanced students, and claimed that: ' when problem-solving strategies are identified and taught, and when students think to use them, the impact on the students problem-solving performance is substantial' (1979, p.185). He hedged this with the observation that even in his closely focused experimental environment, students did not readily use the problem-solving heuristics they had learnt in a post test problem. The key phrase here is 'when students think to use them' which he could ensure within the study itself by limiting the number of heuristics available and ensuring they were relevant for the problems. His concern was that students did not transfer their use to other problem-solving situations. He points out that '[r]eal-life mathematical problem-solving experiences are not nearly as well ordered as they were in this experiment, the likelihood of students picking up the strategies from their experience is small... p.184'. Nevertheless, his work is sometimes taken to be a recommendation for explicitness about mathematical problem-solving heuristics.

Schoenfeld selected the problem-solving strategies that he would teach students and the problems that he would pose, so that he was in control of a limited supply of plausible problem-solving tools. Sweller (1990) critiques this work by pointing out that : 'work on expert-novice differences led directly to the hypothesis that expertise consisted of the accumulation of a large store of domain-specific knowledge and strategies (that is why it takes so long to become an expert) and that there were few differences between experts and novices in general strategies.' In other words, Schoenfeld had inserted his own expertise into the study through his choice of problems and strategies, and Polya's framework can easily be adopted by students, but not necessarily with any improvements in mathematical problem-solving capabilities. The problem with problem-solving was posed as a problem of students transferring learnt heuristics between problems.

The pointlessness of following problem-solving heuristics in an uncritical way can be illustrated using an example from Mason, Burton and Stacey (1982). In the book , Mason at al. offer four mathematical actions to apply to problems: specialise, generalise, conjecture, convince. Consider this problem from the book:

Productive exchange
$27 \times 18 - 28 \times 17 = 10$
$37 \times 18 - 38 \times 17 = 20$
Generalise

Thoughtless application of the actions would be impossible, since we are offered two specialisations of something and asked to generalise - that is the first two actions. Students who have previously succeeded in mathematics by doing exactly as they been asked to do, complying with instructions and applying procedures, would be lost. One problem for an obedient student is 'what does the author expect me to generalise?' or even 'what does the author expect me to notice?' and instead of genuine mathematical enquiry the problem can become one of trying to guess hidden meanings.

What does an experienced problem solver do? I do not recognise any patterns here immediately, so I search for them in the relationships between the particular numbers chosen, while bearing in mind that any relationships I find might not be specific to 7s and 8s. In searching for patterns I am looking at the structure and the relationships expressed in the examples. I am initially torn between re-expressing the numbers in place value (e.g. $28 = 20+8$) and expressing some relationships algebraically (e.g. $nm-(n+1)(m-1)$). The algebraic approach attempt loses some of the specific place value features of the problem statement, so I might start with an obvious conjecture about $47 \times 18 - 48 \times 17$, and so on .... Most of these initial steps are specific to this problem, but could be generalised as, for example, choice of representation. How the problem solver decides what representation to choose depends on past experience of using that representation and past experience of solving similar problems. There is no algorithm for making the choice.

As I have just shown, my own experience of working on mathematical problems would certainly support the view that a good problem solver needs a store of knowledge and strategies, and experience in using these, and a combination of the two that generates awareness of what might be appropriate. However, this does not provide an argument for or against the teaching of problem-solving heuristics. Instead, it provides an argument for students and teachers to imbue their work with a range of problems that require regular application of knowledge and strategies and development of mathematical awareness. As Schoenfeld indicated, a mental list of strategies is no use unless students think to use them in appropriate circumstances. Mason (2000) argues strongly that what is needed is the development of mathematical awareness that can intentionally be brought into action when relevant. Naming appropriate actions is not to provide a list to be learnt and applied, but to make fine distinctions between different actions and draw students into a world of possibilities from which they can consciously or unconsciously choose. This applies both to mathematically-posed problems and also to applications of mathematics in outside contexts.

How does such awareness develop? I have indicated in this paper that problem-solving needs to be an integral part of students' classroom experience, and this would create a mindset towards mathematics as a problem-solving endeavour. I have indicated also the importance of the teacher who does not divert mathematical enterprise towards generic problem-solving strategies, but provides the formats of mathematics to organise mathematical enquiry and mathematical development. To do this well, the teacher also has to have a problem-solving mindset towards mathematics, and therefore needs to have also engaged with the same range of problems, within and outside of mathematics.

The socio-cultural approach to mathematics education draws our attention to the importance of establishing cultural norms in learning. It is not enough to rely on individual creativity in problem-solving; it also has to be the living reality of mathematics classrooms so that students know that their attempts to solve problems are expected to be messy, inconclusive, incomplete and exploratory. In return, the teacher needs to value their attempts and provide the shaping, the formatting, that brings them to completion in standard mathematical forms.

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The role of mathematics and computational science in industries : Case Studies.

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Abstract : Learning to apply Mathematics is very different from learning Mathematics. As "pure" mathematics is usually perceived as a list of specific procedures, techniques, theorems, and rules, "applied" mathematics is used for solving a wide range of problems, many of which do not seem mathematical in nature. The mathematical and computational sciences continue to find many applications, both traditional and novel, in industry. Some of these applications have very dramatic effects on the bottom line of their companies, often in the tens of millions of dollars. Other applications may not have an easily measured impact on the bottom line but simply allow the company to conduct business in a 21st-century data-rich marketplace. Finally, some applications have great value as contributions to science. In this paper, an attempt is made to answer the question "What is mathematics used for, anyway?" . Some basic concepts of mathematical modeling is introduced followed by case studies which bridges the gap between abstract mathematical concepts known to mathematicians and engineers who wish to solve real life problems in industries.

1 Introduction

Role of Mathematics in Industries

The Society for Industrial and Applied Mathematics (SIAM) with support from the National Science Foundation and the National Security Agency conducted a survey for finding out applications of advanced mathematics in Industries and also employment scenario for a significant community of highly trained mathematical scientists[1]. Approximately 500 mathematicians, scientists, engineers, and managers in the United States participated in the survey over a period of three years. The survey was conducted by telephone interviews with 203 recent advanced-degree holders (master's and Ph.D.) in mathematics working in non-academic jobs, followed by telephone interviews with 75 of their managers and 19 in-depth site visits by groups of steering committee members to industrial and governmental organizations in United States. Table 1 shows the distribution of graduates surveyed in five major sectors of industry, based on the Standard Industry Classification codes of the United States Office of Management and Budget.

Table 1 Distribution of mathematics graduates in five major sectors of industry

Non-academic sector	Ph.D.	Master's
Government	28.00%	22%
Engineering research, computer services, software	19.00%	18%
Electronics, computers, aerospace, transportation equipment	17.00%	12%
Services (financial, communications, transportation)	13.00%	22%
Chemical, pharmaceutical, petroleum-related	6.00%	2.00%

Mathematicians and their managers were asked in the telephone survey about the status of advanced mathematics in their overall organizations, where "advanced" means at the level of the respondent's highest degree. Those responses are summarized in Table 2 and show the consistent importance of mathematics not only for its practitioners, but also for their managers.

Table 2 Average perceived importance of mathematics in respondents' overall organizations

Importance of advanced mathematics	Ph.D.	Master's	Managers
Primary	43%	28%	51%
Secondary	43%	40%	37%
Only for general utility	11%	32%	12%

Nearly half (49) characterized mathematics as an underlying requirement or tool for their group's work. Three main functional roles for mathematics were mentioned by managers: development of algorithms and numerical methods (27); modeling and simulation (23); and statistical analysis (15). The site visits, telephone surveys, and experiences of steering committee members in industry build a picture in which mathematics participates in many ways in the overall enterprise of industrial and government organizations. Table 3 indicates selected associations between areas of mathematics and applications encountered in the site visits.

Table 3: Mathematical areas and industrial applications encountered during site visits.

Mathematical Area	Application
Algebra and number theory	Cryptography
Computational fluid dynamics	Aircraft and automobile design
Differential equations	Aerodynamics, porous media, finance
Discrete mathematics	Communication and information security
Formal systems and logic	Computer security, verification
Geometry	Computer-aided engineering and design
Nonlinear control	Operation of mechanical and electrical systems
Numerical analysis	Essentially all applications
Optimization	Asset allocation, shape and system design
Parallel algorithms	Weather modeling and prediction, crash simulation
Statistics	Design of experiments, analysis of large data sets
Stochastic processes	Signal analysis

Nearly every manager interviewed by telephone cited a particular combination of application and mathematics in which mathematics had made a significant contribution. However, the application problems, intended to show the relevance of mathematics, are often concocted in nature confirming students beliefs that mathematics has no relevance to real life.

2 Some Industries Where Mathematics is Applicable

Computer Industry: Software Design, Computer Programming

Software engineers/computer programmers design, write, test and implement software packages for consumer use and other computer applications (for internal use) that help a company perform a task or set of tasks more efficiently.

Cryptography and Security

A cryptographer/cryptanalyst analyzes and deciphers secret coding systems and decodes messages for governmental or law enforcement agencies. They also provide privacy for individuals and companies by keeping hackers out of important data systems. (`www.weusemath.org`)

Pharmaceutical Industry, Biomedical Industry, Public Health

Bio mathematicians and biostatisticians design research studies to analyze data related to human health, animals or plants (e.g. genetic data, disease occurrence data, and medical imaging data.) Many use mathematical and statistical techniques to assess the efficacy of drug treatments and others analyze data for populations exposed to toxic environmental chemicals to understand their health risks and effects. (www.weusemath.org)

Investment and Finance

Financial analysts work for banks, insurance companies, securities firms, and other businesses, helping these companies or their clients make investment decisions. They assess the performance of stocks, bonds, and other types of investments. (`www.business.mtu.edu`, `www.bls.gov`)

Operations Research/Management Science

Operations research analysts are involved in strategizing, planning, and forecasting assignments to help companies make better (profitable) decisions and to solve problems. They help companies allocate resources, measure performance, design production facilities and systems, manage the supply chain, set prices, coordinate transportation and distribution, and analyze large databases. (`www.weusemath.org`, `www.bls.gov`)

Actuarial Science

n actuary deals with the financial impact of risk and uncertainty in the insurance industry. Actuaries compile and analyze data to estimate the probability and likely cost of an event such as death, sickness, injury, disability, or loss of property. (`www.weusemath.org`)

3 Applied mathematics and mathematical modeling:

The mathematical concepts which are used to solve real world problems is generally referred to as Applied Mathematics. In this field, It is essential to be able to apply many different mathematical techniques, be able to handle problems involving data where a knowledge of statistics becomes important. It is also necessary to be able to take a practical problem, from engineering for example, and turn it into a mathematical problem. The process of applying mathematics to a real life situation is often referred to as Mathematical Modeling. Mathematical Modeling is the process of creating a mathematical representation of some physical phenomenon in order to gain a better understanding of that phenomenon. Mathematical equations are based on fundamental laws of physics (conservation principle, transport phenomena, thermodynamics and chemical reaction kinetics).

4 General Methodology of any mathematical modeling problem:

(a) Identify the laws governing the phenomenon

(b) Express these laws as mathematical Equations

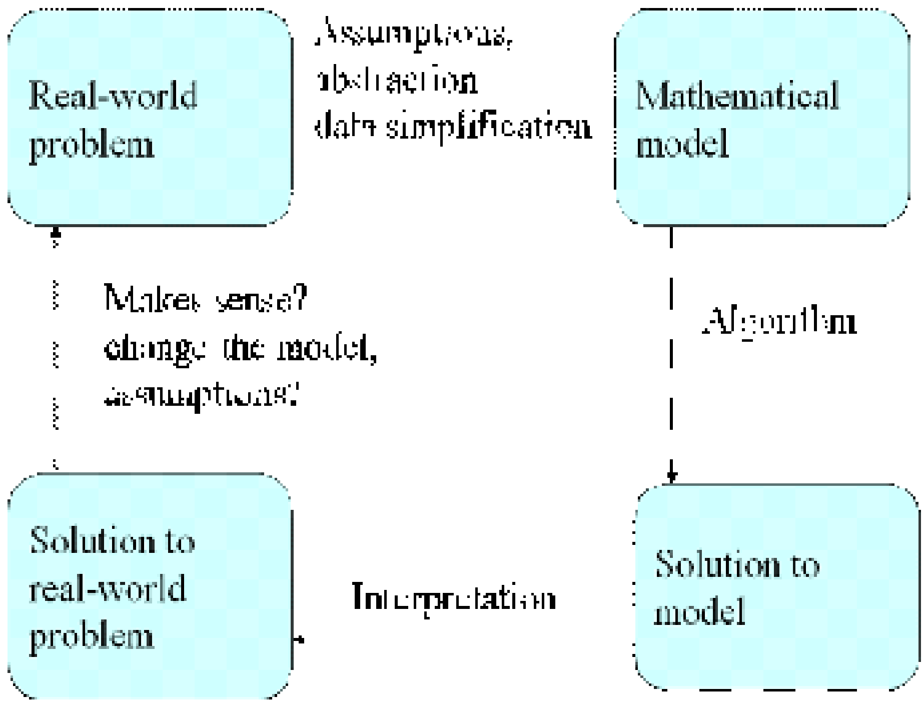
(c) Solve these equations numerically

(d) Display the results graphically

(e) Analyze the results and make necessary interpretation

A schematic view of modeling process is represented in Figure 1.

Figure 1. Schematic Diagram of a Mathematical Model



Mathematical modeling seeks to gain an understanding of science through the use of mathematical models on HP computers. It is a teamwork which involves expertise from the field of Mathematics, Science and Computer Science[2-3]. It is often used in place of experiments when experiments are too large, too expensive, too dangerous, or too time consuming. It can be useful in "what if" studies; e.g. to investigate the use of pathogens (viruses, bacteria) to control an insect population and is a modern tool for *scientific investigation.*

All models contain the same basic elements: some motivating question or purpose, simplifying assumptions that restrict the depth and breadth of the model, an organizational / logical structure, a series of mathematical expressions that followfromthose. Any real world problem has to be first simplified into a conceptual model. In conceptual model, the state variables are first identified. Then, the rates that causes thestatevariables toincreaseordecrease are identified. Finally the feedbacks between the state variables and rates are defined.

5 Classification of mathematical Models:

Mathematical models may be classified according to their subject matter of the models like mathematical models in Physics, mathematical models in Chemistry, mathematical models in Biology etc. It can also be classified according to the mathematical techniques used in solving them like mathematical modeling through classical algebra, mathematical models through matrices, mathematical models through ordinary and partial differential equations etc. Mathematical models may also be classified according to their nature like linear or nonlinear, static or dynamic, deterministic or stochastic, discrete or continuous. Mathematical modeling has emerged as a powerful, indispensable tool for studying a variety of problems in scientific research, product and process development, and manufacturing. **It is often used in place of experiments when experiments are too large,**

too expensive, too dangerous, or too time consuming, where it is extremely difficult to collect data. It can be useful in "what if" studies; e.g. to investigate the use of pathogens (viruses, bacteria) to control an insect population. It has become a modern tool for scientific investigation. Some of the areas where it is being used routinely are

(a) Seismology : oil exploration, earthquake prediction (Parallel computation reduced compute time from weeks to hours)

(b) Climate modeling : global warming, weather prediction

(c) Economics: growth of a local or national economy (Agent-based modeling), management of resources, analysis of tax strategies

(d) Environment: utilization of resources, population modeling, insect control

(e) Material research: design of new materials, smart materials; shape driven by temperature materials; materials aging issues (Stockpile stewardship)

(f) Drug design: design of anti-cancer drugs, etc.

(g) Manufacturing: optimization of manufacturing processes, automation

(h) Medicine: Medical imaging, MRIs

(i) Biology: Applications to understanding and treating disease, design of anti-cancer drugs, etc.

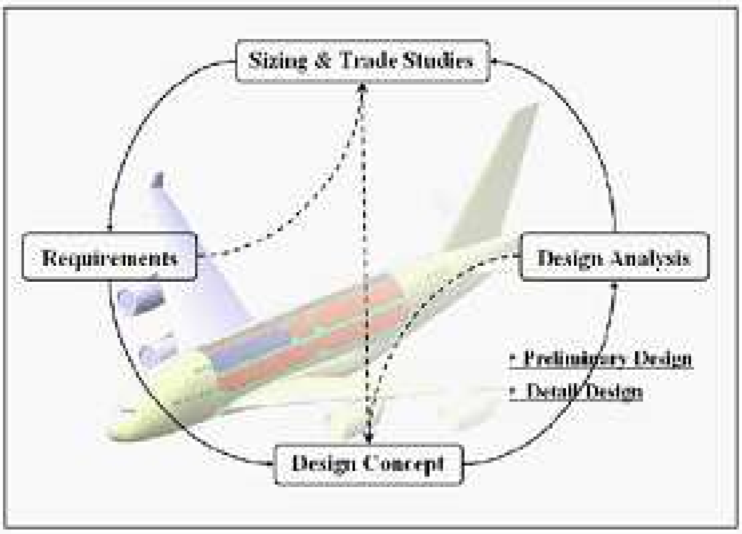
6 Case Studies

6.1 Case study 1: Numerical simulation for aircraft aerodynamic design

The first case study is related to aircraft design using computational fluid dynamics techniques. Computational Fluid Dynamics (CFD) [4-6] is the science of predicting fluid flow, heat transfer, mass transfer, chemical reactions, and related phenomena by solving the mathematical equations which govern these processes using a numerical process (that is, on a computer). The result of CFD analyses is relevant engineering data used in: conceptual studies of new designs, detailed product development, troubleshooting and redesign. The application of CFD today has revolutionized the process of aerodynamic design. CFD has joined the wind tunnel and flight test as primary tools of the trade. Each has its strengths and limitations because of the tremendous cost involved in flight testing, modern air-craft development must focus instead on the use of CFD and the wind tunnel. The wind tunnel has the advantage of dealing with a real fluid and can produce global data over a far greater range of the flight envelope than can CFD. It is best suited for validation and database building.

One major objective for the aircraft industry is the reduction of aircraft development lead-time and the provision of robust solutions with highly improved quality. It will finally be essential to numerically flight-test a virtual aircraft with all its multi-disciplinary interactions in a computer environment and to compile all of the data required for the development and certification with guaranteed accuracy in a reduced time frame. Aerodynamic Design deals with the development of outer shapes of an aircraft, optimizing for its performance, handling qualities and loads. A major ingredient to the design process is the numerical simulation of the external airflow. CFD has made important progress in terms of accuracy of the physical models, robustness and efficiency of the nonlinear solution algorithms and reliability of the overall prediction approach

Figure 2. Aircraft Design Process



The conceptual design process considers what kind of technology will be employed or which new methods will have a possibility to be utilised into the design before moving to the preliminary design stage. A conceptual sketch can be useful to estimate aerodynamics and weight fractions comparing previous designs. This sketch will illustrate the approximate wing and tail geometries, the body shape, and cockpit, payload and passenger compartment of the internal locations of the major components. In the initial sketch design, the aircraft design work is done in full scale using CFD tools. Using the aircraft design on CAD software, the layout is analysed and optimised, with consideration for aerodynamics, structural analysis and the installed propulsion systems. After this performance consideration, the performance capabilities are calculated and optimised compared to the requirements. During the conceptual design and the preliminary design processes the wing design will be finished and analysed as a whole. Therefore, in this phase the wing design will be broken down into surface materials, flats and spoilers, individual ribs, spars, each of which must be separately designed and analysed. Moreover, the production design will determine how the aircraft will be made using the small and simple subassemblies and building up to the final assembly process.

7 Computational process for CFD analysis

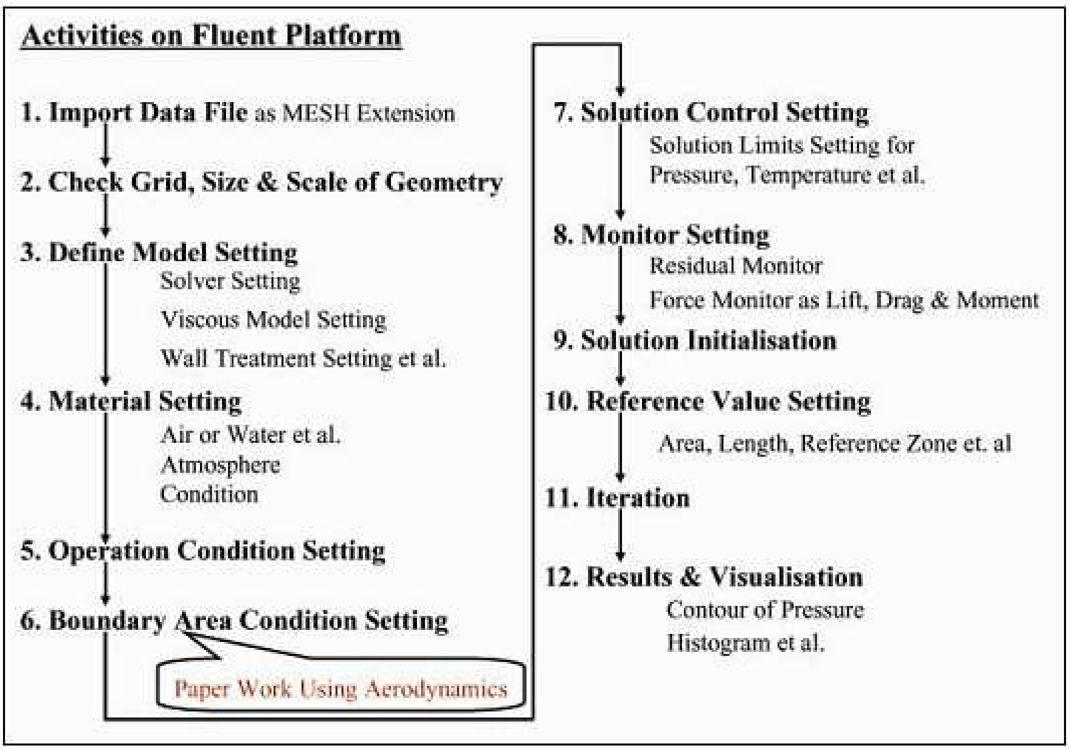
(a) 1st Step: Model is designed on CAD softwares, such as CATIA V5, SolidWorks 2002 and Pro/Engineer 2002.

(b) 2nd Step: The CAD Model is imported from CATIA V5 into Pre-processing software, such as Gambit 2, HyperMesh and TGrid 3, to create meshing surface and meshed volume of boundary area.

(c) 3rd Step: The Meshed Volume design is imported from the Pre-processing software into FLUENT 6 to analyse aerodynamic performance of the configuration.

The flow chart for CFD analysis is shown in Figure 3.

Figure 3. Flow chart for CFD analysis



7.1 Case study 2 Prototype-based Design

Goodyear®, the only major U.S. tire company, was founded in 1898 and is headquartered in Akron, Ohio. Its primary products include theGoodyear®, Dunlop®, Kelly®,, Fulda®, and Sava®, brands. With revenues of 19.6 billion in 2007, the company has more than 60 manufacturing operations in 26 countries and 70,000 employees worldwide. Its two majortechnical centres are located in Akron and Colmar-Berg, Luxembourg. Backin 1992, failed takeover attempt had drained cash reserves of this company and under pressure to reduce R and D expenditures, VP's of Research and Product Development sponsored a study of alternative product development methods. Three alternatives were identified:

- More efficient process of building and testing prototypes
- Extensive use of predictive testing
- Physics-based performance prediction

In 2003 and 2004, the Goodyear Tyre and Rubber Company found itself in a definite slump, suffering declining revenues and losing out to its two main competitors, Michelin and Bridgestone. In response, Goodyear leveraged its high performance computer clusters and its ongoing collaborative relationship with the Sandia National Laboratories to change the way it developed tyres. Rather than designing, building and testing physical prototypes, Goodyear engineers used modeling and simulation to test virtual models and significantly cut time to market. The result was the Assurance® all-weather tyre featuring TripleTred Technology® a huge hit that helped Goodyear not only climb out of the hole it was in, but continue on to launch a flurry of new tyres that resulted in record profits.

8 Conclusions:

Computational scientists create mathematical models and simulations of physical, biological and chemical phenomena and systems, which allow them to better understand these subjects and predict their behavior. Such research has made computational science a third pillar of science, along with theory and experimentation. The way to achieve more computational scientists are, integration of science, technology, engineering, and mathematics into a trans-disciplinary subject where the principles of science and the analysis of mathematics are combined with the design process of technology and engineering. Also introducing scientific computing, computational mathematics and CFD as part of curriculum and introduce the concept of mathematical modeling by assigning projects from real life situations will create computational scientists.

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An existence theorem for mixed iterated function systems in B-metric spaces

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Abstract : The theory of iterated function systems (IFS) and iterated multifunction system (IMS) is extensively studied in the literature. Mihail and Miculescu [15] introduced the notion of generalized iterated function system (GIFS) as an extension of the notion of IFS and obtained interesting results. Our purpose is to define a generalized mixed iterated function system on b-metric spaces and obtain an existence and uniqueness result.

Key-Words: b-metric space, Iterated function system, Meir-Keeler contraction.
2010 Mathematics Subject Classification. 47H10, 54H25.

1 INTRODUCTION

Fixed point theorems are of prime importance in the theory of iterated function systems (IFS). Michael Barnsley and Steven Demko [4] popularized the IFS theory after Hutchinson [10] gave a formal definition of it in 1981. It was born as an application of the theory of discrete dynamical systems and has important applications in image compression, modeling, computer graphics and various other areas of engineering and applied sciences. This basic notion of IFS has been extended and enriched to more general settings by changing the condition on mappings or the space by various authors, see for instance, [1], [3], [9], [13], [15-19], [21-25] and several references thereof. In [1] and [9] contraction maps are replaced by weakly contractive or non-expansive maps. Rus and Triff [27] replaced contraction constant by a comparison function to obtain their results. In [11] and [12] the formulations of the contraction due to Meir and Keeler [14] have been used to generalize the IFS theory. Mihail and Miculescu [15] introduced the notion of generalized iterated function system (GIFS), which is a family of functions in a complete metric space and showed GIFS to be a natural generalization of the notion of IFS (see [16-19]). Llorens-Fuster et al [13] defined mixed iterated function system by taking more general conditions and obtained a mixed iterated function system theory for contraction and Meir-Keeler contraction maps. Our aim is to define generalized mixed iterated function system on b-metric spaces and obtain some existence and uniqueness results. In this respect, our results extend and improve some well known previous results given in [14], [16] and [20].

2 PRELIMINARIES

Definition 2.1 ([8]). Let X be a non empty set and $b \geq 1$ be a given real number. A function $d : X \times X \to R_+$ is said to be a b-metric iff for all $x, y, z \in X$ the following conditions are satisfied

(i) $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$

The pair (X, d) is called a b-metric space.
It is well known that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric space is a metric space when $b = 1$ in the above condition (iii).

Definition 2.2 ([2]). Let (X, d) be a complete b-metric space. Then the Hausdorff distance between points A and B in $K(X)$, the collection of nonempty compact sub sets of X , is defined by
h $(A,\ B)$= max$\{d(A,\ B),\ d(B,\ A)\}$, where $d(A,\ B) = \max\{\min\{d(x,\ y)\colon\ y \in \mathrm{B}\}\colon\ x \in A\}$.

The Hausdorff space $(K(X),\ h)$ is also called as a Fractal space (see Barnsley [2]).
It is known that $(K(X), h)$ is a complete b-metric space provided (X, d) is a complete b-metric space (see Czerwik [8]).

Definition 2.3 ([7]). Let (X, d) be a b-metric space. Then a sequence $\{x_n\}_{n\in N}$ in X is called:

(i) convergent if and only if there exists $x \in X$, such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

(ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Remark 2.1 ([7]). In a b-metric space (X, d), the following assertions hold:

(i) a convergent sequence has a unique limit;

(ii) each convergent sequence is Cauchy;

(iii) in general, a b-metric is not continuous.

Definition 2.4 ([7]). The b-metric space (X, d) is complete if every Cauchy sequence in X converges.

Definition 2.5 ([2]). Let (X, d) be a complete metric space and and $f_n : X \to X$, $n = 1, 2, 3, \ldots, N$ be contractions with the corresponding contractivity factors $b_n, n = 1, 2, 3, \ldots, N$. Then the system $\{X; f_n, n = 1, 2, 3, \ldots, N\}$ is called an iterated function system (IFS) in the metric space (X, d) with contractivity factor $b = \max\{b_n : n = 1, 2, 3, \ldots, N\}$.

Theorem 2.1 ([2])**.** Let (X, d) be a complete metric space. If $w_n : X \to X$ is a contraction with respect to the metric d for $n = 1, 2, \ldots, N$, then there exists a unique non-empty compact subset A of X that satisfies $A = f_1(A) \bigcup f_2(A) \bigcup \cdots \bigcup f_N(A)$. A is called the self-similar set with respect to $\{f_1, f_2, \ldots, f_N\}$.

Definition 2.6 ([2])**.** Let (X, d) be a complete metric space and $\{X; f_n$, n = 1, 2, 3, ..., $N\}$ be an IFS. The Hutchinson-Barnsley operator (HB operator) of the IFS is a function W: $K(X) \to K(X)$ defined by $W(B) = \bigcup_{n=1}^{N} f_n(B)$, for all $B \in K(X)$.

Theorem 2.2 ([2])**.** Let $\{X; f_n$, n = 1, 2, 3, ..., $N\}$ be an IFS of contraction mappings on metric space X and W of the HB operator of the IFS. Then,

(i) The HB operator W is a contraction mapping on $K(X)$.

(ii) There exists only one compact invariant set $A_8 \in K(X)$ of the HB operator W called the attractor (fractal) of IFS or equivalently, W has a unique fixed point namely $A_\infty \in K(X)$.

Definition 2.7 ([22])**.** A mapping $\phi : R_+ \to R_+$ is called a comparison function if it is increasing and $\phi^n(t) \to 0, n \to \infty$ for any $t \in R_+$.

Lemma 2.1 ([22])**.** If $\phi : R_+ \to R_+$ is a comparison function, then:

(i) each iterate ϕ^k of ϕ, $k \geq 1$, is also a comparison function;

(ii) ϕ is continuous at zero;

(iii) $\phi(t) < t$, for any $t > 0$.

The following concept of (c)-comparison function is introduced by Berinde [6].

Definition 2.8 ([6])**.** A function $\phi : R_+ \to R_+$ is called a (c)-comparison function if:

(i) ϕ is increasing;

(ii) there exist $k_0 \in N, a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\phi^{k+1}(t) \leq a\,\phi^k(t) + v_k, \tag{2.1}$$

for $k \geq k_0$ and any $t \in R_+$.

Berinde [6] extended the concept of (c)-comparison to b-comparison functions in the framework of b-metric space in the following manner.

Definition 2.9 ([6])**.** Let $b \geq 1$ be a real number. A mapping $\phi : R_+ \to R_+$ is called a b-comparison function if:

(i) ϕ is monotone increasing;

(ii) there exist $k_0 \in N, a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$b^{k+1}\phi^{k+1}(t) \leq ab^k\,\phi^k(t) + v_k, \tag{2.2}$$

Lemma 2.2 ([5])**.** If $\phi : R_+ \to R_+$ is a b-comparison function, then:

(i) the series $\sum_{k=0}^{\infty} b^k \phi^k(t)$ converges for any $t \in R_+$;

(ii) the function $s_b : R_+ \to R_+$ defined by $s_b(t) = \sum_{k=0}^{\infty} b^k \phi^k(t)$, $t \in R_+$, is increasing and continuous at 0.

Lemma 2.3 ([20])**.** Any b-comparison function is a comparison function.

Definition 2.10 ([26])**.** Let us consider a map $f : X \to X$, we say that f is a

(i) $\phi-$ contraction if $d(f(x), f(y)) \leq \phi\,(d(x, y))$, for any $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is an increasing map and $\phi^p(t) \to 0$ as $p \to \infty$, for every $t \geq 0$, where $\phi^p = \phi \circ \phi^{p-1}$ means the p-times composition of ϕ.

(i) Banach contraction with the contraction factor $a \in [0, 1)$, if it is $\phi-$ contraction, where $\phi(t) = a\,t$;

(i) Contractive if $d(f(x), f(y)) < d(x, y)$, for all $x, y \in X$, $x \neq y$.

(i) Meir-Keeler type mapping if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in X, \varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(f(x), f(y)) < \varepsilon.$$

Definition 2.11 ([22])**.** Let (X, d) be any b-metric space. An operator $f : X \to X$ is a Picard operator if:

(i) Fix $f = \{x^*\}$ where Fix $f = \{x \in X | x = f(x)\}$

(ii) $f^n(x) \to x^*$, as $n \to \infty$, for all $x \in X$.

Now we define generalized mixed iterated functions on the patterns of Llorens-Fuster et al [13] and Mihail et al [15].

Definition 2.12. A function $f : X^m \to X$, is said to be a:

(i) generalized $\phi-$ contraction, if ϕ is a comparison function and for each $x_1, x_2, \ldots, x_m$; $y_1, y_2, \ldots, y_m \in X$, such that $x_i \neq y_i$ for some $i \in \{1, 2, \ldots, n\}$, we have,

$$d(f(x_1, x_2, \ldots, x_m), f(y_1, y_2, \ldots, y_m)) \leq \\ \phi(\max\{d(x_1, y_1), d(x_2, y_2), \ldots, d(x_m, y_m)\})$$

(ii) generalized a-contraction [15], if $a \in [0, 1)$ and for each

$x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_m \in X$, such that $x_i \neq y_i$ for some $i \in \{1, 2, \ldots, n\}$, we have,

$$d(f(x_1, x_2, \ldots, x_m), f(y_1, y_2, \ldots, y_m)) \leq \\ a\, \max\{d(x_1, y_1), d(x_2, y_2), \ldots, d(x_m, y_m)\})$$

where, contractivity factor is defined as follows:

$$a = \sup_{\substack{x_1, \ldots, x_m; y_1, \ldots, y_m \\ \max\{d(x_1, y_1), \ldots, d(x_m, y_m)\} > 0}} \frac{d(f(x_1, \ldots, x_m), f(y_1, \ldots, y_m))}{\max\{d(x_1, y_1), \ldots, d(x_m, y_m)\}}$$

(iii) generalized Meir Keeler contraction if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_m \in X$, we have

$$\max\{d(x_1, y_1), \ldots, d(x_m, y_m)\} < \varepsilon + \delta \Rightarrow d(f(x_1, \ldots, x_m), f(y_1, \ldots, y_m)) < \varepsilon$$

(iv) generalized contractive if for each $x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_m \in X$, such that $x_i \neq y_i$ for some $i \in \{1, 2, \ldots, n\}$, we have

$$d(f(x_1, x_2, \ldots, x_m), f(y_1, y_2, \ldots, y_m)) < \max\{d(x_1, y_1), d(x_2, y_2), \ldots, d(x_m, y_m)\}$$

Now we define generalized mixed iterated function system or GMIFS from $X^m = \times_{k=1}^m X \to X$, rather than contractions from X to itself.

Definition 2.13. Let (X, d) be a complete b-metric space and $m \in N$. A generalized mixed iterated function system or GMIFS on X of order m is defined by $S = (X, (f_k)_{k=1,n})$, consists of a finite family of functions $(f_k)_{k=1,n}$, $f_k : X^m \to X$ such that $f_1, f_2, \ldots, f_n$ are generalized $\phi-$ contraction or generalized Banach contraction or generalized Meir Keeler contraction or generalized contractive.

Definition 2.14 ([16])**.** Let $f : X^m \to X$ and $K(X)$ be the set of all non-empty compact subsets of X. The function $F_f : K(X)^m = \times_{k=1}^m K(X) \to K(X)$ is defined by

$$F_f(K_1, K_2, \ldots, K_m) = f(K_1 \times K_2 \times ... \times K_m)$$
$$= \{f(x_1, x_2, \ldots, x_m) : x_j \in K_j, \forall\, j \in \{1, \ldots, m\}\}$$

for all $K_1, K_2, \ldots, K_m \in K(X)$, is called the set function associated with function f.

Definition 2.15. Let $S = (X, (f_k)_{k=1,n})$ be a GMIFS. The function $F_S : K(X)^m \to K(X)$ is defined by $F_S(K_1, K_2, \ldots, K_m) = \bigcup\limits_{k=1}^{n} F_{f_k}(K_1, K_2, \ldots, K_m)$ for all $K_1, K_2, \ldots, K_m \in K(X)$ is called the set function associated with the GMIFS S.

3 MAIN RESULTS

Theorem 3.1. Let (X, d) be a complete b-metric space such that the b-metric is a continuous functional and $f : X^m \to X$ be a generalized ϕ -contraction with ϕ a b-comparison function or generalized Meir Keeler contraction. Moreover, for any $x_0, x_1, \ldots, x_{m-1} \in X$, the sequence $(x_n)_{n \geq 1}$ defined by $x_{n+m} = f(x_n, x_{n+1}, \ldots, x_{n+m-1})$, $\forall\, n \in N$ has the property that $\lim\limits_{n \to \infty} x_n = \alpha$. Then there exists a unique $\alpha \in X$ such that $f(\alpha, \alpha, \ldots, \alpha) = \alpha$.

Proof. (i) First we prove that if $f : X^m \to X$ be a generalized ϕ -contraction then there exists a unique $\alpha \in X$ such that $f(\alpha, \alpha, \ldots, \alpha) = \alpha$.
Let $x_0, x_1, \ldots, x_{m-1} \in X$. Since the sequence $\{x_n\}$ is defined by
$x_{n+m} = f(x_n, x_{n+1}, \ldots, x_{n+m-1}), \forall\, n \in N$, we have

$$d(x_{n+m}, x_{n+m+1}) = d(f(x_n, x_{n+1}, \ldots, x_{n+m-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+m}))$$
$$\leq \phi\{\max(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \ldots, d(x_{n+m-1}, x_{n+m}))\}$$

which by induction yields

$$d(x_{n+m}, x_{n+m+1}) \leq \phi^{n-m+1}\{\max(d(x_0,x_1), d(x_1,x_2), \dots, d(x_{m-1},x_m)\} \tag{3.1}$$

As d is a b-metric, for $n \geq 0$, $p \geq 1$, we obtain:

$$d(x_{n+m}, x_{n+m+p}) \leq \begin{array}{l} bd(x_{n+m}, x_{n+m+1}) + \\ b^2 d(x_{n+m+1}, x_{n+m+2}) + \cdots + b^p\, d(x_{n+m+p-1}, x_{n+m+p}) \end{array} \tag{3.2}$$

$$d(x_{n+m}, x_{n+m+p}) \leq \begin{array}{l} b\,\phi^{n-m+1}\{\max(d(x_0,x_1), d(x_1,x_2), \dots, d(x_{m-1},x_m))\} + \\ b^2\,\phi^{n-m+2}\{\max(d(x_0,x_1), d(x_1,x_2), \dots, d(x_{m-1},x_m))\} \\ + \cdots + \\ b^p\,\phi^{n-m+p}\{\max(d(x_0,x_1), d(x_1,x_2), \dots, d(x_{m-1},x_m))\} \end{array} \tag{3.3}$$

$$d(x_{n+m}, x_{n+m+p}) \leq \frac{1}{b^{n-m}} \left[\begin{array}{l} b^{n-m+1}\,\phi^{n-m+1} \max\{d(x_0,x_1), d(x_1,x_2), \dots, d(x_{m-1},x_m)\} + \\ b^{n-m+2}\phi^{n-m+2} \max\{d(x_0,x_1), d(x_1,x_2), \dots, d(x_{m-1},x_m)\} \\ + \cdots + \\ b^{n-m+p}\phi^{n-m+p} \max\{d(x_0,x_1), d(x_1,x_2), \dots, d(x_{m-1},x_m)\} \end{array} \right] \tag{3.4}$$

Suppose $S_n = \sum_{k=0}^{n} b^k \phi^k (\max\{d(x_0,x_1), d(x_1,x_2), \dots, d(x_{m-1},x_m\})$, for $n \geq 1$. Then (3.4) becomes

$$d(x_{n+m}, x_{n+m+p}) \leq \frac{1}{b^{n-m}} \left[S_{n-m+p} - S_{n-m}\right].$$

Supposing $d(x_0,x_1) > 0$, $d(x_1,x_2) > 0$, $\dots$,$d(x_{n-1},x_n) > 0$, $n,m,p \geq 1$, the series $\sum_{k=0}^{\infty} b^k \phi^k (\max\{d(x_0,x_1), d(x_1,x_2), \dots, d(x_{n-1},x_n\})$ converges, so there is $S = \lim_{n\to\infty} S_n \in R_+$. Since$b \geq 1$, thus $\{x_n\}$ is a Cauchy sequence in the complete b-metric space(X,d). Therefore, there exists $\alpha \in X$, such that $\alpha = \lim_{n\to\infty} x_n$.
Now we prove that $f(\alpha,\alpha,\dots,\alpha) = \alpha$.
For $n,m \geq 0$, we have

$$d(x_{n+m}, f(\alpha,\alpha,\dots,\alpha)) = d(f(x_n, x_{n+1}, \dots, x_{n+m-1}), f(\alpha,\alpha,\dots,\alpha))$$
$$\leq \phi\{\max(d(x_n,\alpha), d(x_{n+1},\alpha), \dots, d(x_{n+m-1},\alpha))\}$$

But d is a continuous and ϕ is also continuous at 0.
Letting $m,n \to \infty$, we obtain
$d(\alpha, f(\alpha,\alpha,\dots,\alpha)) \leq 0$, which implies $f(\alpha,\alpha,\dots,\alpha) = \alpha$.
Suppose there exist $\beta \in X$ such that $\beta = f(\beta,\beta,\dots,\beta)$ and $\alpha \neq \beta$, then we have
$d(\alpha,\beta) = d(f(\alpha,\dots,\alpha), f(\beta,\dots,\beta)) \leq \phi\{\max(d(\alpha,\beta),\dots,d(\alpha,\beta))\} \leq \phi(d(\alpha,\beta))$

which is a contradiction.
Thus there exists a unique $\alpha \in X$ such that $f(\alpha,\alpha,\dots,\alpha) = \alpha$.
(ii) Now we consider f to be a generalized Meir Keeler contraction.
For $x_1,\dots,x_m, y_1,\dots,y_m \in X$, putting $\varepsilon = \max\{d(x_1,y_1),\dots,d(x_m,y_m)\}$,
we obtain

$$d(f(x_1,\dots,x_m), f(y_1,\dots,y_m)) \leq \max\{d(x_1,y_1),\dots,d(x_m,y_m)\}.$$

So $\{f^n(x_1,\ldots,x_m), f^n(y_1,\ldots,y_m)\}$ is nonincreasing and thus converges.
From the assumption we have,

$$\lim_{n\to\infty} d(f^n(x_1,\ldots,x_m), f^n(y_1,\ldots,y_m)) = 0.$$

Fix $x_0, x_1, \ldots, x_{m-1} \in X$ and define a sequence $\{x_n\}$ in X by

$$x_{n+m} = f(x_n, x_{n+1}, \ldots, x_{n+m-1}),$$

Fix $\varepsilon > 0$, then $b\varepsilon > 0$, $b \geq 1$, is also true and there exists $\delta \in (0, b\varepsilon)$ or $\frac{1}{b}\delta \in (0,\varepsilon)$ such that
$\max\{d(x_1,y_1),\ldots,d(x_m,y_m)\} < \varepsilon + \frac{1}{b}\delta$ implies $d(f(x_1,\ldots,x_m), f(y_1,\ldots,y_m)) < \varepsilon$.
Since $\lim\limits_{n\to\infty} d(x_{n+m}, x_{n+m+1}) = 0$, there exists $p \in N$, with $d(x_{n+p}, x_{n+p+1}) < \delta$.
We shall show,

$$d(x_{n+p}, x_{n+m+p}) < \varepsilon + \delta, \tag{3.5}$$

for $m \in N$ by induction. For $m = 1$, it is obviously true.
We assume that it holds for some $m \in N$. Then we have $d(x_{n+p+1}, x_{n+m+p+1}) < \varepsilon$, but ε be any nonnegative small number, so $d(x_{n+p+1}, x_{n+m+p+1}) < \frac{1}{b}\varepsilon$ is also true and hence

$$d(x_{n+p}, x_{n+m+p+1}) \leq b(d(x_{n+p}, x_{n+p+1}) + d(x_{n+p+1}, x_{n+m+p+1})) \leq \delta + \varepsilon.$$

So, by induction (3.5) holds for every $m \in N$. Therefore we have shown

$$\lim_{n\to\infty} \sup_{m>n} \mathrm{d}(x_n, x_m) = 0.$$

This implies that $\{x_n\}$ or $\{f^n x\}$ is a Cauchy sequence.
Since X is complete, $\{x_n\}$ converges to some point $\alpha \in X$.
Now we prove that $f(\alpha,\ldots,\alpha) = \alpha$. For $n, m \geq 0$, we have

$$\begin{aligned} d(x_{n+m}, f(\alpha,\ldots,\alpha)) &= d(f(x_n, x_{n+1},\ldots,x_{n+m-1}), f(\alpha,\ldots,\alpha)) \\ &\leq \max\{d(x_n,\alpha),\ldots,d(x_{n+m-1},\alpha)\} \end{aligned}$$

But d is continuous, so letting $m, n \to \infty$, we obtain
$d(\alpha, f(\alpha,\ldots,\alpha)) \leq 0$ which implies $f(\alpha,\ldots,\alpha) = \alpha$.
That is, α is a fixed point of *f*. Suppose for every $\beta \in X$, we have
$f(\beta,\ldots,\beta) = \beta$, $\varepsilon = \max\{d(\alpha,\beta),\ldots,d(\alpha,\beta)\}$ and $\alpha \neq \beta$, then we have

$$\begin{aligned} &d(\alpha,\beta) = d(f(\alpha,\ldots,\alpha), f(\beta,\ldots,\beta)) < \varepsilon \\ &= \max\{d(\alpha,\beta),\ldots,d(\alpha,\beta)\} \\ &< d(\alpha,\beta). \end{aligned}$$

This is a contradiction. So the fixed point is unique. □

If we substitute $m = 1$ in part (i) of Theorem 3.1 above, the following result of M. Pacurar [Theorem 4, part 1, 23] is obtained.

Corollary 3.1 ([20]). Let (X,d) be a complete b-metric space such that the b-metric is a continuous functional and $f : X \to X$ be a $\phi-$ contraction with ϕ a b-comparison function. Then there exists a unique $\alpha \in X$ such that $f(\alpha) = \alpha$, that is, f is a Picard operator.

In part (ii) of Theorem 3.1, if we put $m = 1$, $b = 1$ and define $x_{n+1} = f(x_n)$, the following result of Meir and Keeler [14] and Suzuki [28] is obtained.

Corollary 3.2 ([14],[28])**.** Let (X, d) be a complete metric space and $f : X \to X$ be a Meir Keeler contraction, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \varepsilon + \delta \Rightarrow d(T(x), T(y)) < \varepsilon$$

Then there exists a unique $\alpha \in X$ such that $f(\alpha) = \alpha$.

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Egyptian Fractions: Unit Fractions, Hekats and Wages - An update and Innovations

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Abstract : The present paper updates three ancient Egyptian fraction texts, the Akhmim Wooden Tablet, the Egyptian Mathematical Leather Roll, and the Rhind Mathematical Papyrus. The three hieratic texts were written in the Egyptian Middle Kingdom era (2050 BCE to 1550 BCE), a time of innovation. The paper demonstrates that neglected scribal number theory was written in finite arithmetic. The finite arithmetic scaled rational numbers and commodities to exact unit fraction series. Modern scholars transliterated certain hieratic shorthand notes into hieroglyphic. Other original scribal notes were improperly transliterated by modern scholars. This lack of clarity confused the historical record. Despite omissions, scholars showed that the scribes used algebraic methods, common denominators, progressions, inverses, and proportions that calculated areas, quotients, remainders, solutions to second degree equations, slopes, and volumes. The update of three texts repairs scribal shorthand notes. By including missing information, complete ancient arithmetic sentences are written and appreciated in modern arithmetic. The three updated hieratic texts reveal previously unreported scribal skills, properties of ancient number theory, and aspects of ancient economic life that were enhanced by Egyptian fraction innovations.

1 INTRODUCTION

Egyptian fraction arithmetic was closely linked to Old Kingdom mathematics. Old Kingdom numeration, and weights and measures systems were cursive and binary. The hieroglyphic numeration system rounded off infinite series representations of rational numbers to six terms. The Horus-Eye weights and measures system overlooked a 1/64 unit rounding error within a 6-term finite series (Gillings 1972).

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64$$

Note that to exactly sum to unity (1), a 1/64 unit must to be added to the Eye of Horus series

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/64 = 1$$

The binary aspect of the Old Kingdom weights and measures was used in balance beam valuations of commodities, business transactions, and higher math. A significant change in writing binary unit fraction series as Eye of Horus series took place in the Middle Kingdom. The inexact Eye of Horus series was replaced after 2050 BCE with exact series.

Scribes thereafter introduced exact Egyptian fraction innovations. One innovation demonstrated finite arithmetic within balanced algebraic statements that converted rational number n/p to exact unit fraction series. For example, 4/7 was written as $1/2 + 1/14$ (Silverman 1975). Middle Kingdom arithmetic written in Egyptian fraction series corrected. Old Kingdom weights and measure rounding errors. Scribes corrected binary measurements and balanced algebraic statements by including the missing portion of the 1/64 of a unit or portion thereof (Gardner 2006). Other exacting innovations created finite units of measure (Gardner 2008b).

Scholars, from1860 to the present, transliterated hieratic shorthand notes into fragmented sentences. Transliterations converted hieratic script to hiero glyphs and these transliterations were converted into modern arithmetic and sentences. The two-step translation process revealed algebraic methods, common denominators, progressions, inverses, primes, and proportions. Scribes found areas, proofs, quotients, remainders, second-degree equation solutions, slopes, and volumes (Belluck 2010, Gillings 1972).

The deeper aspects of hieratic shorthand were not translated well enough to expose subtle scribal innovations. Scholars worked hard to explain concise 2/n tables and unit fraction statements that relied on the 2/n tables (Peet 1923, Chace 1927). The Egyptian fraction historical record under-reports scribal methods that included rational numbers, least common multiples, common divisors, and **red** auxiliary numbers (Gardner 2011). Poor 20th century transliterations led to inconsistent translations.

This chapter begins to correct the Egyptian fraction historical record by updating six introductory Egyptian Mathematical Leather Roll (EMLR) rules and asks the question, "Howwere the best unit fraction series selected by scribes?" Proposed answers validate additional scribal innovations, and scribal arithmetic skills that implemented each innovation. The scope of this chapter is limited to early number theory and volume topics reported in the EMLR, the Rhind Mathematical Papyrus (RMP), and the Akhmim Wooden Tablet (AWT). Other related metrology topics are not discussed. A future paper may be dedicated to the weighing of bread, gold, silver, tin, and commodities in the unit called debens.

2 UPDATING METHODOLOGY

Egyptian fraction texts are reported in historical context free from modern mathematical metaphors. To achieve readability in modern arithmetic, scribal shorthand conventions are replaced by a seldom-used scribal longhand conven tion. For example, the EMLR began with:

$$1/8 = 1/10 + 1/40$$

The simplest possible method (Occams razor) created an equality likely included LCMs and **red** auxiliary numbers. A scribal longhand convention reported in RMP 36 and RMP 37, scaled 1/8 by LCM 5 to 5/40. The best divisors of 40 were selected by the scribe that summed to numerator 5. Only $4 + 1$ was available in this case. RMP 36 would have recorded $4+1$ in **red**. Applying the seldom used RMP longhand convention to the EMLR, line 1 can be re-written as:

$$1/8(5/5) = 5/40 = (4+1)/40 = 1/10 + 1/40.$$

3 EGYPTIAN MATHEMATICAL LEATHER ROLL AND THE RHIND MATHEMATICAL PAPYRUS

This update focuses on the EMLR and early number theory that connect to the RMP $2/n$ table (Gardner 2002). The EMLR converted 17 rational numbers to 26 unit fraction series. Scribal errors muddled three of the series. Six $1/2n$ rational numbers $1/2, 1/4, 1/8, 1/16, 1/32$, and $1/64$ and ten other rational numbers $1/7, 1/9, 1/10, 1/11\ 1/13, 1/14, 1/15, 1/20,$ and $1/30$ were converted to unit fraction series. One other trivial case reported $1/6+1/6 = 1/3$. Eight of the 17 rational numbers appeared twice, and one rational number, 1/8, appeared thrice times reporting three different unit fraction series. The 1/13 line was unreadable.

A nine year old paper suggested that the EMLR was encoded by six rules:

$$1/2n = (1/A)(A/2n) \qquad \text{(Rule 1.0)}$$
$$1/p = (1/A)(A/p) \qquad \text{(Rule 2.0)}$$
$$1/pq = (1/A)(A/pq) \qquad \text{(Rule 3.0)}$$
$$1/8 = (1/25)(25/8) = 25/200 = 1/25 + 17/200 \qquad \text{(Rule 4.0)}$$
$$17/200 = 1/15 + 1/75 + 1/200 \qquad \text{(Rule 5.0)}$$
$$1/8 = 1/25 + 1/15 + 1/75 + 1/200 \qquad \text{(Rule 6.0)}$$

The approach erroneously suggested that a modern $1/p = (1/A)/(A/p)$ method was used by the EMLR student scribe (Gardner 2002). This update proposes to repair the EMLR historical record by showing that the EMLR scaled rational numbers $1/p$ and $1/pq$ by seven LCMs that applied six rules:

$$(1/2n)(m/m) = m/2mn \qquad \text{(Rule 1.1)}$$
$$(1/p)(m/m) = m/mp \qquad \text{(Rule 2.1)}$$
$$(1/pq)(m/m) = mp/mpq \qquad \text{(Rule 3.1)}$$

Both:

$$(1/8) = (1/25)(25/8) = 25/200 = 1/25 + 17/200 \qquad \text{(Rule 4.1)}$$
$$(1/16) = (1/25)(25/16) = 25/400 = 1/50 + 17/400$$

Both:

$$(17/200)(6/6) = 102/1200 = 1/15 + 1/75 + 1/200 \qquad \text{(Rule 5.1)}$$
$$(17/400)(6/6) = 102/2400 = 1/30 + 1/150 + 1/400$$

Both:

$$1/8 = 1/25 + 1/15 + 1/75 + 1/200 \qquad \text{(Rule 6.1)}$$
$$1/16 = 1/50 + 1/30 + 1/150 + 1/400$$

The EMLR cited two out-of-order series without demonstrating a calculation method. Line 8 of the EMLR converted $1/8$ to $1/25 + 1/15 + 1/75 + 1/200$. Searching for the scribal calculation method, Rule 4.1 scaled $1/8$ by LCM 25 to $25/200$, which subtracted

$1/25$ obtained $17/200$. Rule 5.1 scaled $17/200$ by LCM 6 to the unit fraction series $1/15 + 1/75 + 1/200$. Rule 6.1 reported the total. Line 9 of the EMLR converted $1/16 to 1/50 + 1/30 + 1/150 + 1/400$. Rule 4.1 scaled $1/16$ by LCM 25 to $25/400$ and subtracted $1/50$, which obtained $17/400$. Rule 5.1 scaled $17/400$ by LCM 6 and obtained a unit fraction series. Rule 6.1 shows the total. Seven LCMs in total were used to scale EMLR rational numbers $1/2n, 1/p$ and $1/pq$. Final EMLR series were sometimes scaled to awkward unit fraction series. But, how were the best unit fraction series selected by the student scribe? To discuss the best unit fraction series question, another modern splitting proposal recommended:

$$1/(ab) = [1/(a + b)](1/a + 1/b) \qquad \text{(proposed rule 7.0)}$$

Using the example:

$$1/(4)(7) = (1/11)(4 + 7) = 1/44 + 1/77$$

a recent proposal asked if the EMLR student understood the rule (Malkevich 2011) ?

The seven LCMs scaled rational numbers $1/2n, 1/p$ and $1/pq$ to sometimes awkard unit fraction series. Restated EMLR statements include LCMs and real auxiliary numbers show that the student converted $1/8$, the thrice repeated rational number, by two sigle LCMs- 3 and 5- and a pair of LCM- 25 and 6 per :

1. $\frac{1}{8}\left(\frac{3}{3}\right) = \frac{3}{24} = \frac{(2+1}{24} = \frac{1}{12} + \frac{1}{24}$ (line 13)
2. $\frac{1}{8}\left(\frac{5}{5}\right) = \frac{5}{40} = \frac{(4+1)}{40} = \frac{1}{10} + \frac{1}{40}$ (line 1)
3. $\frac{1}{8}\left(\frac{25}{25}\right) = \left(\frac{25}{200}\right)$

(a) $25/200 = (8 + 17)/200 = (1/25 + 17/200)$

(b) $17/200(6/6) = 102/1200 = (80 + 16 + 6)/1200 = 1/15 + 1/75 + 1/200$

(c) $1/8 = 1/25 + 1/15 + 1/75 + 1/200$ (line 8)

The 26 restated EMLR statements used seven LCMs $2, 3, 5, 6, 7, 10$, and 25. Two $1/8$ and $1/16$ sentences used four LCMs- 3, 5, 25, and 6 and the remaining 24 sentences used six LCMs 2, 3, 5, 6, 7, and 10 (Gardner 2007). To answer the proposed question, based on the EMLR scaling of $1/14$ by LCM 3, consider:

$$1/14(3/3) = 3/42 = (2 + 1)/42 = 1/21 + 1/42 \text{(line 21)}$$

and the EMLR-like series:

$$1/28(3/3) = 3/84 = (2 + 1)/84 = 1/42 + 1/84$$

Suggests the EMLR student would have scaled $1/28$ by LCM 3. Moreover, the EMLR student would have considered LCM 6, 8, 10, 12, and 14, such that:

(a) $1/28(6/6) = 6/168 = (3 + 2 + 1)/168 = 1/56 + 1/84 + 1/168$

(b) $1/28(8/8) = 8/224 = (7 + 1)/224 = 1/32 + 1/224$

(c) $1/28(10/10) = 10/280 = (7 + 2 + 1)/224 = 1/40 + 1/140 + 1/280$

(d) $1/28(12/12) = 12/336 = (7+3+2)/336 = 1/49 + 1/112 + 1/168$

(e) $1/28(14/14) = (14/392) = (7+4+2+1)/392 = 1/56 + 1/98 + 1/196 + 1/292$

(f) $1/28(16/16) = 16/448 = (8+7+1)/448 = 1/56 + 1/64 + 1/448$

Note that alternate EMLR-like series last-term denominators are larger than the LCM 3 series last term denominator.

Going on, was the best EMLR series the one with smallest last-term de- nominator? Or, was the best EMLR series the one with shortest series and the smallest first-term denominator? The question of the best EMLR conversion of 1/n and 1/p to a unit fraction series can be answered by discussing the best RMP 2/n table series (Gardner 2002, 2009b). To begin, 87 RMP problems used three rational number conversion methods:

(a) $2/n(m/m) = 2m/mn$ (Rule 8.0)

(b) $n/p = (n-2)/p + n/p$ (Rule 9.0)

(c) $p/p =$ (numerators summed to p)$/p$ (Rule 10.0)

The 2/n table scaled 2/3, 2/5, 2/7 to 2/101 to the best unit fraction series (Gardner 2008a) The 51 rational numbers were scaled by 15 LCMs2, 3, 4, 6, 8, 10, 12, 20, 24, 30, 36, 40, 56, 60, and 70followed by 28 sets of red auxiliary numbers; facts are included in Appendix II.

Red auxiliary numbers did not appear in the majority of the RMPs short- hand notes. As scholars reported, red auxiliary numbers were cited in RMP 7 through RMP 20 as completion problems (Gillings 1972). This paper suggests that scribal constructions of the 2/n table must include LCMs and red auxiliary numbers. By considering the best LCMs and the best red auxiliary numbers presented in RMP 36 and 37, wider views of 2/n table calculations and scribal skills are offered. Scholarly reviews of scribal shorthand notes suggest that LCMs and red auxiliary numbers were not understood beyond RMP 7 to RMP 20 completion problems (Gillings 1972). Appendix II is written in scribal longhand that cites the best LCMs and the best red auxiliary numbers. The scribal longhand offers an additional scribal innovation.

Rule 8.0 scaled n/p by LCM m to mn/mp that defined the scribal longhand innovation. The best divisors of denominator mp were summed to numerator mn. Each red auxiliary number wasdivided by mp that calculated a unit fraction. The finite sum of the unit fractions equaled the initial rational number n/p.

A breakdown of 28 sets of red auxiliary numbers report one set, $(3+1)$, that appeared 16 times; a second set, $(5+1)$, that appeared four times; a third set, $(7+1)$, that appeared three times; and a fourth and fifth set, $(11+1)$ and $(19+3+2)$, that each appeared twice. Twenty- three (23) sets of red number numbers appeared once. The 23 single red number series followed Rule 1 were summed to numerator 2m.

Rule 8.0 did not work for 30/53, 28/97, and other rational numbers. The second method (Rule 9.0) solved otherwise impossible n/p conversions. The method was a parallel to the EMLR lines that scaled 1/8 and 1/16 by LCM 25 and LCM 6. Rule 8.0 replaced n/p with (n -2)/p + 2/p and solved (n -2)/p by one LCM with the best red auxiliary numbers, and 2/p by a second LCM with the red auxiliary number series (Gardner 2009a).

Ahmes provided two proofs in RMP 36 and RMP 31. The first replaced 30/53 with 28/53 + 2/53, and solved 28/53 by LCM 2 and 2/53 by LCM 30 (in RMP 36). The second proof replaced 28/97 with 26/97 + 2/97, and solved 26/97 by LCM 4 and 2/97 by LCM 56 (in RMP 31) (Gardner 2009a).

Rule 9.0 points out an important use of 2/n tables, a method that converted difficult rational numbers by solving two rational numbers by Rule 8.0 LCMs and red auxiliary numbers. The third RMP rule (Rule 10.0) partitioned unity (1) into vulgar fractions such as 53/53 into:

$$53/53 = 2/53 + 3/53 + 5/53 + 15/53 + 28/53$$

Rule 10.0 was mentioned in RMP 36. Ahmes partitioned the identity 53/53 into six Rule 8.0 unit fraction series:

$$53/53 = (2 + 3 + 5 + 15 + 28)/53 = 2/53 + 3/53 + 5/53 + 15/53 + 28/53$$

Five LCMs30, 20, 12, 4, and 2and three sets of red auxiliary numbers $(53 + 5 + 2), (53 + 4 + 2 + 1)$, and $(53 + 2 + 1)$ scaled the five $n/53$ rational numbers per:

$$2/53 \text{ by LCM } 30 = 60/1590 = (53 + 5 + 2)/1590 = 1/30 + 1/318 + 1/795$$
$$3/53 \text{ by LCM } 20 = 60/1060 = (53 + 4 + 2 + 1)/1060 = 1/20 + 1/265 + 1/530 + 1/1060$$
$$5/53 \text{ by LCM } 12 = 60/636 = (53 + 4 + 2 + 1)/636 = 1/12 + 1/159 + 1/106 + 1/212$$
$$15/53 \text{ by LCM } 4 = 60/212 = (53 + 4 + 2 + 1)/212 = 1/4 + 1/53 + 1/106 + 1/212$$

The importance of the rule 10.0 offers a fail-safe third conversion method that always scaled rational number n/p to a concise unit fraction series (Gardner 2009b). The method defined a virtual table. Virtual unity tables were used as an alternative to the second RMP n/p conversion rule. Focusing on RMP 36, 2/53 was scribal long hand scaled by LCM 30 to 60/1590 within the balanced sentence:

$$2/53 = 60/1590 = (53 + 5 + 2)/1590 = 1/30 + 1/318 + 1/795$$

Divisors of numerator $60 = (53 + 5 + 2)$ were recorded in red. Red made it clear that the best divisors had been selected. The first RMP conversion rule was used 24 times in the EMLR. Egyptian scribes tried to write the best unit fractions series that were available. Rephrasing scribal notes begins to parse the scribal skills topics. Considering Ahmes best 2/35 and 2/91 series by using an EMLR-type rule reports:

(a) $2/35 = 1/30 + 1/42$

  (a) as per proposed rule 7.0
  $a = 5$ and $b = 7$ such that

  (b) $2/35 = (1/6)(1/5 + 1/7) = 1/30 + 1/42$

(b) $2/91 = 1/70 + 1/130$

  (a) as per proposed rule 7.0 $a = 7$ and $b = 13$ such that

  (b) $2/91 = (1/10)(1/7 + 1/13) = 1/70 + 1/130$

Ahmes 2/n table data reported

$$2/35 = 1/30 + 1/42$$
$$2/91 = 1/70 + 1/30$$

There is little hard evidence to refute a claim that proposed rule 7.0 was used in the RMP 2/n table. One theme offers a contrary view that Ahmes may have scaled 2/35 by LCM 30, and 2/91 by LCM 70. A small fragment of 2/35 shorthand data mentions 6/210. Had 12/210 beenmentioned, full agreement with 2/35(6/6) = 12/210 reported in Appendix II would have closed this issue.

Since the issue is open to a small degree, readers are free to choose. Of course, had proposed rule 7/0 been known to Ahmes,

$$2/99 = [(9 + 11)(1/9 + 1/11) = 1/90 + 1/110$$

may have been the best 2/99 unit fraction series. Yet, Ahmes recorded:

$$2/99 = 1/66 + 1/198$$

It appears that LCM 3 was considered.

4 AKHMIM WOODEN TABLET AND THE RHIND MATHEMATICAL PAPYRUS

Egyptian scribes scaled rational numbers n/p to unit fraction series within practical statements. One hekat of grain was scaled to 64/64 of a hekat, and one hekat of grain was scaled to 320 ro1/320 a hekat. Thus, 64/64 hekat and 320 ro both meant 1 hekat. In RMP 36, (3/53)ro was scaled by LCM 20 to (1/20 + 1/265 + 1/530 + 1/1060) hekat

Scribal rational numbers, LCMs, red auxiliary numbers, 2/n tables, al- gebra, geometry, unit fraction series, and hekat (volume) units jump-started Middle Kingdom finite arithmetic. A developing economy was the beneficiary. Scribes created finite quotient and exact remainders data, combining theoretical methods that precisely valued commodities. Commodities, including beer and bread, were economically allocated for a range of purposes. One allocation paid pre- determined wages to a diverse labor force.

The AWT reports five exact divisions of a hekat by a quotient and scaled remainder method. The AWT detailed a hekat unity 64/64 divided by 3, 7, 10,11, and 13 using: (64/64)/n = Q/64 + (5R/n)ro

Example 4.1. $(64/64)3 = 21/64 + (5/3)(1/320) = (16 + 4 + 1)/64 + (5/3)ro = (1/4 + 1/16 + 1/64)hekat + (1 + 2/3)$ro

Each answer was proven by inverting the divisor and multiplying:

Example 4.2. $[(1/4 + 1/16 + 1/64)$hekat $+ (1 + 2/3)ro]$ times $3 = [21/64 + 5/3(1/320)]$ hekat times $3 = (63/64 + 1/64)$hekat $= 1$ hekat

Surviving records report that absentee landlords grew grain and flax for clothing. The hekat was used from the field to the commodities consumed as wages. Workers were paid at standard Middle Kingdom levels, from two to eight hekats a month. During flood years, low crop yields managed pay rates to be proportionally reduced (Ezzamel 2002).

The RMP and the Kahun Papyrus include three arithmetic progression allocations (Gardner 2008c). Several Middle Kingdom texts included pesu and hekat calculations. The RMP converted hekats into different strengths of pesu beer and bread by applying an inverse proportion (Clagett 1999). Distributions of commodities were achieved by arithmetic proportions discussed in RMP 40 and RMP 64 (Gardner 2008c). Scribal longhand included rational numbers, least common multiples, red auxiliary numbers, and other innovations. Scribal multiplication and division sentences reveal inverse operations with ancient number theory properties. Longhand volume sentences reveal that the hekat was scaled to 64/64, 320 ro, and a pesu unit (Gardner 2006).

Pesu sentences show that an inverse arithmetic method scaled strengths of loaves of bread, jugs of beer, and other commodities (Gillings 1972).

The improved Egyptian fraction numeration, and weights and measures assisted Pharaoh and absentee landlords to control granary inventories. Longhand sentences can be read in modern arithmetic. Unit fraction data included practical valuations of commodities double-checked inventories and allocations as wages. Farm productions and commodity inventories were often decentralized. Productions of bread, beer, and other grain-based product inventories were monitored. Inventory withdrawals were scaled in grain units to pay wages (Ezzamel 2002). Scribal algebraic geometry created linear cubits, square cubits, volume khar unit, and other hekat unit formulas. A cubit times a squared-cubit was transcribed as a cubit-cubit (Peet 1923, Gillings 1972). The cubit-cubit contained 3/2 khar. The khar contained 20 hekat. A hekat contained close to $4800\,cm^3$ when translated into modern metrics. Theoretical and practical cubit measurements of cubit-cubits, khar, 400-hekat, 100-hekat, 4-hekat, 2-hekat, 1-hekat, 4-ro, 2-ro, and 1-ro report hekat sub-divisions (Gardner 2011).

Scribes converted rational numbers to concise unit fraction series for several purposes. To value commodities, scribes usually scaled the hekat three ways. The first replaced one hekat with64/64 and applied a multiplier, 1/n, that created a finite quotient Q/64 plus a remainder (5R/n)ro. This class of hekat substitution was used in the AWT and over 40 times in the RMP (Gardner 2006). The second hekat form replaced one hekat with 320 ro. The 320 ro form created a multiplier 1/n and rational number quotients plus unit fraction remainder answers. Answers were double- checked by an inverse arithmetic operation (Gardner 2009a).
In RMP 38, a multiplier 7/22 reported:

$$\text{320ro times}7/22 = 2240/22 = (101 + 9/11)ro.$$

Rational number 9/11 scaled to a unit fraction series by LCM 4 such that:

$$9/11 \times (4/4) = 36/44 = (22 + 11 + 2 + 1)/44 = (1/2 + 1/4 + 1/22 + 1/44)$$

Ahmes recorded $(101 + 1/2 + 1/4 + 1/22 + 1/44)ro$ by writing ciphered sound symbols, denoted as fractions by placing a line over the symbol, writing from right to left, omitting plus (+) signs, not making it clear that that 101 was included.

Scribes recorded weights and measures in double-entry bookkeeping systems that made scribes accountants and mathematicians. Inventory control answers were proven. In RMP

38, Ahmes proved $(101 + 1/2 + 1/4 + 1/22 + 1/44)ro$ by multiplying by 22/7, the inverse of the 7/22 multiplier such that:

$$(101 + 1/2 + 1/4 + 1/22 + 1/44)(22/7)ro = (2204 + 22 + 11 + 2 + 1)/7ro = 320ro$$

In returning 320 ro, Ahmes commented that an exact hekat had been found. The scribal multiplication and division operations were inverse operations. Transliterations incorrectly reported duplation multiplication and single false position division as historical inverse operations (Chace et al. 1927, Clagett 1999, Peet 1923). Scribal division was based on a well- known definition, invert the divisor n to 1/n and multiply. Longhand scribal arithmetic reported multiplication and division as inverse operations to one another. To prove a multiplication answer, scribes inverted multiplicand n to 1/n and multiplied. To prove a division answer, scribes inverted divisors n to 1/n and multiplied. Practical hekat measurement units were recorded in wage payments and implemented management controls in finite Egyptian fraction quotients and remainders (Gillings 1972, Simpson 1973). Scribes precisely scaled grain to bread, beer,and other products in Egyptianfraction statements, an Appendix III topic. Pesu and besha, and des-jugs sub-units further scaled grain products for practica wage distributions.

One RMP problem reported 5 hekats of grain produced 200 loaves of bread. The balance of the hekats10 hekat produced different pesu strengths of beer. Each hekat produced one, two, and three types of beerlabeled from 8/3 pesu to 6 pesudenoted by grain content by an inverse relationship to the product. A product with 3/8 hekat of grain reported 8/3 pesu, enhancing everyday Middle Kingdom life by another Egyptian fraction innovation (Gardner 2009b).

5 CONCLUSIONS

This update of EMLR and AWT papers applies a scribal longhand conven- tion. A student EMLR scribe converted 1/p and 1/pq rational numbers to 26 unit fraction series by six introductory rules. Ahmes converted 2/p and 2/n to the 50 best unit fraction series by three advanced rules. Scribes scaled rational numbers n/p and n/pq to unit fraction series and selected the best LCMs with the best red auxiliary numbers. Scribal longhand exposes early number theory. The number theory was finite and algebraic.

Middle Kingdom scribes recorded unity objects in unit fraction series. Four unities were associated with the hekat. The unities assisted scribes in calculating wages paid in bread, beer, meat and other commodities. Wage payments were audited in unit fraction innovations (unities), summarized by:

(a) 1 hekat = 64/64 hekat

(b) 1 hekat = 320 ro

(c) 1/n hekat = n pesu, and n hekat = 1/n pesu

The fourth RMP unity innovation was based in Rule 10.0

$$p/p\text{hekat} = (1/p + 2/p + 3/p + ... + n/p)\text{hekat}$$

Scribes reported scribal multiplication and division that anticipated the modern definition of divisioninvert the divisor and multiply.

6 ACKNOWLEDGMENTS

This chapter is dedicated to the memory of Professor B.S. Yadav. Professor Yadav edited EMLR and AWT papers that provide this papers predicate. Pam Belluck, Bruce Friedman, and Cora Dillard pro- vided themes that made this paper possible. Pam Belluck, a New York Times writer, captured the oldest known Egyptian puzzles in everyday language. Bruce Friedman improved early number theory discussions that reported puzzle solutions in everyday language. Cora Dillard provided important editing for the enjoyment of the reader.

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Appendix I. Egyptian Mathematical Leather Roll (EMLR): adapted from Gardner (2007).

(a) $(1/8)(5/5) = 5/40 = (4 + 1)/40 = 1/10 + 1/40$

(b) $(1/4)(5/5)= 5/20 = (4 +1)/20 = 1/5 +$

(c) (1/3)(3/3)) = 3/9 = (2 + 1)/9 = 1/4 + 1/12

(d) (1/5)(2/2)) = 2/10 = 1/10 + 1/10

(e) (1/3)(2/2) = 2/6 = 1/6 + 1/6

(f) (1/2)(3/3) = 3/6 = 1/6 + 1/6 + 1/6

(g) 2/3 = 1/3 + 1/3

(h) (1/8)(25/25) = 25/200 = (8 +17)/200 = 1/25 + (17/200)(6/6) = 1/25 + (80 + 16 + 6)/1200 = 1/8 = 1/25 + 1/15 + 1/75 + 1/200

(i) (1/16)(25/25) = 25/400 = (8 +17)/400 = 1/50 + (17/2400)(6/6) = 1/50 + (80 + 16 + 6)/2400 = 1/16 = 1/50 + 1/30 + 1/150 + 1/400

(j) (1/15)(10/10) = 10/150 = (6 + 3 + 1)/150 = 1/25 + 1/50 + 1/150, (1/6 was initial term)

(k) (1/6)(3/3) = 3/18 = (2 + 1)/18 = 1/9 + 1/18

(l) (1/4)(7/7) = 7/28 = (4 + 2 + 1)/28 = 1/7 + 1/14 + 1/28

(m) (1/8)(3/3) = 3/24 = (2 + 1)/24 = 1/12 + 1/24

(n) (1/7)(6/6) = 6/42 = (3 + 2 + 1)/42= 1/14 + 1/21 + 1/42

(o) (1/9)(6/6) = 6/54 = (3 + 2 + 1)/54 = 1/18 + 1/27 + 1/54

(p) (1/11)(6/6) = 6/66 = (3 + 2 + 1)/66= 1/22 + 1/33 + 1/66

(q) (1/13)(?) = 1/28 + 1/49 + 1/196 (corrected by?)
(1/13) (6/6) = 6/78 = (3 + 2 + 1)/78 =1/26 + 1/39 + 1/78

(r) (1/15)(6/6) = 6/90 = (3 + 2+ 1)/90 = 1/30 + 1/45 + 1/90

(s) (1/16)(3/3) = 3/48 = (2 + 1)/48 = 1/24 + 1/48

(t) (1/12)(3/3) = 3/36 = (2 + 1)/36 = 1/18 + 1/36

(u) (1/14)(3/3) = 3/42 = (2 + 1)/42 = 1/21 + 1/42

(v) (1/30)(3/3) = 3/90 = (2 + 1)/90 = 1/45 + 1/90

(w) (1/20)(3/3) = 3/60 = (2+ 1) 60 = 1/30 + 1/6024.

(x) (1/10)(3/3) = 3/30 =(2 + 1)/30 = 1/15 + 1/30

(y) (1/32)(3/3) = 3/96 = (2 + 1)/96= 1/48 + 1/96

(z) (1/64)(3/3) = 3/192 = (2 + 1)/192 + 1/96 + 1/92

Information in this Appendix is further discussed by Gardner (2002, 2005).

Appendix II. Rhind Mathematical Papyrus (RMP) 2/n Table (Gardner 2008a, 2009a,2009b)

1. 2/3 = 1/3 + 1/3)
2. 2/5(3/3) = 6/15 = (5+ 1) = 1/3 + 1/15
3. 2/7(4/4) = 8/28 = (7 + 1)/28 = 1/4 + 1/28
4. 2/9 (2/2) = 4/18 = (3 + 1)/18 = 1/6 + 1/18
5. 2/11(6/6) = 12/66 = (11 + 1)/66 = 1/6 + 1/66
6. 2/13(8/8) = 16/104 = (13 + 2 + 1)/104 = 1/8 + 1/52 + 1/104
7. 2/15(2/2) = 4/30= (3 + 1)/30 = 1/10 + 1/30
8. 2/17(12/12) = 24/204 = (17 + 4 + 3)/204 = 1/12 + 1/51 + 1/68
9. 2/19(12/12) = 24/228 = (19 + 3 + 2)/228 = 1/12 + 1/76 + 1/114
10. 2/21((2/2) = 4/42 = (3 + 1)/42 = 1/14 + 1/42
11. 2/23(12/12) = 24/276 = (23 +1)/276 = 1/12 1/276
12. 2/25(3/3) = 6/75 = (5 + 1)/75 = 1/15 + 1/75
13. 2/27(2/2) = 4/54 = (3 + 1)/54 = 1/18 + 1/54
14. 2/29(24/24) = 48/696 = (29 + 12 + 4 + 3)/696 = 1/24 + 1/58 + 1/174 + 1/232
15. 2/31(20/20) = 40/1620 = (31 + 5 + 4)/1620 = 1/20 + 1/124 + 1/155
16. 2/33(2/2) = 4/66 (3 + 1)/66 = 1/22 + 1/66
17. 2/35(30/30) = 60/1050 = (35 + 25)/1050 = 1/30 + 1/42
18. 2/37(24/24) = 48/888 = (37 + 8 + 3)/888 = 1/24 + 1/111 + 1/296
19. 2/39(2/2) = 4/78 = (3 + 1)/78 = 1/26 + 1/78
20. 2/41(24/24) = 48/984 = (41 + 4 + 3)/984 = 1/24 + 1/246 + 1/328
21. 2/43(42/42) = 84/1806 = (43 + 21 + 14 + 6)/1806 = 1/42 + 1/86 + 1/129 + 1/301
22. 2/45(2/2) = 4/90 = (3 + 1)/90 = 1/30 + 1/90
23. 2/47(30/30) = 60/1410 = (47 + 10 + 3)/1410 = 1/30 + 1/141 + 1/470
24. 2/49(4/4) = 8/196 = (7 + 1)/196 = 1/28 + 1/196
25. 2/51(2/2) = 4/102 = (3 + 1)/102 = 1/34 + 1/102
26. 2/53(30/30)= 60/1590 = (53 + 5 + 2)/1590 = 1/30 + 1/318 + 1/795
27. 2/55(6/6) = 12/330 = (11 + 1)/330 = 1/30 + 1/330

28. $2/57(2/2) = 4/114 = (3 + 1)/114 = 1/38 + 1/11429.$

29. $2/59(36/36) = 72/2124 = (59 + 9 + 4)\ /2124 = 1/36 + 1/236 + 1/531$

30. $2/61(40/40) = 80/2440 = (61\ 10 + 5 + 4)/2440 = 1/40 + 244 + 1/488 + 1/610$

31. $2/63(2/2) = 4/126 = (3 + 1)/126 = 1/42 + 1/126$

32. $2/65(3/3) = 6/195 = (5 + 1)/195 = 1/39 + 1/195$

33. $2/67(40/40) = 80/2680 = (67 + 8 + 5)/2680 = 1/40 + 1/335 + 1/536$

34. $2/69(2/2) = 4/138 = (3 + 1)/138 = 1/46 + 1/138$

35. $2/71(40/40) = 80/2840 = (71 + 5 + 4)2840 = 1/40 + 1/568 + 1/710$

36. $2/73(60/60) = 120/4380 = (73 + 20 + 15 + 12)/4380 = 1/60 + 1/219 + 1/292 + 1/365$

37. $2/75(2/2) = 4/150 = (3 + 1)/150 = 1/50 + 1/150$

38. $2/77(4/4) = 8/388 = (7 + 1)/388 = 1/44 + 1/308$

39. $2/79(60/60) = 120/4740 = (79 + 20 + 15 + 6)/4740 = 1/60 + 237 + 1/316 + 1/790$

40. $2/81(2/2) = 4/162 = (3 + 1)/162 = 1/54 + 1/162$

41. $2/83(60/60) = 120/4980 = (83 + 15 + 12 + 10)/4980 = 1/60 + 1/332 + 1/415 + 1/498$

42. $2/85(3/3) = 6/255 = (5 + 1)/255 = 1/51 + 1/255$

43. $2/87(2/2) = 4/174 = (3 + 1)/174 = 1/58 + 1/74$

44. $2/89 = (60/60) = 120/5340 = (89 + 15 + 10 + 6)/5340 = 1/60 + 1/356 + 1/534 + 1/890$

45. $2/91(70/70) = 140/6370 = (91 + 49)/6370 = 1/70 + 1/130$

46. $2/93(2/2) = 4/186 = (3 + 1)/186 = 1/62 + 1/186$

47. $2/95(60/60) = 120/5700 = (95 + 15 + 10)/5700 = 1/60 + 1/380 + 1/570$

48. $2/97(56/56) = 112/5432 = (97 + 8 + 7)/5432 = 1/56 + 1/679 + 1/776$

49. $2/99\ (2/2) = 4/198 = (3 + 1)/198 = 1/66 + 1/198$

50. $2/101(6/6) = 12/606 = (6 + 3 + 2 + 1)/606 = 1/101 + 1/202 + 1/303 + 1/606$

Appendix III. Akhmim Wooden Tablet (AWT) and the Rhind Mathematical Papyrus(RMP)

The AWT was written in 1925 BCE. The text was transliterated in 1906 (Daressy 1906). An improved transliteration was published in 2002 (Vymazalova 2002). One hekat was scaled by 1/3, 1/7, 1/10, 1/10, and 1/13 to strings of binary quotients and remainders that were returned to a hekat unity written as (64/64). The Rhind Mathematical Papyrus (RMP), a 1650 BCE text, used a related quotient and remainder method over 60 times (Gardner 2006). The binary quotients plus scaled 1/320 of a hekat(ro) remainder was first reported to the modern era by the Akhmim Wooden Tablet (AWT). The scaled ro remainder was implicitly amended by Ahmes, in his own shorthand notes, to include 2-ro, 3-ro, and 4-ro remainders in RMP 47 (Gardner 2011).

Early in the 20th century, Ahmes bird-feeding problem (RMP 83) was re- ported by unclear additive patterns (Chace et al. 1927). Ahmes listed seven grain (hekat) portions within the AWTs hekat unity (64/64) divided by divisor n quotient and scaled remainder pattern, asking how much grain did the seven birds eat in one day, and how much did all the birds eat in 1, 10, 20, and 30 days. Corrected AWT quotient and remainder patterns report RMP 83 by:

1. 2 geese and a crane each ate (1/8 + 1/32 hekat + (3 + 1/3) ro
2. a set-duck ate (1/32 + 1/64) hekat + 1 ro, and
3. a set-goose, dove, and quail each ate (1/64) hekat + 3 ro

Ahmes reported seven portions of grain recorded within a hekat unity (64/64) a scribal context was misunderstood by scholars such as Peet (1923), Chace et al. (1927), Clagett (1999), Gillings (1982), Neugebauer (1962), Pommerening (2002), and Robins (1987). Ahmes reported 1/6 of a hekat (three times), 1/20 of a hekat (once), and
1/40 of a hekat (three times such that:
(3/6 + 1/20 + 3/40) hekat
(20/40 + 2/40 + 3/40) hekat
(25/40 = 5/8) of a hekat (of grain)
was eaten by seven birds in one day.

Vulgar fractions were scaled by LCMs and red auxiliary numbers before unit fraction answers (Gardner 2008a).The bird feeding method was extended to wage valuations by pesu and other methods (Gardner 2008c).

Some continuous maps into linear groups

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Abstract : In this paper we give a proof of that if X is a topological space and $f : X$ x $\mathbb{R} \to \mathbb{R}$ is a map such that for each $x \in X$, the map $f_x : \mathbb{R}^n \to \mathbb{R}^m$ given by $f_\times(v) = f(x, v)$ is a linear transformation, then f continuous if and only if the map $f : X \to L(\mathbb{R}^n, \mathbb{R}^m)$ defined by $f(x) = f_x$ is continuous. Also we give a proof of that if X is a topological space and $f : X \times \mathbb{R}^n \to \mathbb{R}^n$ is a map such that for each $x \in X$, the map $f_x : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_x(v) = f(x, v)$ is a linear isomorphism and Y is a topological space with a homeomorphism $g : X \to Y$, then the map $h : X \times \mathbb{R}^n \to Y \times \mathbb{R}^n$ given by $h(x, v) = (g(x), f(x, v))$ is homeomorphism if and only if f is continuous.

Keywords : topological space, continuous map, linear transformation.
AMS 2010 Mathematics Subject Classification: 26B05; 58C05; 58C25; 58C99

1 Introduction

We know that (see [1],[2] and [4]) the set of all mn real matrices $M_{m\times n}(\mathbb{R})$ and set of all linear transformations $L(\mathbb{R}^n, \mathbb{R}^m)$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ are smooth manifolds whose smooth structures are obtained by identifying with the Euclidean space $\mathbb{R}^{m\times n}$ and the general linear group $GL(n, \mathbb{R})$ the set of invertible $n \times n$ real matrices and the set of non-singular linear transformations $Aut(\mathbb{R}^n)$ are also smooth manifolds being the open subsets of $M_{n\times n}$ $(\mathbb{R})$ and $End(\mathbb{R}^n)$ respectively. If G is a matrix Lie group then a finite dimensional real presentation of G is given by a smooth map $g : G \to GL(n, \mathbb{R}) \cong Aut(\mathbb{R}^n)$ which is also a group homomorphism for some positive integer n (See [2], [4], [3] and [5]). This gives raise to a smooth map $f : G\times\mathbb{R}^n \to \mathbb{R}^n$ given by $f(x, v) = (g(x))(v)$. Conversely if $f : G \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map such that for each $x \in M$, the map $f_x : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_x(v) = f(x, v)$ for all $v \in \mathbb{R}^n$ is a linear isomorphism then the map $g : G \to Aut(\mathbb{R}^n)$ given by $g(x) = f_x$ is a finite dimensional real presentation of G if g is a homomorphism. In this paper we consider an arbitrary topological space X instead of a Lie group G and $L(\mathbb{R}^n, \mathbb{R}^m)$ instead of $Aut(R^n)$ and study the continuity properties of f and g.

We start with the following proposition:

Proposition 1.1. Let X be a topological space and $f : X \times \mathbb{R}^n \to \mathbb{R}^m$ be a map such that for each $x \in X$, the map $f_x : \mathbb{R}^n \to \mathbb{R}^m$ given by $f_x(v) = f(x, v)$ is a linear transformation. Then f is continuous if and only if the map $f : X \to L(\mathbb{R}^n, \mathbb{R}^m)$ defined by $f(x) = f_x$ is continuous.

Proof. Let $E_n = (e_1, \cdots, e_n)$ and $E_m = (e_1, \cdots, e_m)$ be the standard ordered base of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let g be the map from $L(\mathbb{R}^n, \mathbb{R}^m)$ to $M_{m\times n}(\mathbb{R})$ which maps to each linear transformation $T \in L(\mathbb{R}^n, \mathbb{R}^m)$, the $m \times n$ matrix $m[T]$ of T with respect to the base E_1, E_2 and h be the map from $M_{m\times n}(\mathbb{R})$ to $R_{m\times n}$ which maps to each $m \times n$ matrix $A = (aij)$, the $m \times n$ tuple $(a_{11}, a_{21}, \cdots, a_{m1}, \cdots\cdots, a_{1n},$

$a_{2n}, \cdots, a_{mn})$ in $\mathbb{R}_{m\times n}$. For each $x \in$, the matrix of the linear transformation f_x with respect to the base E_1, E_2 has the form

$$\begin{bmatrix} f_1(x,e_1) & \cdots & f_1(x,e_n) \\ \cdots & \cdots & \cdots \\ f_m(x,e_1) & \cdots & f_m(x,e_n) \end{bmatrix}$$

and image of this matrix under h is
$(f_1(x,e_1), \cdots, f_m(x,e_1), \cdots, f_1(x,e_n), \cdots, f_m(x,e_n)) = (f(x,e_1), \cdots, f(x,e_n))$ in $\mathbb{R}^{m\times n}$
Suppose f is continuous. Then for $i = 1, \cdots, n$ the function $k_i : X \to \mathbb{R}^m$ given by $k_i(x) = f(x,e_i)$ is continuous as it is the composition of the inclusion $X \to X \times \{e_i\}$ and the restriction of f on $X \times \{e_i\}$. From this we see that the map $hogof : X \times R^{mn}$ is continuous as $(h \text{ o } g \text{ o } f)(x) = (k_1(x), \cdots, k_n(x))$. Since g and h are homeomorphism we get that $f : X \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous map.

Conversely suppose $f : X \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous map. Then $k_i : X \to R^m$ is continuous for $i = 1, \cdots, n$ as $h \text{ o } g \text{ o} f$ is continuous map. Now for $(x,v) \in X \times \mathbb{R}^n$, we have $f(x,v) = f_x(v) = f_x(\sum_{i=1}^n v_i e_i) = \sum_{i=1}^n v_i f_x(e_i) = \sum_{i=1}^n v_i f(x,e_i) = \sum_{i=1}^n v_i k_i(x)$ which is a sum of continuous map. Therefore $f : X \times \mathbb{R}^n \to \mathbb{R}^m$ is a continuous map. □

As an corollary we have :

Corollary 1.1. If $g : X \to \mathbb{R}^m$ is a continuous map and $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the map $h : X \to L(\mathbb{R}^n, \mathbb{R})$, where $h(x) : \mathbb{R}^n \to \mathbb{R}$ as the dot product in $\mathbb{R}^m$ is distributive over addition. So h is well defined map and $h = \hat{f}$ where $f : X \times \mathbb{R}^n \to \mathbb{R}$ is given by $f(x,v) = g(x)\cdot T(v)$, for each $x \in X, f_x : \mathbb{R}^n \to \mathbb{R}$ given by $f_x(v) = f(x,v)$ and $\hat{f}(x) = f_x$. Since f is the composition of the continuous maps $\xrightarrow{g\times T} \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ by the above proposition we see that h is continuous.

Similarly as $\mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ multiplication of a vector by a scalar and the cross product $\mathbb{R}^3 \times \mathbb{R}^3\mathbb{R}^3$ are continuous maps, we also have the following corollaries:

Corollary 1.2. If $g : X \to \mathbb{R}$ is a continuous map and $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then the map $h : X \to L(\mathbb{R}^n, \mathbb{R}^m)$, where $h(x) : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation given by $h(x)(v) = g(x)T(v)$, is continuous.

Corollary 1.3. If $g : X \to \mathbb{R}^3$ is a continuous map and $T : \mathbb{R}^n \to \mathbb{R}^3$ is a linear transformation then the map $h : X \to L(\mathbb{R}^n, \mathbb{R}^3)$, where $h(x) : \mathbb{R}^n \to \mathbb{R}^3$ is the linear transformation given by $h(x)(v) = g(x) \times T(v)$, is continuous.

Another corollary which follows from the above proposition is :

Corollary 1.4. If $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m$ is a bilinear map then f is continuous.

Proof. Since f is linear on the second component, the map $\hat{f} : \mathbb{R}^p \to L(\mathbb{R}^q, R^m)$ given by $f(v) = f_v$, where $f_v(w) = f(v,w)$, is well defined. Again since f is linear on the first component, f is a linear transformation and so it is continuous. Therefore f is continuous. □

Corollary 1.5. Let X be a topological space and $f : X \times \mathbb{R}^n \to \mathbb{R}^n$ be a map such that for each $x \in X$, the map $f_x : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_x(v) = f(x,v)$ is a linear isomorphism. Let Y be a topological space and $g : X \to Y$ be a homeomorphism. Then the map $h : X \times \mathbb{R}^n \to Y \times \mathbb{R}^n$ given by $h(x,v) = (g(x), f(x,v))$ is homeomorphism if and only if f is continuous.

Proof. By the (1.1), we know that f is continuous if and only the map $\hat{f} : X \to L(\mathbb{R}^n, \mathbb{R}^n)$ defined by $\hat{f}(x) = f_x$ is continuous. Therefore h is continuous if and only $\hat{f}$ is continuous. To check that h is one to one, let (x_1, v_1) and (x_2, v_2) be two elements of $X \times \mathbb{R}^n$ such that $h(x_1, v_1) = h(x_2, v_2)$. Then we have $g(x_1) = g(x_2)$ and $f(x_1, v_1) = f(x_2, v_2)$. Since g is one to one, we get $x_1 = x_2$. Now $f(x_1, v_1) = f(x_1, v_2)$ implies that $f_{x_1}(v_1) = f_{x_1}(v_2)$. Since f_{x_1} is an isomorphism, we have $v_1 = v_2$. To check that h is onto, let (y, w) be an element of $Y \times \mathbb{R}^n$. Since g is onto, there exists an element x of X such that $g(x) = y$. Again since f_x is onto, there exists $v \in \mathbb{R}^n$ such that $f_x(v) = w$ i.e. $f(x, v) = w$. So we have $h(x, v) = (y, w)$ and hence h is bijective and $h1(y, w) = (g^1(y), (fg^1(y))^1(w))$. We notice here that the second component of h^1 is the composition of the maps $Y \times \mathbb{R}^n \xrightarrow{g^{-1}\times Id_{\mathbb{R}^n}} X \times \mathbb{R}^n \xrightarrow{k} \mathbb{R}^n$ where $k(x, w) = f_x^{-1}(w)$ and for $x \in X$, the map $k_x : \mathbb{R}^n \to \mathbb{R}^n$. given by $k_x(w) = k(x, w)$ is an element of $Aut(\mathbb{R}^n)$ as $k_x = f_x^{-1}$. So again by the (1.1) we see that k is continuous if and only if the map $\hat{k} : X \to Aut(\mathbb{R}^n)$ given by $\hat{k}(x) = k_x$ is continuous. But $\hat{k}$ is the composition of the maps $X \xrightarrow{\hat{f}} Aut(\mathbb{R}^n) \xrightarrow{l} Aut(\mathbb{R}^n)$, where l is the inversion map. As l is continuous, $\hat{k}$ is continous if and only if $\hat{f}$ is and so h^1 is continuous if and only if f is. Therefore h is homeomorphism if and only if f is continuous. □

Proposition 1.2. Let $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form such that for $v = (v_1, \cdots, v_n) \neq 0, g(v, w) = 0$ for all $w \in \mathbb{R}^n\{0\}$. Then the map $h : (\mathbb{R}^n\{0\}) \times \mathbb{R}^n \to (L(\mathbb{R}^n, \mathbb{R})\{0\}) \times \mathbb{R}^n$ given by

$$h(v, w) = (g_v, (v_1g(e_1, w), \cdots, v_ng(e_n, w)))$$

is a homeomorphism, where $e_1, \cdots, e_n$ is the standard ordered basis of $\mathbb{R}^n$ and $g_v : \mathbb{R}^n \to \mathbb{R}$ is the linear transformation given by $g_v(w) = g(v, w)$.

Proof. Since g is bilinear, the map $\hat{g} : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}) given by \hat{g}(v) = g_v$ where $g_v(w) = g(v, w)$ is linear transformation and $\hat{g}(v) = 0$ implies that $v = 0$. So $\hat{g}$ is one to one. As the dimensions of $\mathbb{R}^n$ and $L(\mathbb{R}^n, \mathbb{R})$ are same, we see that $\hat{g}$ is a homeomorphism.

Let $f : (\mathbb{R}^n\{0\}) \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$f(v, w) = (v_1g(e_1, w), \cdots, v_ng(e_n, w)),$$

where $e_1, \cdots, e_n$ is the standard ordered basis of $\mathbb{R}^n$ and $v = (v_1, \cdots, v_n)$. Clearly f is continuous as g is continuous. Again since g is bilinear, for $v \in \mathbb{R}^n$, the map $f_v : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_v(w) = f(v, w)$ is a linear transformation. Let w be an element of $\mathbb{R}^n$ such that $f_v(w) = 0$. This implies that $v_1g(e_1, w) = \cdots = v_ng(e_n, w) = 0$. This gives us that $v_1g(e_1, w) + \cdots + v_ng(e_n, w) = 0$. Since g is bilinear we have $g(v_1e_1 + \cdots + v_ne_n, w) = 0$ which gives us that $g(v, w) = 0$. Since $v \neq 0$, by the hypothesis we should have $w = 0$. This shows that f_v is one to one and also onto by the dimension property. So by the Proposition [6] the map $(\mathbb{R}^n\{0\}) \times \mathbb{R}^n \to (L(\mathbb{R}^n, \mathbb{R})\{0\}) \times R^n, (v, w) \to (g_v, f(v, w))$ is a homeomorphism hence we complete the proof. □

As a corollary we have:

Corollary 1.6. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a non degenerate bilinear form. Then the map $h : (\mathbb{R}\{0\}) \times \mathbb{R} \to (L(\mathbb{R}, \mathbb{R})\{0\}) \times \mathbb{R}$ given by $h(v, w) = (f_v, f(v, w))$ is a homeomorphism, where $f_v : \mathbb{R} \to \mathbb{R}$ is the linear transformation given by $f_v(w) = f(v, w)$.

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"O! Playful One !"- on humanizing mathematics education

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Abstract : In the historical mathematical texts like the Rhind Papyrus, Śulbha Sūtra or Bhāskara's Līlāvatī, we see that the problems are posed in lively, engaging and possibly lyrical manner. This is in stark contrast to the seemingly dogmatic abstractions, structured methods and procedures that Mathematics education has been commonly reduced to today. In the core subjects of engineering, the biography and anecdotes of the scientists have been used to enhance the effectiveness of teaching. In modern subjects like software engineering, it is partly the personality cult associated with pioneers that grabs the attention of the student.
Recent advances in ethnomathematics and the historical research into the lives of the mathematicians offer a wonderful opportunity to enrich mathematics education.
Such a humanized approach can also serve to highlight the contributions of local historical genius like the Kerala School of Mathematics. Popularizing such historical information by their inclusion in the curriculum will make it easier for the students to personalize and associate with the imparted knowledge and that in turn can fan the innovative spirit in the young minds.

1 Introduction

Figure 1: A portion of the Rhind mathematical papyrus. Image Courtesy: Wikipedia

When the Rhind Papyrus (1650 BC) was brought to light in 1858 by antiquarian Alexander Henry Rhind, the modern world got the first glimpse of how mathematics was taught in the ancient world. In the Egyptian civilization, the stress was on practical problems that directly concerned the civil governance of the kingdom. Similarly the Śulab Sūtras in India list problems meant to convey various mathematical concepts to students. However in the Indian texts, imaginary and animalistic scenarios are invoked to create mathematics problems. A classic example is the owl and mouse problem that reduces to the calculation of a chord length.

A quick perusal of textbooks and question papers at the school and university levels today quickly demonstrates that fiction has all but disappeared. Nietzche said, “No more fiction for us; we calculate; but that we may calculate, we had to make fiction first.” Mathematics and its problems are presented to student community in a language of symbols with concern for abstract formulations and equations. As variables and functions dominate, the connection with the real world is thinned.

Recent survey shows that among the three vital Rs for education namely Reading, wRiting and aRithmetic, students both in India and the United States perform mostly disappointingly in arithmetic. The recently unveiled PISA report spread over 74 countries including the Plus nations (10 countries were added to the original 64), the two Indian states (Tamil Nadu & Himachal Pradesh) came up 72nd and 73rd out of 74 in both reading and mathematics. The latest set of results from the 2012 data collection (PISA 2012) focusing on mathematics will be released in December 2013.

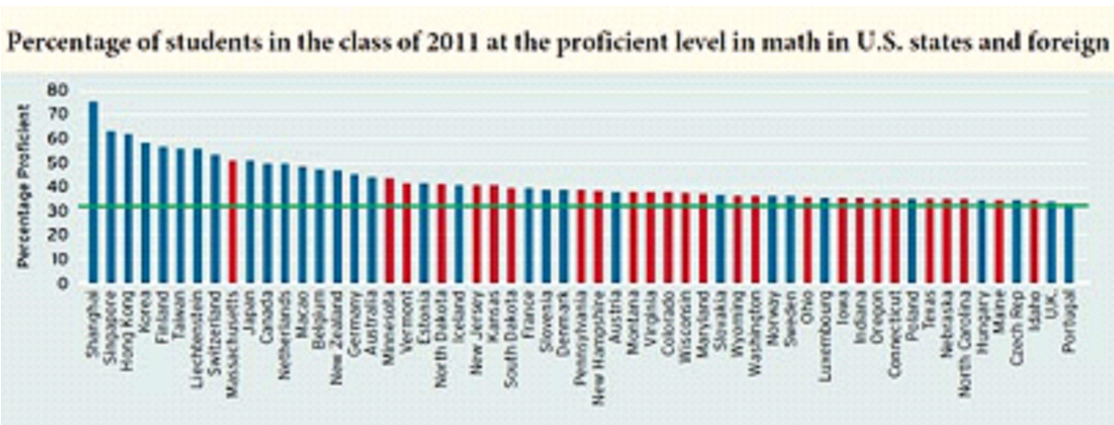

Figure 2 : Mathematics proficiency 2011, *Globally Challenged: Are U.S. Students Ready to Compete?*, Harvard University’s Program on Education Policy and Governance (PEPG), 2011. Image Courtesy: `www.asianscientist.com`

It is obvious that we lose plenty of potential mathematicians and engineers by the time they get to higher education. Human beings have a natural ability to count. At school this skill is horned but rapidly gives way to a capacity to follow step by step procedure. It can be argued that such adherence to method reduces the ability for mathematical thinking and modeling which are the essential ingredients of the engineering mind.

2 ‘Real’ Mathematics

Tobias Dantzig’s classic text: Number that fascinated Einstein defines Number as the language of science in the subtitle itself. In the case of engineering education, mathematical fluency is imperative. An engineering graduate must be bilingual in this sense. He or she needs to seamlessly be able to look at the world and communicate with each other in the language of mathematics.

But the falling standards of education states that the case is otherwise. What makes it difficult for mathematics to be the "underlying" language of other entire engineering subjects just as English is the language in which Economics is taught? What factors in the higher education system prevent such deep penetration and imbibing?

In the 2008 scheme of Kerala University, the first sections that first year students encounter are conic sections and matrices. The powerful nature of these fields when it comes to real world application is left to the keenness of the student. The problems posed in the university question papers do not invoke any real world applications but are symbolic manipulations. An analysis of AICTE-CII Survey of Industry-Linked Technical Institutes 2013 shows only 1% of technical institutes do consultancy work or research projects for industry. This scenario creates an unbridgeable gap in the minds of the fresh graduates. Constant exposure to tangible, immediate problems is a must to create both an appreciation for the powerful nature of the tools learnt as well as readiness to take up challenging realistic situations that can be mathematically reduced to solutions.

As mentioned earlier, the tendency to present mathematics as an abstraction abounds. Budding engineers are taught existing methods and techniques with scant attention to enhance their ability to approach the world mathematically. It is often developed as a subject consolidating techniques. The strong underlying usefulness and universality is never brought forth or glossed over.

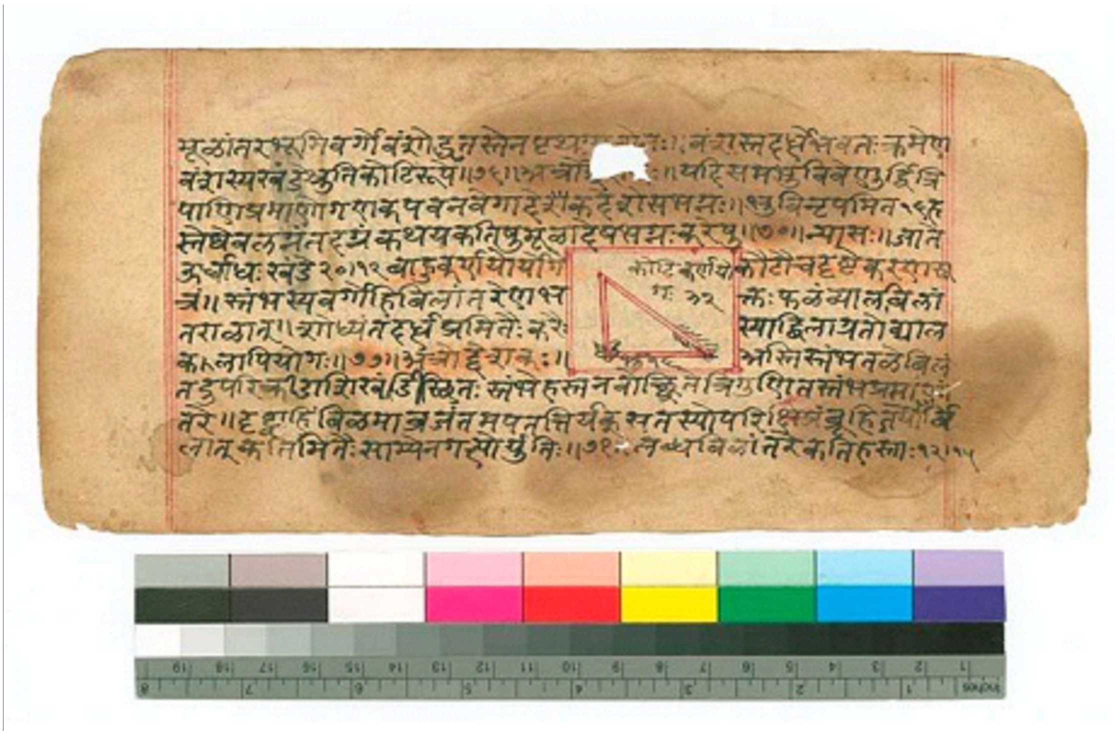

Figure 3 : Page from Lilavati showing illustration of Pythagoras theorem. Image Courtesy: Mathematical Association of America.

Stark contrast can be found in the way Bhaskaracharya poses problems in his legendary work titled Lilavati (Playful One). The work was a main source of learning the then state-of-the-art arithmetic and algebra. The work had immense influence in the Middle-East and that a translation was rendered by Abul Fazal, a vizier of the Mughal emperor Akbar. The 12^{th} century mathematician Bhaskara wrote the text to immortalize his daughter, named Lilavati, after she was distraught about missing the auspicious hour for wedding.

A couple of examples from the work will illustrate how poetically close to life the posing of problems were :

(a) *Whilst making love a necklace broke.*

A row of pearls mislaid.
One sixth fell to the floor.
One fifth upon the bed.
The young woman saved one third of them.
One tenth were caught by her lover.
If six pearls remained upon the string
How many pearls were there altogether?
And

(b) In a lake swarming with geese and cranes,
the tip of a bud of lotus was seen one span above the water.
Assaulted by the wind, it moved slowly,
and was submerged at a distance of two cubits.
O mathematician, find quickly the depth of the water.

3 A Computational Paradigm

We live in a rapidly digitizing world. Our mental realm has become more or less digital more than we may like to give credit to. Mutual appreciation in the Facebook era is measured as the number of 'likes' and in the number of comments. Movie reviews are summed up in number of stars or percentage of rottenness. We have become comfortable with objective numbers rather than subjective statements in many areas of life. Memory, bandwidth, encoding etc are all mathematical concepts pervade our personal and professional life.

Government of India has embarked on the ambitious project to bring cheap computational power to its billions. The government is going ahead with plans for a new and improved low-cost Aakash tablet for students to be priced under $25 (Rs.1,500). The aim of the Aakash project was to link 25000 colleges and 400 universities through e-learning. IT@ School initiative of the Kerala government remodeled teaching learning techniques through the use of information technology.

Computers are fundamentally computational tools. Mathematics is the core language in which they operate and can be used. As more power software become available to students at younger stages, it becomes imperative for them to reduce problems into software compatible format.

But it appears that the increasing mathematisation of life is coupled with an increasing alienation of the subject in the curriculum format from life. 'Mathematical Anxiety' is a genuine problem. Even as mathematics pervades every sphere of modern life, the population comfortable in applying valuable mathematical concepts is shrinking.

There is an unfortunate effect of students questioning the usefulness of mathematics prematurely. Unless the reality of mathematics is conveyed continuously, students will not appreciate its beauty. It is imperative that the language of mathematics that has led to all the technological advancement ingrains itself as something much more fundamental to the human mind than just another subject that is open to being loved or hated in school.

The interest aroused in the students will definitely improve with vigorous treatment that links the techniques immediately with real relevant engineering problems is established right from the beginning. We can see that this is in sharp contrast with how computer languages like C++ are taught. The students learn by writing programs that solve actual problems. Those problems might be mathematical. So here C++ becomes the computational language that can solve a mathematical problem. If we take it one step higher by making mathematics the language to solve engineering problems, the effectiveness would be enhanced. The students in a C++ class are trained to reduce any problem into that particular language.

Similarly all engineers must inculcate an ability to reduce any engineering problem into a mathematical language. Thus mathematics can progress through three stages. It can start off as one of the subjects of the core curriculum. In the second stage, it must become a powerful tool that can help solve several real world problems. In the third, it should transform into a language, fluency in which allows the student to view the world in a clear perspective.

To further illustrate the idea, we can consider category theory that can be applied throughout science to create qualitative models. Once such a qualitative model is formed as a category, its basic structure can be meaningfully compared with that of any other category, be it mathematical, linguistic, or other. Like a biological system, the basic building blocks of a category are simple, but the networks that can be formed out of them are as complex as mathematics itself. If we compare logic and set theory to the "machine language" of computers, we can regard category theory as an extremely useful universal programming tool.

Category theory is the theory of structure-preserving transformations. This theory provides us with a language for describing complex problems in an elegant way, and tools to give elegant, and above all simple, proofs. The language of category theory allows us to consider the essence of some problem, without the burden of /often numerous and complicated/ non-essential aspects.

4 An "Ethnomathematical" Route

Walter Fisher's Narrative paradigm of communication theory puts forth that all meaningful communication is a form of storytelling. It is worthwhile to examine whether the near total absence of stories in a mathematical classroom causes the disconnect the subject tends to have in young minds. Affirming mathematical techniques using real world framing of problems, as done in the ancient and classic texts, is a method to bring some narrative element back into the classroom.

Another effective method would be the use of rich stories from the history of mathematics and ethnomathematics. History of mathematics is full of captivating characters and events that can provide a fertile framework and colorful backdrop through which different topics can be introduced.

Niccoloa Fontana, known as Tartaglia (the stammerer) left the method he had devised to solve cubic equation in his rival, Cardano's house in the form of an Italian poem. The mention of the rather intriguing origins of L'Hospital's rule that is taught in undergraduate program will help with exploration and memorization of the technique. The case of amateur Marquis de L'Hospital "freely" borrowing from Johann Bernaulli and Leibnitz is surely a story that can create interest in a classroom.

Periodically, there are serious discussions about the unfounded bias across the world that turns girls away from mathematics. Perhaps, active introduction of the stellar contributions of the female mathematicians like Sophie Germain, Ada Lovelace, Sofia Kovalevskaya, Emmy Noether et al can help female engineers get more involved. The study of the lives, struggles, disappointments and successes of the mathematicians can help the students become more comfortable with their own insecurities and errors in the subject matter rather than the feel of the subject as a cold, unforgiving, lifeless abstraction.

Much closer to home, in Kerala, the 'Kaṭapay$\bar{a}$di$'$ system attributed to Vararuchi in 4^{th} century, sparkles with wit and creativity as it transforms mathematical formula, equations, lists and figures into metrical poetry in Sanskrit and Malayalam that serve as memory aides. A brilliant example being "Ayur Arogya Soukhyam", the closing blessing in Melpathoor's devotional masterpiece, Narayaneeyam, that via Kadapayadi system actually captures the date of finishing the composition of the work.

Harking back to the greatest interest with which computer education has penetrated the system, it can be seen that personality cult plays a significant part in student motivation. The business successes of Bill Gates, Steve Jobs, Sergei Brin, Larry Page et al go a long way in generating and sustaining the attraction of the field. Even in other subjects, we know that exemplary teachers have sparked lifelong interest in students like Feynman in physics. With the help of Youtube today it is possible for students anywhere in the world to have access to the best of the teachers and practitioners of any field. The internet has truly ‘localised’ all fame.

Mining the history of mathematics makes it amply clear that there is no dearth of colorful personalities as well as captivating teachers in the field of mathematics. The huge success of some of the massively open online courses (MOOCs) in mathematics through websites like Coursera§show how much influence a great professor can have on the popularity of a field.

While ethnomathematics can serve to increase the sense of close association, care must be taken of the criticism that going overboard with the local pride can only lead to proliferation of pseudoscience without rigorous testing and experimentation.

5 Conclusion

The diminishing mathematical ability among students of school and higher education raises alarm. The lack of interest in the subject could be attributed to the non-narrative, abstract way it is presented most of the time. Using the paradigm of computer languages, it might be possible to introduce mathematics as a powerful tool as well as language to navigate the real world. Also, adding a healthy dose of storytelling element to the subject based on its history as well as the involved personalities could create deeper impact in a classroom.

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On ring of δ-continuous and δ-perfectly continuous functions

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Abstract : The notion of δ-continuous function was introduced by T. Noiri and the notion of δ-perfectly continuous function was introduced by Kohli and Singh. In this note we show that if X and $X \times X$ has almost partition topology then the space of all real valued δ-continuous functions on X from a ring under the point wise addition and multiplication and if X and $X \times X$ have δ- partition topology then the space of all real valued δ-perfectly continuous functions on X from a ring under the point wise addition and multiplication.

Keyword : δ-continuous functions, δ-perfectly continuous functions, almost partition topology, δ-partition topology.
AMS 2010 Mathematics Subject Classification: 54C40, 54C35

1 Introduction

Several weak, strong and other variants of continuity occur in mathematics literature. The classes of discontinuous functions possess interesting properties. A lot of work has been done recently on dynamical systems generated by discontinuous functions. Even simpler operations lead one out of the class. In this note we try to see under what condition the classes of all real valued δ-continuous functions and the classes of all δ-perfectly continuous functions on a topological space X form a ring under the pointwise addition and multiplication. The notion of δ-continuous function has been introduced by T. Noiri [2] and the notion of δ-perfectly continuous function has been introduced by Kohli and Singh [1].

Let X, Y be topological spaces and $f : X \to Y$ be a single valued function. A subset A of X is said to be regular open if $A = int(cl(A))$ and regular closed if X is regular open. A subset A of a space X is said to be δ-open if it is union of regular open sets. A function $f : X \to Y$ is δ-continuous if for each regular open set V in Y, $f^1(V)$ is a clopen set in X and $f : X \to Y$ is δ -perfectly continuous if for each δ-open set V in Y, $f^1(V)$ is a clopen set in X. The notion of δ-perfectly continuous functions in general are independent of continuous functions but coincide with perfect continuity if Y is a semiregular space.

2 On ring of δ-continuous functions

Proposition 2.1. Let $f : X \to Y$ be continuous and $g : Y \to Z$ δ-perfectly continuous. Then gof is δ-perfectly continuous.

Proof. Let W be δ-open in Z. Since g is δ-perfectly continuous $g^{-1}(W)$ is clopen in Y. As f is continuous, $f^{-1}(g^{-1}(W)) = (gof)^{-1}(W)$ is clopen in X. Hence, gof is δ-perfectly continuous function.

□

Definition 2.1. A space X is said to be endowed with a δ-partition topology if every δ-open set in X is closed.

Proposition 2.2. Let X be a topological space and $X \times X$ has δ-partition topology. If $f : X \to Y$ and $g : X \to Z$ are δ-perfectly continuous, then $(f, g) : X \times X \to Y \times Z$ is δ-perfectly continuous.

Proof. Let A be regular open in $y \times Z$. Then $A = \cup U_i \times V_i$ where U_i is regular open in Y and V_i is regular open in Z. So, $(f, g)^{-1}(A) = \cup (f, g)^{-1}(U_i \times V_i) = \cup f^{-1}(U_i) \times g^{-1}(V_i)$.
Since f and g are δ-continuous and U_i, V_i are regular open, $f^{-1}(U_i)$ and $g^{-1}(V_i)$ are clopen. So, $(f, g)^{-1}(A)$ is regular open in $X \times X$ and since $X \times X$ has almost partition topology, $(f, g)^{-1}(A)$ is closed. Also, $(f, g)^{-1}(A)$ is open. Hence, (f, g) is δ-continuous. □

Proposition 2.3. Let X and $X \times X$ have almost partition topology. Then the classes of all real valued continuous functions $D(X, \mathbb{R})$ is a ring.

Proof. Let $h : \mathbb{R} \times \mathbb{R} to \mathbb{R}$ denote either of the functions $h(x, y) = x + y$ or $h(x, y) = xy$.

Let U be a regular open set. Then U is open in $\mathbb{R}$. Since h is continuous,$h^{-1}(U)$ is open in $\mathbb{R} \times \mathbb{R}$. Any open set in $\mathbb{R} \times \mathbb{R}$ is union of product of basic open subsets of $\mathbb{R}$. So, $h^{-1}(U) = \cup U_i \times V_i$. Since basic open subsets in $\mathbb{R}$ are regular open, $h^{-1}(U)$ is regular open in $\mathbb{R} \times \mathbb{R}$. Now let $f, g : X \to \mathbb{R}$ be δ-continuous. By proposition (2.2) (f, g) is δ-continuous and so $(f, g)^{-1}h^{-1}(U)$ is clopen in $X \times X$. If $d : X \to X \times X$ is the diagonal map then $(f + g)^{-1}(U)$ or $(fg)^{-1}(U)$ is $d^{-1}((f, g)^{-1}h^{-1})(U) = \cup (f^{-1}(U_i) \cap g^{-1}(Vi))$ which is clopen in X as X has almost partition topology.

□

3 On ring of δ-perfectly continuous functions

Proposition 3.1. Let $f : X \to Y$ be continuous and $g : Y \to Z$ δ-perfectly continuous. Then gof is δ-perfectly continuous.

Proof. Let W be δ-open in Z. Since g is δ-perfectly continuous $g^{-1}(W)$ is clopen in Y. As f is continuous, $f^{-1}(g^{-1}(W)) = (gof)^{-1}(W)$ is clopen in X. Hence, gof is δ-perfectly continuous function. □

Definition 3.1. A space X is said to be endowed with a δ-partition topology if every δ-open set in X is closed.

Proposition 3.2. Let X be a topological space and $X \times X$ has δ-partition topology. If $f : X \to Y$ and $g : X \to Z$ are δ-perfectly continuous, then $(f, g) : X \times X \to Y \times Z$ is δ-perfectly continuous.

Proof. Let A be δ-open in $Y \times Z$. Then $A = \cup U_i \times V_i$ where $U_i \times V_i$ is regular open in $Y \times Z$. Then U_i is regular open in Y and V_i is regular open in Z. So, $(f, g)^{-1}(A) = \cup (f, g)^{-1}(U_i \times V_i) = \cup f^{-1}(U_i) \times g^{-1}(Vi)$.
Since f and g are δ-perfectly continuous and U_i, V_i are δ-open, $f^{-1}(U_i)$ and $g^{-1}(V_i)$ are clopen. So, $(f, g)^{-1}(A)$ is δ-open in $X \times X$ and since $X \times X$ has δ-partition topology, $(f, g)^{-1}(A)$ is closed. Also, $(f, g)^{-1}(A)$ is open. Hence, (f, g) is δ-perfectly continuous. □

Proposition 3.3. Let X and $X \times X$ have δ-partition topology. Then the classes of all real valued δ-perfectly continuous functions $P(X, \mathbb{R})$ is a ring.

Proof. Let $h : \mathbb{R} \times \mathbb{R} \to R$ denote either of the functions $h(x, y) = x + y$ or $h(x, y) = xy$.
Let U be a δ-open set. Then U is open in $\mathbb{R}$. Since h is continuous, $h^{-1}(U)$ is open in $\mathbb{R} \times \mathbb{R}$. Any open set in $\mathbb{R} \times \mathbb{R}$ is union of product of basic open subsets of $\mathbb{R}$. So, $h^{-1}(U) = \cup U_i \times V_i$. Since basic open subsets in $\mathbb{R}$ are regular open, $h^{-1}(U)$ is δ-open in $\mathbb{R} \times \mathbb{R}$. Now let $f, g : X \to \mathbb{R}$ be δ-perfectly continuous. By proposition (3.2), (f, g) is δ-perfectly continuous and so $(f, g)^{-1}h^{-1}(U)$ is clopen in $X \times X$. If $d : X \to X \times X$ is the diagonal map then $(f + g)^{-1}(U)$ or $(fg)^{-1}(U)$ is $d^{-1}((f, g)^{-1}h^{-1})(U) = \cup(f^{-1}(U_i) \cap g^{-1}(V_i))$ which is clopen in X as X has δ-partition topology. □

Proposition 3.4. Let X be a topological space. Let $f : X \to \mathbb{R}, \mathbb{R}$ with usual topology, be a δ-perfectly continuous map. then f is continuous.

Proof. Let U be open in $\mathbb{R}$. Then U is δ-open as each open set in $\mathbb{R}$ is countable union of disjoint segments in $\mathbb{R}$ and each segment is regular open. Now $f^{-1}(U)$ is clopen as f is δ-perfectly continuous. Hence f is continuous. □

Proposition 3.5. Let X be a space with δ-partition topology. If f is continuous then it is δ-perfectly continuous.

Proof. Let U be δ-open. Then U is open and so f^{-1} is open as f is continuous. Again, since U is δ-open and X has δ-partition topology, U is closed. Since f is continuous, $f^{-1}(U)$ is closed. Thus $f^{-1}(U)$ is clopen and hence f is δ-perfectly continuous. □

Corollary 3.1. If X and $X \times X$ have δ-partition topology then $P(X, \mathbb{R}) = C(X, \mathbb{R})$.

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Numerical ability in isolated students

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Abstract : Mathematics is an abstract subject and students find it difficult to comprehend. There are more failures in Mathematics than in any other subject. An annual national survey in 2012 called the "Annual Status of Education Report" revealed that the students from Maharashtra are the weakest in Mathematics in the country. It was found that 77.4% of fifth standard students couldn't do simple problems taught in third standard. Hence this study was conducted in Maharashtra to analyze the causes of poor performance in Mathematics. In schools there are students found to be isolated with very less interaction with peers, their parents and teachers. They appear to be unhappy, tense, nervous, frustrated, operate independently, quiet, sober, easily discouraged, abandon tasks if it is difficult, do not initiate or volunteer and are mostly employed in day dreaming. Some of them are poor in numerical ability and hence they are weak in Mathematics. The purpose of this study was to examine the correlation between numerical ability and math scores of isolated students of eighth standard. It was found that home isolation and home rejection are the major causes for poor numerical ability. Parents need to take a proactive role in the education of their children. Computer Assisted Instruction can be a supplement for these students. With the parents' involvement a mathematics teacher can change the behavior of isolated students and can bring them to the main stream of society.

Key words : Numerical ability, isolated students, Computer Assisted Instruction

1 INTRODUCTION

The French philosopher and mathematician Ren Descartes (1596-1650) said "Mathematics is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency." Mathematics is the oldest of all sciences that has been developed through the ages of mankind. Mathematics originated with number system which is also the base for every mathematical concept. Mathematics is an abstract subject and symbols occupy an important position. Numerical ability influences pupils' achievement in Mathematics. A significant number of students find it difficult to comprehend the symbols and so find it difficult to learn Mathematics and there are more failures than in any other subject. [1] It is a subject most feared by students of primary school and secondary school levels. There is an increasing concern about the number of learners who drop mathematics in the later years of high school. Mathematics is used as an essential tool in many fields including Natural Science, Engineering, Medicine, Information Technology, Astronomy and Social Science. In today's world, it is important that children grow to become confident in their ability to do Mathematics in this modern high-tech competitive society.[2]Understanding numbers and mathematics is so critical that a deficit in basic mathematical abilities has been found to have greater negative effect on employment opportunities than reading difficulties. In spite of this there is not only a lack of awareness about the indispensability of mathematics but instead there is a tendency to ignore mathematics. Many students avoid studies that involve Mathematics in their higher education. [3]Avoidance of Mathematics leads to a limit of career choices, eroding a country's resource base in science and technology.

[4]For many students passing examinations to secure certificates either for admission into institutions of higher learning or to secure a good job is the main goal of education and not the acquisition of knowledge and skills through studying. [5]The National Policy of Education (1986) suggested that Mathematics should be visualized as the vehicle to train a child to think, reason, analyze and to articulate logically. [6]The National Curriculum Framework (NCF) 2005 proposes that the aim of mathematics teaching and learning is mathematisation of a child's thought processes instead of mathematics learning being loaded with content. Emphasis is to learn meaningful Mathematics and to focus on developing the inner resources of the child to bring clarity of thought while pursing logical conclusions with an ability to handle abstractions.

According to [7] the characteristics of mathematics are: It is the science of numbers and space, it is the science of calculation, it is the science of measurement, quantity and magnitude, it deals with quantitative facts and relationships, it is the abstract form of Science, it is the science of logical reasoning. [8] School mathematics today takes place in a situation where: children learn to enjoy Mathematics, children learn important Mathematics, Mathematics is a part of children's life experience which they talk about, children pose and solve meaningful problems, children use abstractions to perceive relationships and structure, children understand the basic structure of Mathematics and teachers expect to engage every child in class.

2 RATIONALE OF THE STUDY

[9] *"Modern nations see value in building a mathematically literate society and hope for a strong mathematical elite that can shape the knowledge economy of the 21st century. At the same time, mathematical proficiency is universally considered hard to achieve. India, with its strong mathematical traditions, may be expected by the world to produce excellence in mathematics"*. But the [10] Annual Status of Education Report (2012) has disturbingly revealed that basic arithmetic skills of students are declining. In its national survey among school-going children's ability in Mathematics, it was found that at the national level 75.2% of the students from fifth standard could not do problems taught in third standard and students from Maharashtra are the weakest in the country with 77.4% students unable to solve mathematical problems. Such astounding facts led to this study.

Isolated Students

All children need a connection with their peers but some students are alienated from others. [11]It refers to a sense of social estrangement and absence of social support. Within the context of school, alienation is related to the negative student behaviors such as self-isolation, failure, absenteeism, and drop outs. [12]To be alienated is to lack a sense of belonging and to feel cut off from the family, friends and school. [13Those who are rejected by their peers have a two to eight times greater chance of dropping out of school. [14]Failure in school often stems from feeling disconnected from the teacher, other students, or the school community at large. Social isolation, according to [15] is the feeling of loneliness, even when in the company of others, due to a lack of meaningful, intimate relationships with peers, family, and the wider community. Students who feel isolated tend to be separated from mainstream groups, feel a lack of connection to others, and feel no one cares or pays attention to them. Even if a child is enjoying academic success in the classroom, his attitude to school will be determined by the degree of social success that he experiences. The children who feel isolated from their peers tend to have increasing social and academic problems especially in Mathematics.

Hypothesis

The null hypothesis for this study was:
H_0 : There is no correlation between numerical ability and achievement in Mathematics among isolated students of eighth standard students.

Sampling Method and Participants

Two schools each from rural, urban and metro areas of Maharashtra were selected randomly. 910 students of eight standard participated in the survey and by sociometric method 223 were found to be socially isolated students. The isolated students were selected for this study along with their parents and Mathematics teachers.

Instruments

[16]Sociometry, [17] Mathematics Anxiety Scale-India, [18] Students Liking Scale, [19] Home Environment Inventory, [20] Parent Involvement Scale and [21] Teachers' Attitude Scale towards Teaching and Teacher Student Relationship questionnaires were used for this study.

The study investigates the numerical ability of isolated students of eighth standard towards Mathematics and their achievements in Mathematics. Despite its importance, Mathematics seemed to be an unpopular subject among most students. Several dimensions of attitudes were considered. They are: Numerical ability, Home-isolation, Home-rejection, Student - Teacher understanding, Teacher-Student understanding, Parent-Teacher understanding, Teacher-Parent understanding, Parent-Child at home, Parent-Child about school and Maths Mark.

Correlation coefficients between the variables

	Numerical ability	Home-isolation	Home-rejection	St.-Tr. understanding	Tr.-St. understanding	Par-Tr understanding	Tr-Par understanding	Parent-Child at home	Parent-Child @ school	Maths Mark
Numerical ability	1									
Home-isolation	0.290	1								
Home-rejection	0.321	0.613	1							
St.-Tr. understanding	−0.125	−0.06	−0.047	1						
Tr.-St. understanding	−0.211	−0.3	−0.261	0.047	1					
Par-Tr understanding	−0.024	0.062	0.043	0.06	0.041	1				
Tr-Par understanding	−0.222	−0.25	−0.3	−0.008	0.378	−0.103	1			
Parent-Child at home	−0.079	−0.1	−0.153	0.028	0.072	0.613	−0.001	1		
Parent-Child@school	−0.108	−0.18	−0.247	0.129	0.08	0.549	0.014	0.759	1	
Maths Mark	−0.541	−0.22	−0.3	0.209	0.257	−0.034	0.066	0.033	0.065	1

The above table indicates there is negative correlation between numerical ability and mathematics achievement and there is positive correlation between numerical ability, home isolation and home rejection. Also the correlation between numerical ability and other variables are negative. Hence null hypothesis is rejected and it is concluded that there is correlation between numerical ability and mathematics achievement.

3 CONCLUSION

The development of mathematical knowledge is a gradual process. It needs a lot of practice and understanding; the students who don't remain in touch with it regularly find it difficult in examinations which results in fear towards Mathematics. Poor numerical ability is an important factor for the backwardness in mathematics. Students who are weak in numerical ability have difficulty in fractions and decimals as well as arithmetic operations which in turn hinders them from learning algebra. Algebra is considered to be a gateway to higher Mathematics. Hence we can conclude that a student's poor numerical ability is the root cause of all his/her problems connected with Mathematics.

Home isolation and home rejection are the major causes for poor numerical ability. This could be minimized if teacher-student understanding and teacher-parent understanding are improved with interaction of the parent and child about school activities.

Computer Assisted Instruction can be a supplement to these students, as it provides wide range of visuals, graphics and pictures to make learning more interesting. It helps the students to learn at their own pace and at their own convenience. It enables the students to practice as many times as they like to achieve the required competencies. It interacts with them by providing self-evaluation and immediate feedback to the students.

[8]"Our vision of excellent mathematical education is based on the twin premises that **all students can learn Mathematics and that all students need to learn Mathematics**. Curricula that assume student failure are bound to fail; we need to develop curricula that assume student success. We are at a historic juncture when we wish to guarantee education for all. It is therefore a historic imperative to offer our children the very highest quality of Mathematics education possible"

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Effect of striking angle parameter of oblique nanofluid flow on a stretching sheet

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Abstract :
An examination has been made for the study of steady two-dimensional flow of a viscous and incompressible nanofluid striking at some angle of incidence on a stretching sheet. Fluid is considered in the influence of transverse magnetic field. The stream function splits into a Hiemenz and a tangential component. Using similarity variables, the governing partial differential equations have been transformed into a set of three nondimensional ordinary differential equations. These equations have been then solved numerically using Runge-Kutta Fehlberg method. In the present reported work the effect of striking angle on nanofluid parameters on heat and mass transfer characteristics have been discussed. The reported results are in good agreement with the published work in the literature.

Keywords. Nanofluid, stretching sheet, stagnation point flow, magnetic field

1 Introduction

The conventional heat transfer in nanofluid has become a coeval topic of interest in modern times. The term nanofluid was coined by Choi (see [4]). A nanofluid is a novel class of heat transfer fluids which comprises a base fluid and nanoparticles. They possess very high thermal conductivity and unit phase heat transfer coefficient with their base fluids. Nanoparticles have dimension in the range of 1 to 100 nm in diameter. Nanoparticles used in nanofluids are typically composed of metals (Al, Cu), oxides (Al_2O_3), carbides (SiC), nitrite(AlN, SiN) and carbon nano tubes (allotrope of carbon nonmetal). Nanoparticles have the intrinsic peoperty of enhancing the thermal conductivity when mixed with other substrate. Conductive fluids such as water, ethylene glycol, oil, biofluids and polymer solution can be pondered as base fluid. The thermal conductivity of the ordinary heat transfer fluids is not adequate enough to apt the cooling rate requirement of modern industries. Nanofluid coolant exhibiting an ameliorate thermal performance is being considered as novel technology to secure nuclear safety. Nanoparticle frequently owning up to meager 5 percent volume fraction of nanoparticles to ensure serviceable heat transfer enhancement. Experimental study exhibit that even with small volumetric fraction of nanoparticle, the thermal conductivity of the base fluid is enhanced tremendously by 10-50 percent with a remarkable proficiency in the convective heat transfer coefficient.

A comprehensive survey of convective transport in nanofluids was fashioned by Buongiorno (see [3]). He marked that the absolute velocity of the nanoparticle can be assumed as a sum of velocity of base fluid and the slip velocity (relative velocity). He examined seven slip mechanisms viz. inertia, Brownian motion, thermophoresis, diffusiophoresis, Magnus effect, fluid drainage and gravity setting. After proceeding through he concluded that only Brownian motion and thermophoresis are momentus in the absence of turbulence. Buongiorno proposed a new model based on the mechanics of the nanoparticle / base fluid relative velocity.

The flow of an incompressible viscous fluid over a stretching surface is pressing in various processes. Due to its astounding applications the present field has attracted many researchers in modern times. Crane (see [5]) examined the application of uniform stress on a two dimensional steady flow of a viscous and incompressible fluid and concluded that the velocity of the stretching of the elastic flat sheet vary linearly with the distance from the fixed point. Gupta and Mahapatra (see [7]) analyzed stagnation point flow towards the stretching surface. They reported in their research work that a boundary layer is formed when stretching velocity is less than the free stream velocity. As the stretching velocity surpasses the free stream velocity than an inverted layer is formed. Singh et. al (see [13]), (see [14]),(see [15]),(see [16]) communicated the effect of magnetic parameter and radiations on various aspects of the stretching sheet.

The study of magneto hydrodynamic flow of an electrically conducting fluid is of prime interest in recent metallurgical and metal working process. Magneto hydrodynamic flow is induced by the deformation of the wall of a vessel containing the fluid. Magnetic nanofluid is a rare material containing both the liquid and magnetic properties. Many of the physical properties can be controlled by varying the magnetic field. Hamad (see [8]) examined the convective flow and heat transfer of nanofluid past a semi-infinite vertical stretching sheet in the absence of magnetic field. Bachok et. al (see [1]), (see [2]), studied the boundary layer stagnation point flow towards a stretching/ shrinking sheet in a nanofluid.

The objective of the present study is to analyze the development of the steady boundary layer flow, heat transfer over a stretching sheet in oblique flow of a nanofluid. This problem is extension of Singh et al., (see [13]), [flow model.

2 Mathematical Analysis

The mathematical model considered here consists of a viscous, incompressible, steady two-dimensional flow of an electrically conducting nanofluid striking at some angle of incidence γ on stretching sheet. Nanofluid is considered in the influence of transverse magnetic field. The stretching sheet has uniform temperature T_w and moving with non-uniform velocity $u_w = A_x$

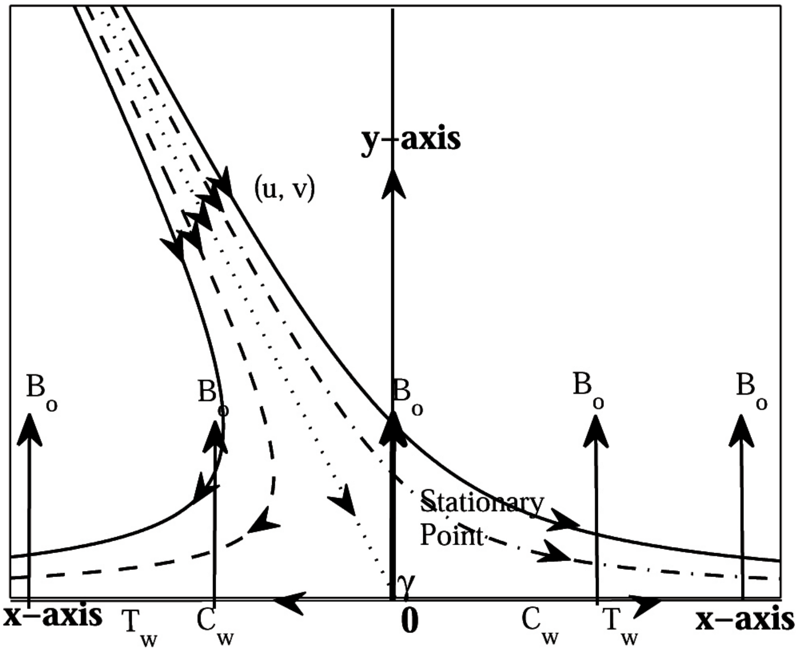

Figure 1: Physical picture of the Problem

where A is a positive constants with dimension $(time)^1$. Stretching sheet is placed in the plane $y = 0$ and $xaxis$ is taken along the sheet. The fluid occupies the upper half plane i.e. $y > 0$ as shown in Figure 1. The viscous dissipation, Joule heating and induced magnetic field are neglected. The governing equations of conservation of mass, momentum, thermal energy and nanoparticles under above assumptions are given by (see [9])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho b}\frac{\partial p}{\partial y} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^u}{\partial y^2}\right) - \frac{\sigma B_0^2}{\rho}u \tag{2.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho b}\frac{\partial p}{\partial y} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^v}{\partial y^2}\right) \tag{2.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \infty\left(\frac{\partial^2 T}{\partial x^2} + v\frac{\partial^2 T}{\partial y^2}\right) + \tau\left(D_b\left[\frac{\partial C}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial C}{\partial y}\frac{\partial T}{\partial y}\right] + \frac{D_1}{T_\infty}\left[(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\right]\right) \tag{2.4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_b\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) + \frac{D_1}{T_\infty}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{2.5}$$

where u, v are velocity components along x and y axes, respectively, v is kinematic viscosity, σ is electrical conductivity, T is the temperature, ρb is density of the base fluid, p is the fluid pressure, α is the thermal diffusivity, D_b is the Brownian diffusion coefficient, D_t is the thermophoresis diffusion coefficient and $\tau = (\rho c)_p/(\rho c)_b$ is the ratio between the effective heat capacity of the nanoparticle material and heat capacity of fluid with ρ being the density, c is the volumetric volume expansion coefficient. Boundary conditions for the given model are:

$$\begin{aligned} &u = u_w(x) = Ax, v = 0, T = T_w, C = C_w \text{ at } y = 0 \\ &u = azsiny + bycosy, v = -aysiny, T = T_\infty, C = C_\infty \text{ as } y \to \infty \end{aligned} \tag{2.6}$$

where a, b and A are positive constants of dimension $(time)^1$ and T_∞, C_∞ are the constant temperature and concentration of the fluid far away from the sheet and y is angle parameter.

Introducing the stream function $\psi(x,y)$ as defined by $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$ the dimensionless temperature $\theta = \frac{T-T_\infty}{T_w-T_\infty}$, the dimensionless concentration $\phi = \frac{C-C_\infty}{C_w-C_\infty}$ and the similarity variable $\eta = \sqrt{\frac{A}{v}}y, \xi = \sqrt{\frac{A}{v}}x$. Using these in boundary condition (2.6)

$$\begin{aligned} &\psi = 0, \frac{\partial\psi}{\partial\eta} = \xi, \theta(0) = 1, \phi(0) = 1 \text{ at } \eta = 0 \\ &\psi = \lambda\xi\eta siny + \frac{k}{2}\eta^2 cosy, \theta \to 0, \phi \to 0 as\eta \to \infty \end{aligned} \tag{2.7}$$

where $\lambda = A$ is stretching sheet parameter and $k = \frac{b}{A}$ is some positive constant. We seek solution of equations (2.2) and (2.3) in the form of $\psi = \xi f(\eta) + g(\eta)$ where the function $f(\eta)$ and $g(\eta)$ are referring to the normal and tangential component of the flow. Using above the governing equations reduces to (as explained in Singh et al., (see [13])

$$f'''(\eta) + f(\eta)f''(\eta) - f'(\eta)^2 - M(f'(\eta) - \lambda\sin y) + (\lambda\sin y)^2 = 0 \tag{2.8}$$

$$g'''(\eta) + f(\eta)g''(\eta) - f'(\eta)g'(\eta) - M(g'(\eta)) - k\eta\cos y) - km\cos y = 0 \tag{2.9}$$

$$\theta''(\eta) + Prf(\eta)\theta'(\eta) + PrN_b\phi'(\eta)\theta'(\eta) + PrN_1\theta'(\eta)^2 = 0 \tag{2.10}$$

$$\phi''(\eta) + Lef(\eta)\phi'(\eta) + \frac{N_t}{N_b}\theta''(\eta) = 0 \tag{2.11}$$

Boundary condition reduces to

$$\begin{aligned} f(0) = 0, f'(0) = 1, g(0) = 0, g'(0) = 0, \theta(0) = 1, \phi(0) = 1 \\ f'(\infty) = \lambda\sin y, g''(\infty) = k\cos y, \theta(\infty) = 0, \phi(\infty) = 0 \end{aligned} \tag{2.12}$$

where primes denotes differentiation with respect to η and, $M = \frac{\sigma B_0^2}{A\rho_b}$ is the magnetic parameter, m is some real constant, $Pr = \frac{v}{\infty}$ is the Prandtl number, $Le = \frac{v}{D_b}$ is the Lewis number, $N_b = \frac{(c\rho)_p D_b(C_w-C_\infty)}{c\rho}_b v$ is the Brownian motion parameter and $N_t = \frac{(c\rho)_p D_t(T_w-T_\infty)}{(c\rho)_b T_\infty v}$ is the thermophoresis parameter.
The quantities of practical interest are the Nusselt number N_u and the Sherwood number Sh which are defined as $Nu = fracxq_w K(T_w - T_\infty)$, $sh = \frac{xq_m}{D_b(C_w-C_\infty)}$ where q_w and q_m are the wall heat and mass fluxes, respectively and K is the thermal conductivity of the fluid. Using Similarity variables, we obtain $Re_x^{-1/2}Nu = -\theta'(0), Re_x^{-1/2}Sh = -\theta'(0)$ where $Re_x = u_w(x)x/v$ is the local Reynolds number based on the stretching velocity $u_w(x)$. Kuzetsov and Nield [11]). referred $Re^{1/2}Nu$ and $Re_x^{1/2}Sh$ as the (2.8) and (2.11) subject to the boundary conditions (2.12) constitute a two-point boundary value problem. In order to solve these equations numerically, we follow most efficient numerical shooting technique with Runge integration scheme. It is worth mentioning here that equations (2.8) and (2.9) with boundary condition (2.12) has been solved numerically by Singh et al., 2010.

3 Results and Discussion

The Runge-Kutta Fehlberg method with the help of shooting technique has been used to solve equations (2.8) to (2.11) subject to the boundary conditions (2.12) for different values of M, λ, n_b, n_t and Le taking step size 0.001. While numerical simulation, step size 0.002 and 0.003 were all checked and values of $f''(0), g''(0), \theta'(0)$ and $\phi(0)$ found in each case were correct up to six decimal places. Hence the scheme used in this paper is stable and accurate. It has been observed from Table 1 that, the numerical values of (0) in the present paper for different value of Pr when

$n_b = 0, n_t = 0, \lambda = 0, y = \pi/2$ in absence of magnetic field are in good agreement with results obtained by Khan and Pop[10]), Wang [17]) and Gorla and Sidawi [6]).

Table 1: Comparison of results for the reduced Nusselts number $\theta'(0)$

Value of Pr	Khan and Pop [10].	Wang [17].	Gorla and Sidawi [6]).	Present Paper
0.70	0.4539	0.4539	0.5349	0.53487
2.00	0.9113	0.9114	0.9114	0.91166
7.00	1.8954	1.8954	1.8905	1.89034
20.00	3.3539	3.3539	3.3539	3.34994
70.00	6.4621	6.4622	6.4622	6.45879

Table 2: Values of reduced Nusselt number $-\phi'(0)$ and reduced Sherwood number $\phi'(0)$ when $\lambda = 2$ and $M = 3$

$-\theta(0)$	$-\phi'(0)$	$f''(0)$	Pr	γ	Le	n_b	n_r
−1.1492982	−2.9234527	2.653930	10	$\pi/2$	10	0.1	0.1
−1.1168558	−2.8116037	1.885775	10	$\pi/3$	10	0.1	0.1
−1.0770794	−2.6708548	1.027422	10	$\pi/4$	10	0.1	0.1
−1.0461960	−2.5580143	0.422444	10	$\pi/5$	10	0.1	0.1
−1.3815730	−2.8150402	2.653930	10	$\pi/2$	10	0.1	0.05
−1.3410427	−2.7181180	1.885775	10	$\pi/3$	10	0.1	0.05
−1.2911287	−2.5971296	1.027422	10	$\pi/4$	10	0.1	0.05
−1.2521525	−2.5010682	0.422444	10	$\pi/5$	10	0.1	0.05
−1.6063255	−2.8403950	2.653930	10	$\pi/2$	10	0.1	0.01
−1.5578009	−2.7518132	1.885775	10	$\pi/3$	10	0.1	0.01
−1.4978592	−2.6420914	1.027422	10	$\pi/4$	10	0.1	0.01
−1.4508710	−2.5557919	0.422444	10	$\pi/5$	10	0.1	0.01
−1.5395909	−2.3714272	2.653930	10	$\pi/2$	10	0.05	0.1
−1.4960066	−2.2509261	1.885775	10	$\pi/3$	10	0.05	0.1
−1.4425316	−2.0969317	1.027422	10	$\pi/4$	10	0.05	0.1
−1.4009731	−1.9712720	0.422444	10	$\pi/5$	10	0.05	0.1
−0.7728973	−2.7627232	2.653930	0.7	$\pi/2$	10	0.1	0.1
−0.8790425	−2.7384882	2.653930	1	$\pi/2$	10	0.1	0.1
−1.2451040	−2.7292317	2.653930	5	$\pi/2$	10	0.1	0.1
−0.7342667	−2.6768300	1.885775	0.7	$\pi/3$	10	0.1	0.1
−0.8365913	−2.6522284	1.885775	1	$\pi/3$	10	0.1	0.1
−1.2005324	−2.6318047	1.885775	5	$\pi/3$	10	0.1	0.1
−0.6465449	−2.4867638	0.422444	0.7	$\pi/5$	10	0.1	0.1
−0.7387362	−2.4607583	0.422444	1	$\pi/5$	10	0.1	0.1
−1.1003736	−2.4125200	0.422444	5	$\pi/5$	10	0.1	0.1
0.1444885	−1.8881674	2.653930	10	$\pi/2$	1	0.1	0.1
−1.6511198	−0.6249025	2.653930	10	$\pi/2$	2	0.1	0.1
−1.5114309	−1.1323003	2.653930	10	$\pi/2$	3	0.1	0.1
−1.3446019	−1.8342790	2.653930	10	$\pi/2$	5	0.1	0.1
−1.1492982	−2.9234527	2.653930	10	$\pi/2$	10	0.1	0.1
−1.7464306	0.29883752	0.422444	10	$\pi/5$	1	0.1	0.1
−1.5260678	−0.3957912	0.422444	10	$\pi/5$	2	0.1	0.1
−1.3933471	−0.8646115	0.422444	10	$\pi/5$	3	0.1	0.1
−1.2335648	−1.522038	0.422444	10	$\pi/5$	5	0.1	0.1
−1.0461960	−2.5580143	0.422444	10	$\pi/5$	10	0.1	0.1

Our main purpose is to explore the influence of the parameters n_b, n_t, Pr, Le with striking angle γ on the heat and mass flux characteristics. The variations of the dimensionless temperature and concentration with Brownian motion parameter n_b and γ for $n_t = 0.1, M = 3, \gamma = 2, Pr = 10, Le = 10$ have been presented through Figs. 2 and 3 and Table 2. As expected, the boundary layer profiles for the temperature are of the same form as in the case of regular heat transfer fluids. The temperature in the boundary layer increases with the increase in the Brownian motion parameter n_b. The nanoparticle volume fraction profile, decreases with the increase in the n_b. Brownian motion serves to warm the boundary layer and simultaneously exacerbates particle deposition away from the fluid regime (onto the surface), thereby accounting for the reduced concentration magnitudes (Rana and Bhargava (see [12]). It is also noticed from figs that, as the striking angle decreases, the temperature and concentration in the boundary layer increases with the Brownian motion parameter. Reduced Nusselt number $\theta(0)$ and re duced Sherwood number $\psi'(0)$ increase with decrease in striking angle which has been shown in Table 2. It is also observed from Table 2 that reduced Nusselt number decrease and reduced Sherwood number increase with decrease in Brownian motion parameter.

It is observed from Fig. 4 that the dimensionless temperature decreases with an increase in the Prandtl number Pr for any value of striking angle and stretching parameter $\lambda = 2$. This is in agreement with the physical fact that at higher Prandtl number, fluid has a thinner thermal boundary layer and this increases the gradient of temperature. Fig. 5 shows that concentration

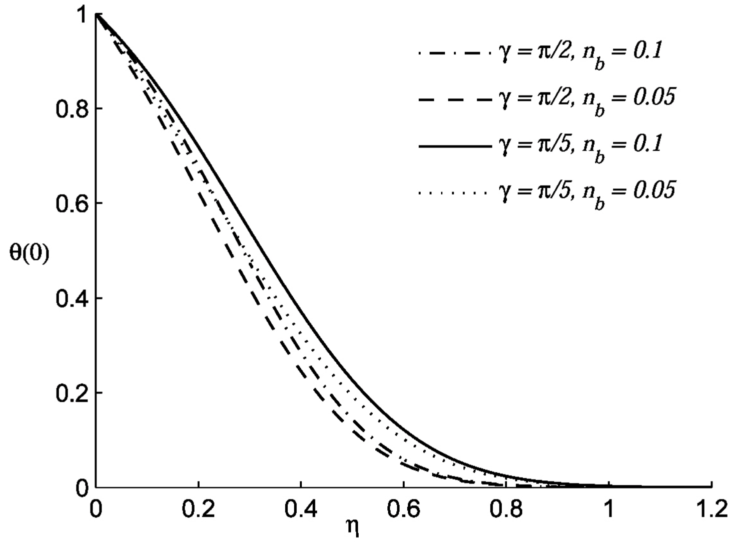

Figure 2: Effect of Brownian motion parameter n_b on temperature distribution for specified parameters.

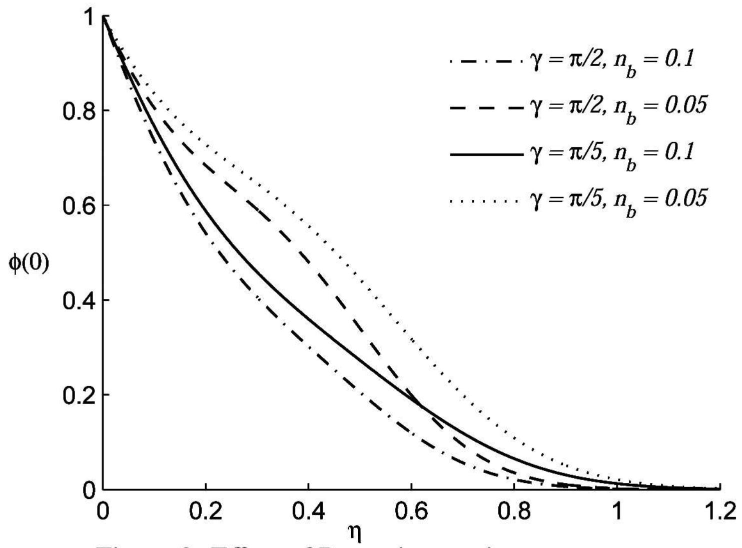

Figure 3: Effect of Brownian motion parameter n_b on concentration distribution for specified parameters.

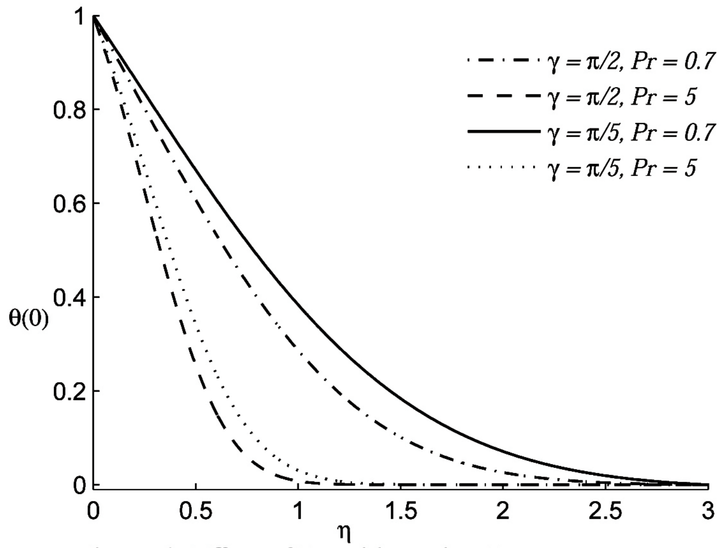

Figure 4: Effect of Prandtl number Pr on temperature distribution for specified parameters.

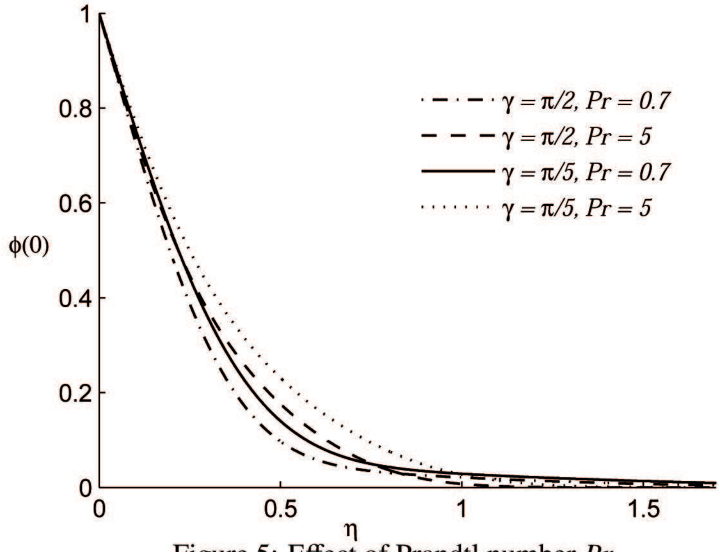

Figure 5: Effect of Prandtl number *Pr* on concentration distribution for specified parameters.

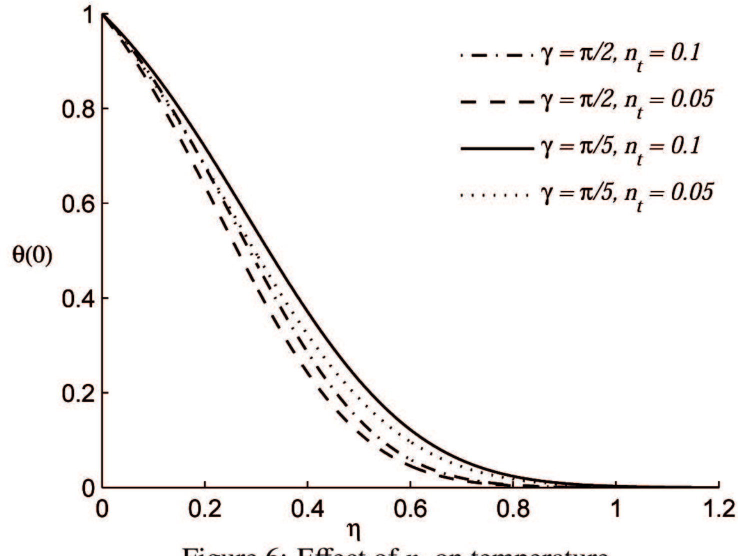

Figure 6: Effect of n_t on temperature distribution for specified parameters.

distribution increases with an increase in Pr. The temperature and concentration in the boundary layer increases with decrease in striking parameter for any value of Pr. Reduced Nusselt number decrease and reduced Sherwood number increase with increase in Prandtl number which has been shown in Table 2.

Fig. 6 and 7 depicts the dimensionless temperature and concentration distribution for different values of striking angle and thermophoresis parameter for $n_b = 0.1, \lambda = 2, Pr = 10, Le = 10$ and $M = 3$. It is observed that the decrease in striking angle leads to increase in dimensionless temperature and concentration distribution. It also exhibits that dimensionless temperature and concentration distribution reduces with thermophoresis parameter. Reduced Nusselt number decrease with thermophoresis parameter, whereas and reduced Sherwood number increase with increase in thermophoresis parameter which has been shown in Table 2.

Fig. 8 and 9 portrays the dimensionless temperature and concentration distribution for different values of Lewis number taking $n_b = 0.1, \lambda = 2, Pr = 10, n_t = 0.1$ and $M = 3$. It has been observed that the dimensionless temperature reduces with Lewis number. It shows the concentration distribution increases with decrease in Lewis number. Table 2 exhibits that the reduced Nusselt number increases with Lewis number, whereas and reduced Sherwood number decrease with increase in Lewis number. For $Le = 1$ an opposite trend is observed for reduced Nusselt number and reduced Sherwood number.

4 Conclusions

An analysis has been made for the steady two-dimensional flow of a viscous and incompressible nanofluid striking at some angle of incidence on a stretching sheet. Influence of the parameters n_b, n_t, Pr with striking angle γ on the heat and mass flux characteristics has been studied. The main results of the paper can be summarized as follows:

(a) The temperature in the boundary layer increases with the increase in the Brownian motion parameter.

(b) The nanoparticle volume fraction profile, decreases with the increase in the Brownian motion parameter.

(c) Decrease in striking angle leads to increase in temperature and concentration in the boundary layer with the Brownian motion parameter.

(d) Dimensionless temperature decreases whereas concentration distribution increases with the increase in the Prandtl number.

(e) Dimensionless temperature and concentration distribution reduces with thermophoresis parameter.

(f) Dimensionless temperature decreases whereas concentration distribution increases with the decrease in the Lewis number.

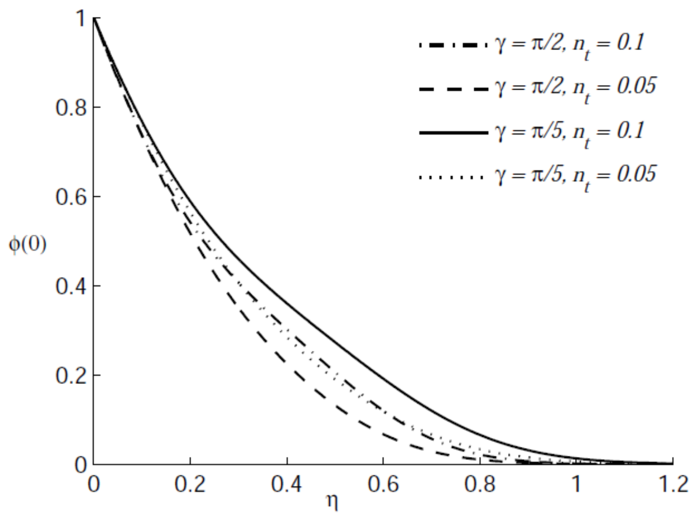

Figure 7: Effect of n_t on concentration distribution for specified parameters.

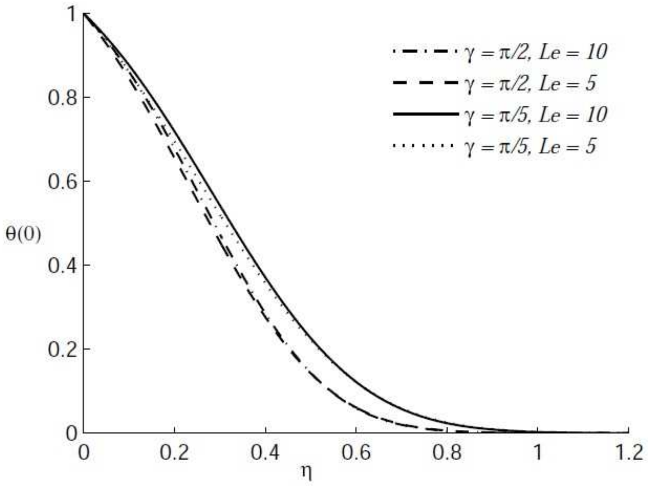

Figure 8: Effect of Le on temperature distribution for specified parameters.

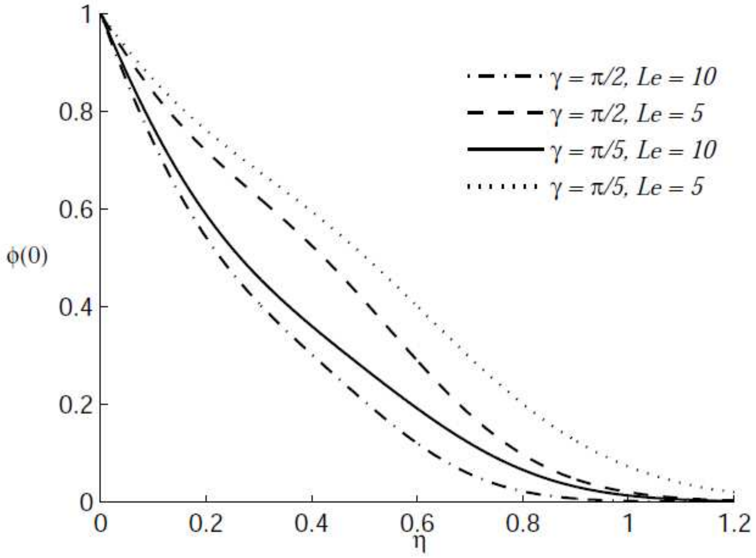

Figure 9: Effect of Le on concentration distribution for specified parameters.

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Fractal and time series analysis of a deformed chaotic map

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Abstract : The logistic model of Verhulst forms the basis of the modern chaos theory and thus represents the simplest cases of chaotic system. The significance of this model is due to its strange conduct for changing values of the parameter. It can exhibit stable, periodic and chaotic behaviours for the successive values of the growth parameter. Our aim is to study the stability analysis of this map for Mann iterates and visualize the fractal patterns for changing values of the parameters. The Matlab tools are used for the computational and graphical purpose.

Keywords: Complex logistic map, Mann iteration, fractals.
2010 Mathematics Subject Classification. 37B25, 37C25.

1 INTRODUCTION AND PRELIMINARIES

The logistic growth model was proposed by the Belgian mathematician Verhulst [20] in the year 1845. In 1976, R. May [9-10] recognized the significance of this model specially for demographic modelling. He observed that the continuous time model may not be suitable to reflect the realities in most of the cases and constructed a discrete version of this model. The extreme sensitivity to the initial condition is the most interesting aspect of this model due to which it exhibits a variety of behaviors from stable to chaotic. Dettmer [3] pointed out that the regularity and stability disappears from the system once it becomes chaotic and therefore it is important to recognize and possibly avoid it. Kint et al [6] explored the graphical potential of this map and generated fractal figures named as Verhulst fractals (see also [11]). The authors of this paper studied the stability of the logistic and complex logistic map for different iterative schemes in [14-18]. Jaganathan and Sinha [7] proposed the deformed nonlinear maps for expoloring the interestingly wide spectrum of behaviours of some of the physical systems. They realized that q–deformation effectively takes into account the interactions in physical systems and thus shows the rare phenomena of the co-existence of the normal and chaotic nature (see also [1-2], [4]). Our aim is to study the stability of the deformed logistic map in complex domain and visualize the fractal patterns for varying values of the parameters under Mann iteration scheme.

First, we summarize the basic concepts required for our results.

Definition 1.1. Let X be a non-empty set and $f : X \to X$. Then for any point x_0 in X, the iterative scheme $x_{n+1} = f(x_n), n = 0, 1, 2, \dots$ is called Picard iterate and the Picard orbit is defined as follows

$$O(f, x_0) := \{x_n \ : \ x_n = f(x_{n-1}), n = 0, 1, 2, \dots\} \tag{1.1}$$

Definition 1.2 ([8])**.** Let X be a non-empty set and $f : X \to X$. For a point x_0 in X, construct a sequence $\{x_n\}$ in the following manner:

$$x_{n+} = \alpha_n f(x_n) + (1 - \alpha_n)x_n \tag{1.2}$$

for $n = 0, 1, 2, 3, \ldots$, where $0 < \alpha_n \leq 1$.

The sequence $\{x_n\}$ generated as above is called Mann iterate of a point x_0 and it is denoted by $MO(f, x_0, \alpha_n)$. It is also called a two-step feedback scheme. In all our studies, we consider $\alpha_n = \alpha$, for $n = 0, 1, 2, 3, \ldots$, where $0 < \alpha \leq 1$.

It is remarked that (1.2) with $\alpha_n = 1$ generates the Picard orbit (1.1).

2 DISCUSSIONS AND RESULTS

Verhulst postulated that the growth rate at any time should be proportional to the fraction of the environment that is not yet used up by the population at that time. Verhlust's model was further expressed by R. May [9] in the following manner

$$x_{n+1} = rx_n(1 - x_n), \tag{2.1}$$

where x_n (a real number between 0 and 1) represents population density at time $n = 1, 2, 3, \ldots$ and r is the combined rate for reproduction and starvation [5].

We study the stability of the logistic map with q-deformation proposed by Jaganathan and Sinha [7] under Mann iterative scheme. A deformed version of the logistic map (2.1) is proposed in [7] as follows:

$$x_{n+1} = r[x_n](1 - [x_n]) \tag{2.2}$$

where $[x] = \frac{x}{1+\varepsilon(1-x)}$ and $-1 < \varepsilon < \infty$ for x in the interval [0, 1]. The q-deformed logistic map (2.2) with Mann iteration (1.2) is given by

$$x_{n+1} = \alpha r[x_n](1 - [x_n]) + (1 - \alpha)x_n \tag{2.3}$$

where $0 < \alpha_n \leq 1, [x_n] = \frac{x_n}{1+\varepsilon(1-x_n)}$ and $-1 < \varepsilon < \infty$ for x in the interval $[0, 1]$.

The quadratic transformations of the type $z \to z^2 + c$, where z and c both are from complex plane C, are widely studied in the literature, see, for instance [17], [19] and several references of them. Our aim is to study the map $z \to [z]^2 + c$. The interest is to know the behaviour of the structure of the orbit of the iterates of z when z and c vary. For $n = 0, 1, 2, \ldots$, the iteration scheme for such a map is

$$z_{n+1} = [z_n]^2 + c. \tag{2.4}$$

Following Peitgen et al. [12-13], one can easily find that (2.2) and (2.4) are identical for $c = r(2 - r)/4$, $[z_n] = r/2 - r[x_n]$ and $z_{n+1} = r/2 - rx_{n+1}$. This functional equivalence shows that the fractal patterns shown by the map (2.2) are similar to that of the map (2.4).

2.1 Stability analysis through time series

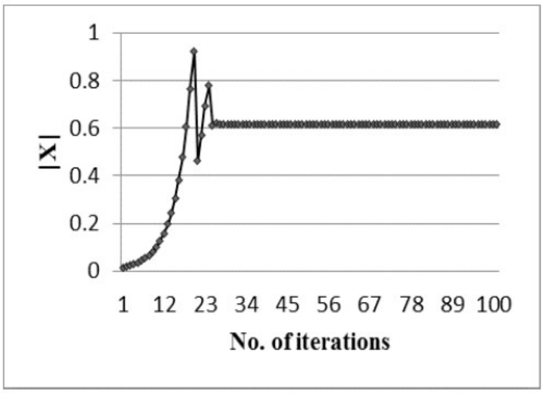
We study the behavior of q-deformed logistic orbits generated by (2.3) using time series. For this, we assume $x_n = x_{x_n} + \mathrm{i}x_{y_n}$ and $r = r_x + \mathrm{i}r_y$. Now we compute the values of x_{x_n} and x_{y_n} at different iteration levels using the scheme (2.3) and find the optimum value of $|r|$ for various choices of parameter α and deformation parameter ε. We consider $X_0 = X_{X_0} + \mathrm{i}X_{y_0} = 0.01 + 0.01\mathrm{i}$ as the initial choice for our experimental study of the q-deformed complex logistic map (2.2) with Mann iteration scheme. Our study involves two cases for the values of the parameter r.

Case I: When r is purely real, we compute the orbits of the map for fixed α and ε and go on varying r until the iterate of the map remains bounded. These threshold values of r are computed under 2000 iterations and shown in the Table 1.

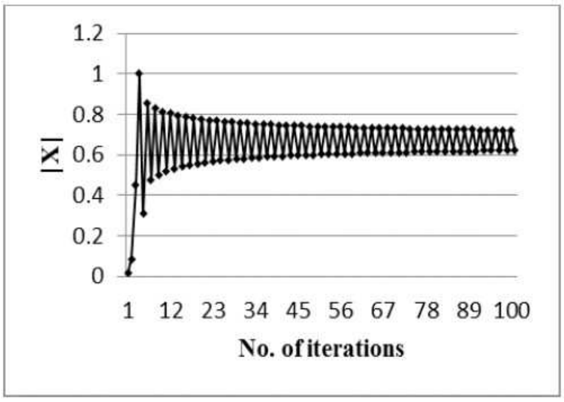
Table 1. The optimum values of $|r|$ (when $r_y = 0$)

ε \ α	1	0.7	0.4	0.1
-0.50	2.96	<u>4.27</u>	6.58	<u>23.37</u>
1	<u>2.49</u>	2.62	2.84	3.36
10	15.541	16.16	<u>17.37</u>	19.20

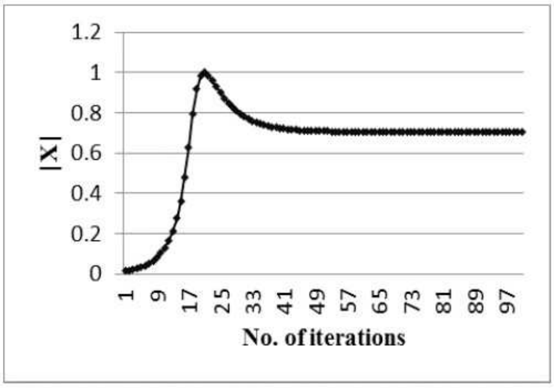
In this case it is observed that for a fixed ε and varying α (from 1 towards zero), the optimum value of the control parameter r increases surprisingly to a maximum of 23.37. The corresponding fractal patterns and time series analysis showing the behaviours of the map for some random values of r (shown underlined) are drawn, although the same could be drawn for all the tabulated values of r (see figures 1, 3).



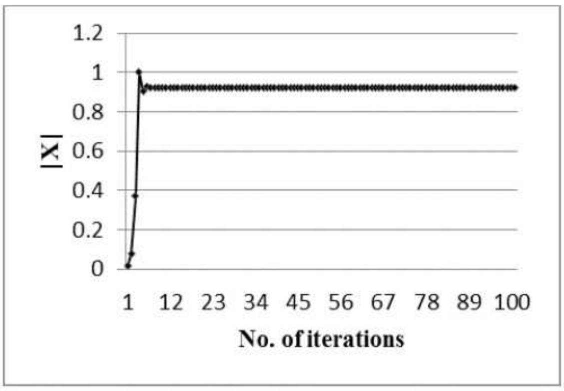
(a) $\alpha = 1, \varepsilon = 1, r = 2.49$



(b) $\alpha = 0.7, \varepsilon = -0.5, r = 4.27$



(c)$\alpha = 0.4, \varepsilon = 10, r = 17.37$



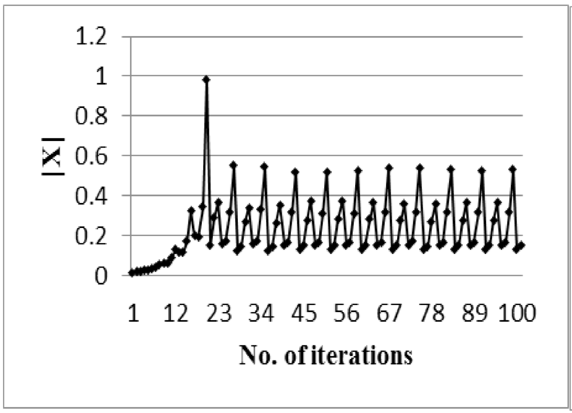
(d)$\alpha = 0.1, \varepsilon = -0.5, r = 23.37$

Figure 1: Time series at different values of α, ε (when $r_y = 0$).

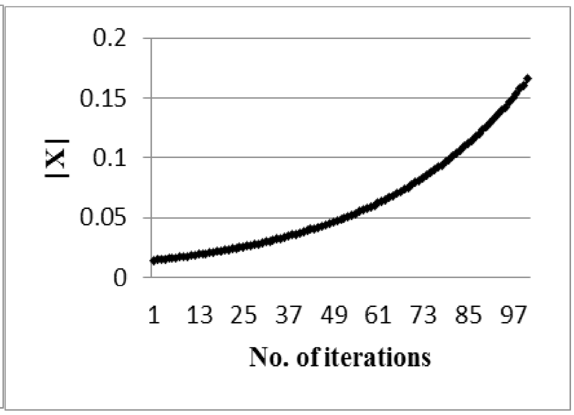
Case II: In this case, we obtain the optimal values of a purely imaginary r for the same choices of the parameters α and ε. We observe that for a fixed ε and varying α (from 1 towards zero), the optimum value of the control parameter r increases to a maximum of 48.22 for the same choice of α and β (see Table 2).

Table 2. The optimum value of $|r|$ (when $r_x = 0$)

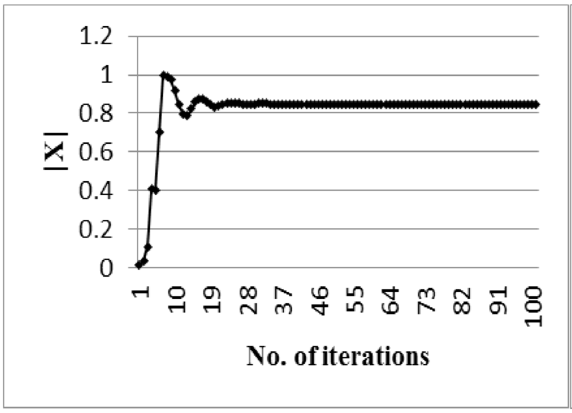
ε \ α	1	0.7	0.4	0.1
-0.5	<u>0.60</u>	2.08	<u>3.32</u>	3.33
1	2.00	<u>2.80</u>	4.01	8.81
10	11.00	15.00	22.02	<u>48.22</u>



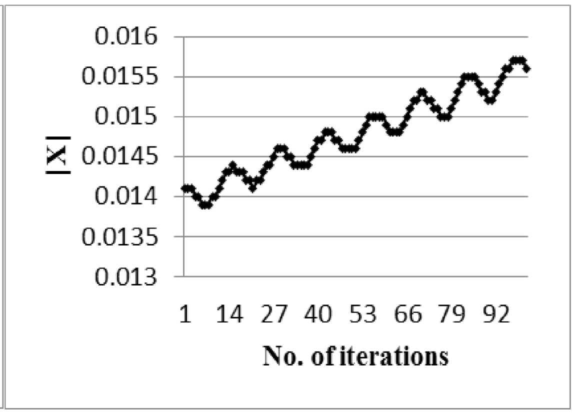
(a) $\alpha = 1, \varepsilon = -0.5, r = 0.60$



(b) $\alpha = 0.7, \varepsilon = 1, r = 2.80$



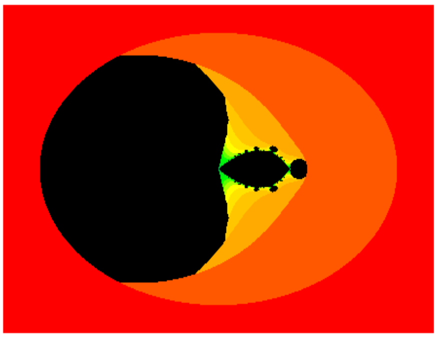
(c)$\alpha = 0.4, \varepsilon = 0.5, r = 3.32$

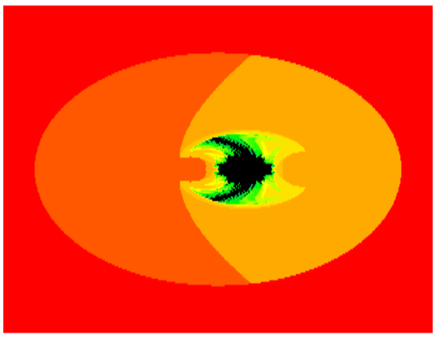


(d)$\alpha = 0.1, \varepsilon = 10, r = 48.22$

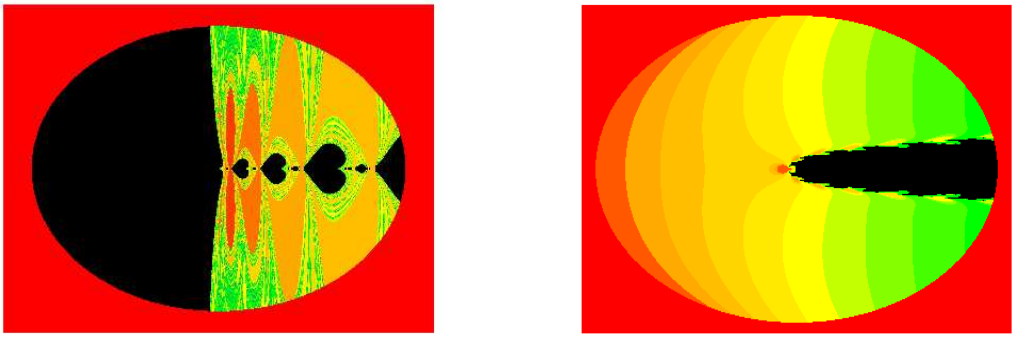
Figure 2: Time series at different values of α , ε (when $r_x = 0$).

The corresponding fractal patterns and time series analysis for specific choices of r (shown underlined) showing the behaviours of the map are drawn, although the same could be drawn for all the tabulated values of r (see figures 2, 4).



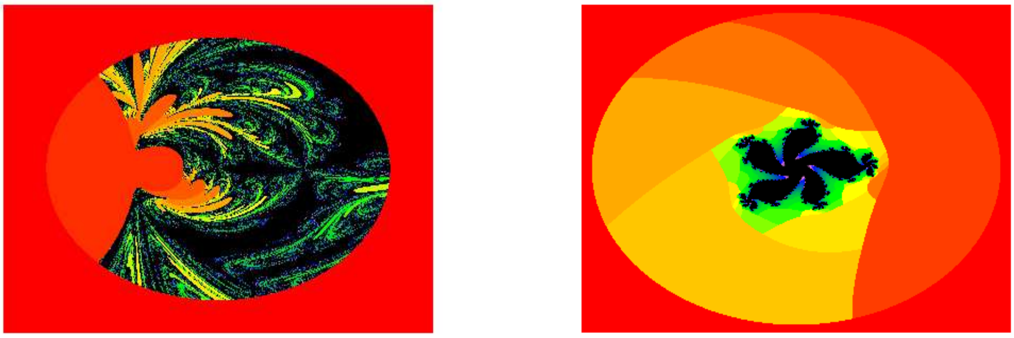


(a) $\alpha = 1, \varepsilon = 1, r = 2.49$ (b) $\alpha = 0.7, \varepsilon = -0.5, r = 4.27$

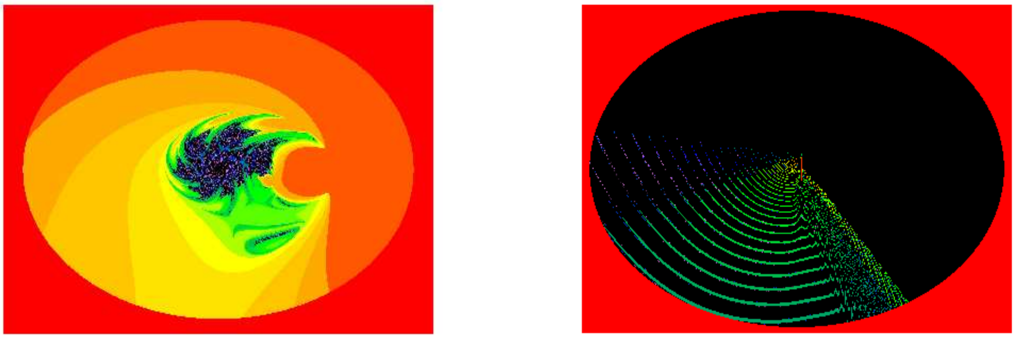


(c) $\alpha = 0.4, \varepsilon = 10, r = 17.37$ (d) $\alpha = 0.1, \varepsilon = -0.5, r = 23.37$

Figure 3: Fractal patterns for real r.



(a) $\alpha = 1, \varepsilon = -0.5, r = 0.60$ (b) $\alpha = 0.7, \varepsilon = 1, r = 2.80$



(c) $\alpha = 0.4, \varepsilon = 0.5, r = 3.32$ (d) $\alpha = 0.1, \varepsilon = 10, r = 48.22$

Figure 4: Fractal patterns for purely complex r.

3 CONCLUSION

We observe that for a fixed ε and varying α (from 1 towards zero), the optimum value of the control parameter r increases to a maximum of 48.22 whereas it increases to a maximum of 23.37 in case of purely real r for the same choices of α and β (see Table 1-2). The fractal patterns of q-deformed logistic map for Mann iterative scheme are also plotted for same specific choices of the parameters α and ε . The chaotic behavior of the mapping shown for Picard scheme is found to be stable when we use the two step Mann iterative scheme for fixed α and all values of the parameters ε.

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The effect of mathematics curriculum reform in Turkey

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Abstract : During the last decade, Turkish Ministry of National Education (MNE) has undertaken many fundamental development and improvement efforts to improve the quality of Turkish education system owing to educational reform movements of other nations, Turkey's candidacy for European Union (EU), and low performances of Turkish students in international studies such as TIMSS, PISA, and PIRSL. One of these fundamental changes is redeveloping the primary mathematics curriculum and this newly developed primary mathematics curriculum has been implemented since 2005. The aim of the present study was to compare mathematics performances of Turkish eighth graders participated in TIMSS 2011 and 2007. The comparisons were made from different perspectives such as, average mathematics achievement scores (AMAS), AMAS in content and cognitive domains, percentages of students reaching international benchmarks of mathematics achievement as well as percentages of correct responses for released items. The results displayed that in each handled area, the mathematics performances of the eighth graders participated in TIMSS 2011 were better than those of eighth graders participated in TIMSS 2007. It was concluded that the new mathematics curriculum had a positive effect on students' mathematics performance.

Keywords: *Curriculum reform, mathematics achievement, TIMSS*

1 Introduction

Over the last decade, Turkish Ministry of National Education (MNE) has undertaken many development and improvement efforts to improve the quality of Turkish education system. The influential reasons for these efforts of Turkish Ministry of National Education are educational reform movements of other nations [4], Turkey's candidacy for European Union (EU) [3], declaration of World Bank [18] with regard to unsatisfactory level of education in Turkey, and finally low performances of Turkish students in international studies such as Trends in International Mathematics and Science Study (TIMSS) [7; 8; 9], Programme for International Student Assessment (PISA) [11; 12; 13], and Progress in International Reading Literacy Study (PIRLS) [10]. In 1997, the compulsory education was increased from five to eight. Curriculum reform in elementary grades (grades 1 to 8) was started in metricconverterProductID2003 in2003 in five school subjects; life science, mathematics, science and technology, social science, and Turkish. The duration of secondary education was extended from three to four in metricconverterProductID2005. In2005. In 2012 the compulsory education was increased from eight to twelve.

One of these efforts is redeveloping the primary mathematics curriculum taking into consideration that there are problems with regard to teaching mathematics in Turkey. The new primary mathematics curriculum was prepared based on the mathematics education research studies conducted in national and international areas, mathematics curriculum of developed countries as well as experience of mathematics education in Turkey. The principle "every child can learn mathematics" is the main focus of the new curriculum. This new curriculum stresses the mathematical concepts, the relationships among these concepts, the meaning under procedures as well as procedural skills to be acquired. One of the important approaches of the new curriculum is that it considers mathematics as an active process. Thus, the importance of learning environments in which the students investigate, discover, solve problem, share and discuss their solutions and approaches is strongly emphasized [5; 6].

The effect of new mathematics curriculum that has been implemented since 2005 has not been investigated with respect to students' academic performance up to now. Some studies have been conducted from different perspectives to investigate whether the new primary mathematics curriculum has provided positive improvements and developments. They engaged with different aspects of the new curriculum such as the effects of new mathematics curriculum on teachers, students and parents [17], evaluation of new mathematics curriculum based on teachers' views [15], and analyses of new mathematics curriculum based on opinions of 5^{th} grade students and teachers with respect to classroom management, instruction and strengths and weaknesses of the curriculum [2]. Differently from previous studies, this study is the first study investigating the effect of new mathematics curriculum with regard to students' academic achievement in mathematics. The purpose of this study is to investigate the effect of new primary mathematics curriculum by comparing the mathematics performances of eighth grade Turkish students participated in TIMSS 2007 and TIMSS 2011. It is evident that investigating the effect of a new curriculum with regard to academic performances of students required long running comprehensive research studies; however TIMSS provides unprecedented opportunity to make such comparisons by providing extensive information on the performance of students in mathematics as well as sub-domains in mathematics. Turkish eighth graders participated in TIMSS 2011 had been taught by the new primary mathematics curriculum since they were at second grade whereas Turkish eighth graders participated in TIMSS 2007 had been taught by the new primary mathematics curriculum since they were at seventh grade. Although, the mathematics performances of these students are lower than the international average of TIMSS 2011 it is clear that their mathematics performances are higher than those of eighth grade Turkish students participated in TIMSS 2007. Based on this difference the current study is aimed to compare mathematics performances Turkish eighth grade students participated in TIMSS 2007 and 2011 from different perspectives such as, mathematics performances at Benchmarks, content and cognitive domains. Additionally, released items of both TIMSS 2007 and 2011 were scrutinized and percentages of correct responses of Turkish eighth graders participated in TIMSS 2007 and 2011 were compared.

2 Procedure

2.1 Participants

Since the major aim of TIMSS is to provide comparative information about educational achievement across and within countries, the sample design of TIMSS also aims to provide accurate measures of changes in student achievement from cycle to cycle [14]. Eighth grade population of TIMSS is that the mean age at the time of testing is at least 13.5 years. The sampling design of TIMSS is two-stage stratified cluster sample design schools as the first stage and intact classes as the second stage [14]. As a result, 4498 and 6928 eighth grade Turkish students were participated in TIMSS 2007 and 2011, respectively.

2.2 Method

In the current study, mathematics performances of eighth grade Turkish students participated in TIMSS 2007 and 2011 were compared descriptively. TIMSS mathematics assessment framework is organized around two dimensions; a content dimension and a cognitive dimension. The content dimension specifies the subject matter to be assessed within mathematics; number, algebra, geometry and data and chance whereas the cognitive dimension describe the sets of behaviors expected of students as students engage with the mathematics content; knowing, applying, and reasoning. The mathematics performances compared in the present study include overall mathematics achievement average scores, average achievement scores in the number, algebra, geometry, and data and chance mathematics content domains, average achievement scores in the knowing, applying, and reasoning cognitive domains as well as the percentages of students reaching international benchmarks of mathematics achievement.

Since TIMSS that is repeated in each 4-year period provides trend data, assessment policy of TIMSS provides for retaining some of the items for the measurement of trends and releasing some items into the public domain. In addition to these comparisons, the released items of TIMSS 2007 and 2011 were investigated with regard to items' specific mathematics subject matter and the cognitive skills the items required. The aim of this investigation was to find comparable items to reveal mathematics performance differences between two participant groups.

3 Results

The scaling methodology of TIMSS summarizes the achievement on a scale with a mean of 500 and a standard deviation of 100. Thus, this methodology enables comparable trend measures from assessment to assessment. The results displayed that Turkey ranked as 31^{th} in TIMSS 2007 out of 50 countries with a score of 432 [8], whereas Turkey ranked as 24^{th} in TIMSS 2011 out of 45 countries with a score of 452 [7]. When it is thought that the average scale of TIMSS is 500, Turkey ranked under the international average in both TIMSS 2007 and 2011. However, it can be concluded that there is a slight improvement (a score of 20) in the overall mathematics achievement score between eighth graders participated in TIMSS 2007 and 2011.

The average achievement scores of Turkish eighth graders in four content domains for TIMSS 2007 and 2011 are presented in Table 1.

Table 1. Average achievement scores in content domains

TIMSS	Content Domain			
	Number	Algebra	Geometry	Data and Chance
2007	429	440	411	445
2011	435	455	454	467
Difference	6	15	43	22

As seen in Table 1, average achievement scores of Turkish students participated in TIMSS 2011 were better than those of students participated in TIMSS 2007 in all of the four content domains. When comparisons were made across the content domains, it is observed that the biggest difference is in the geometry content domain with a difference score of 43. The other content domains; data and chance, algebra, and number follow the geometry content domain, respectively from the biggest difference to the smallest difference.

The average achievement scores of Turkish eighth graders in three cognitive domains for TIMSS 2007 and 2011 are presented in Table 2.

Table 2. Average achievement scores in cognitive domains

TIMSS	Cognitive Domain		
	Knowing	Applying	Reasoning
2007	439	425	441
2011	441	459	465
Difference	2	34	24

As seen in Table 2, average achievement scores of Turkish students participated in TIMSS 2011 were better than those of students participated in TIMSS 2007 in all of the three cognitive domains. When comparisons were made across the cognitive domains, it is observed that the biggest difference is in the applying cognitive domain with a score of 34. The difference in the reasoning cognitive domains is 24 whereas there is only 2-score difference in the knowing cognitive domain. The results indicated that in TIMSS 2011 more students are able to apply mathematical knowledge of facts, skills, and procedures and understand mathematical concepts to create representations. Additionally, in TIMSS 2011 more students have the capacity for logical, systematic thinking, and are able to use intuitive and inductive reasoning to arrive at solutions to non-routine problems.

The percentages of students reaching international benchmarks of mathematics achievement in TIMSS 2007 and 2011 are presented in Table 3.

Table 3. Percentages of students reaching international benchmarks

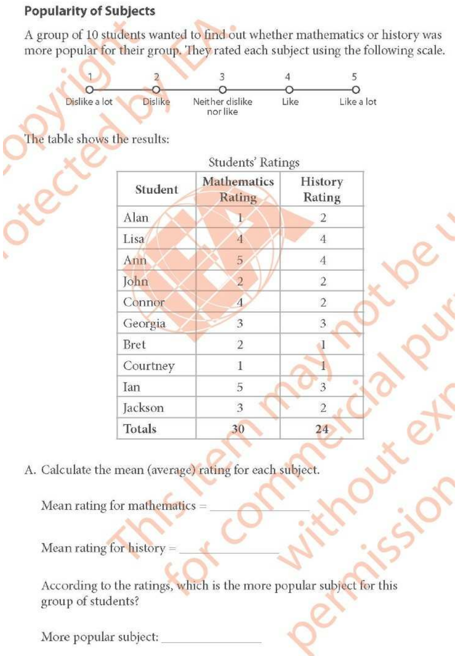
TIMSS	International Benchmarks			
	Advanced	High	Intermediate	Low
2007	5	15	33	59
2011	7	20	40	67

As seen in Table 3, the percentages of students reaching international benchmarks in TIMSS 2007 are 5, 15, 33, and 59 for advanced, high, intermediate, and low, respectively. However, the percentages of students reaching international benchmarks in TIMSS 2011 are 7, 20, 40, and 67 for advanced, high, intermediate, and low, respectively. The percentages of students participated in TIMSS 2011 are more than those of TIMSS 2007 in all of the international benchmarks.

91 released items of TIMSS 2007 and 90 released items of TIMSS 2011 were scrutinized to find comparable items to compare their percentages of correct responses. The content domain, topic area, cognitive domain and the context of the items were investigated. In order to make accurate comparisons very similar item pairs should be found among released items. Fifteen items; seven items from TIMSS 2007 and eight items from TIMSS 2011 are found to compare students' percentages of correct responses. These items are included in the content domain of number (whole number, fractions and decimals), algebra (patterns), geometry (geometric measurement), and data and chance (data interpretation). The comparison results display that in all of the item pairs students participated in TIMSS 2011 have higher percentages of correct responses than those of students participated in TIMSS 2007. Table 4 displays one of these item pairs. These comparison results provide additional support for the previous comparisons displaying better performances for students participated in TIMSS 2011.

In the current study mathematics performances Turkish eighth grade students participated TIMSS 2007 and TIMSS 2011 were compared from different perspectives such as, mathematics performances at Benchmarks, content and cognitive domains and percentages of students reaching Benchmarks. In addition, released items of both TIMSS 2007 and 2011 were scrutinized and percentages of correct responses of Turkish eighth graders participated in TIMSS 2007 and 2011 were compared. All of the comparison results displayed that Turkish eighth graders participated in TIMSS 2011 displayed better mathematics performance than those of eighth graders participated in TIMSS 2007.

Table 4. One sample item-pair

<table>
<tr><td colspan="2">Content Domain: Data and Chance</td><td colspan="2">Topic Area: Data Interpretation</td></tr>
<tr><td colspan="2">TIMSS 2007</td><td colspan="2">TIMSS 2011</td></tr>
<tr><td colspan="2">
Popularity of Subjects
A group of 10 students wanted to find out whether mathematics or history was more popular for their group. They rated each subject using the following scale.
1 Dislike a lot — 2 Dislike — 3 Neither dislike nor like — 4 Like — 5 Like a lot
The table shows the results:
Students' Ratings
<table>
<tr><th>Student</th><th>Mathematics Rating</th><th>History Rating</th></tr>
<tr><td>Alan</td><td>1</td><td>2</td></tr>
<tr><td>Lisa</td><td>4</td><td>4</td></tr>
<tr><td>Ann</td><td>5</td><td>4</td></tr>
<tr><td>John</td><td>2</td><td>2</td></tr>
<tr><td>Connor</td><td>4</td><td>2</td></tr>
<tr><td>Georgia</td><td>3</td><td>3</td></tr>
<tr><td>Bret</td><td>2</td><td>1</td></tr>
<tr><td>Courtney</td><td>1</td><td>1</td></tr>
<tr><td>Ian</td><td>5</td><td>3</td></tr>
<tr><td>Jackson</td><td>3</td><td>2</td></tr>
<tr><td>Totals</td><td>30</td><td>24</td></tr>
</table>
A. Calculate the mean (average) rating for each subject.
Mean rating for mathematics = ________
Mean rating for history = ________
According to the ratings, which is the more popular subject for this group of students?
More popular subject: ________</td>
<td colspan="2">The Real Burger Company owns 5 restaurants. The number of staff members employed in their 5 restaurants are : $12, 18, 19, 21$ and 30 people

A. What is the mean number of staff members in the 5 restaurants ?

Answer:____</td></tr>
<tr><td>International Average</td><td>Turkish Students Average</td><td>International Average</td><td>Turkish Students Average</td></tr>
<tr><td>36</td><td>31</td><td>43</td><td>48</td></tr>
</table>

4 Discussion and Conclusion

The purpose of the current study was to compare mathematics performances Turkish eighth students participated in TIMSS 2007 and 2011. The comparison results indicated that eighth graders participated in TIMSS 2011 have better mathematics performances than those of students participated in TIMSS 2007. The difference between these performances may be attributed to the new primary mathematics performance. The compared items also support this idea that the consistent differences observed in overall mathematics scores, mathematics achievement scores in content and cognitive domains and percentages of students reaching Benchmarks as well may not be incidental. When the comparisons across cognitive domains were considered, it was observed that the biggest difference is in applying cognitive domain and then difference in the reasoning cognitive domain is in the second order. These findings were supported by the idea that the new primary mathematics curriculum emphasizes the development of higher order thinking skills as well as the application skills of students in a range of contexts [5; 6]. Consistent with this idea, [1] indicated that new mathematics curriculum can improve students' higher order thinking level.

It is believed that the current study is valuable in terms of investigating the effect of new mathematics curriculum with respect to students' mathematics performances based on the results of TIMSS. Differently from previous studies, this study is the first study investigating the students' academic achievement in mathematics. Although investigating the effect of a new curriculum with respect to students' academic achievement requires long running studies it is a good idea to use results of TIMSS for such purposes. Based on the results of the current study, it is concluded that the new mathematics curriculum has displayed a positive change with respect to students' mathematics performance. It is predicted that mathematics performances of Turkish eighth graders will improve gradually as the development processes proceed, problems with regard to implication of the new curriculum are minimized, and above all new mathematics teachers are trained as best implementers of the new mathematics curriculum.

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Pulsatile flow of a dusty fluid between permeable beds in the presence of magnetic field

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Abstract : The problem of pulsatile flow of a dusty fluid between permeable beds is analyzed in the presence of magnetic field for both steady and unsteady cases. The flow between the beds is governed by the Navier Stokes' equations for the momentum, for the fluid phase and the particle phase. The flow through the beds is governed by the Darcy's law. The influence of Hartmann number, permeability parameter, the phase angle and the dust parameters namely mass concentration parameter and time relaxation parameter on the velocity for both fluid and particles are discussed graphically. Also the Skin friction coefficient is discussed through numerical values.

Keywords :

Dusty fluid, Magnetic field, Permeable beds , Pulsatile flow.

Nomenclature :

u, v	-	The velocity component of the fluid in the x and y direction
u_p, v_p	-	The velocity component of the Dust in the x and y direction
V	-	The suction/injection velocity
ρ	-	density
p	-	pressure
$Nm = \rho_p$	-	density of the particle
$K = 6\pi\mu a B_0$	-	The magnetic Induction
R	-	Reynolds number
λ	-	Non dimensional relaxation parameter
h	-	width of the channel
Q_1	-	$k_1/\mu\left(\frac{\partial p}{\partial x}\right)$ Darcy's velocity
Q_2	-	$k_2/\mu\left(\frac{\partial p}{\partial x}\right)$ Darcy's velocity

1 Introduction

It is of great importance for researchers in fluid dynamics to study the influence of inert particles in the motion of fluids. This leads to the problems of mechanics of systems having two phases which has been developing rapidly in recent years. The fluid flow embedded with dust particles is encountered in a wide variety of engineering problems concerned with atmospheric fall out, dust collection, nuclear reactor cooling, powder technology, acoustics, sedimentation, performance of solid fuel, rock nozzles, batch settling, rain erosion, guided missiles, paint spraying etc.

Pulsatile flow is composed of a steady component and a superimposed periodical time varying component called oscillation. Pulsatile flow has wide applications in the field of medicine like respiratory system, circulatory system etc ., and in the field of engineering such as reciprocating pumps, IC engines, pulse combustors etc.

Flows through porous media are very much prevalent in nature and hence their study is of principal interest in many scientific and Engineering applications. To study the under ground water resources and seepage of water in river beds, one needs to investigate the flows of fluids through porous media. Saffman [1] was the pioneer in studying the stability of the laminar flow of a dusty gas with uniform distribution of dust particles. Ramamurthy et.al studied the unsteady flow of a dusty fluid in a channel and a pipe[2] .A.J.Chamkha and J.Peddieson Jr.[3] considered the unsteady flow of a dusty viscous fluid through a channel when the axial pressure gradient is an arbitrary function of time values in magnitude but not in direction. B.J.Gireesha et.al, [4], investigated the laminar flow of an unsteady viscous liquid with uniform distribution of dust particles through a rectangular channel under the influence of pulsatile pressure gradient in Frenet Frame field system. Also he considered the fluid and dust particles to be at rest initially and obtained the analytical expressions for velocities of fluid and dust particles. N.Datta and D.C.Dalal [5] developed Laminar flow and heat transfer of a dusty fluid in an infinite annular pipe with a pulsatile pressure gradient. They [6] also considered the the problem of unsteady heat transfer to pulsatile flow of a dusty fluid in a parallel plate channel . They had shown that the unsteady part of the fluid velocity as well as the particle velocity has a phase lag which increases with increase of volume fraction.

Hazem.A.Attia [7] obtained numerical solution to the unsteady couette flow and heat transfer of an electrically conducting viscous, incompressible dusty fluid with temperature dependent viscosity with the assumption of very small magnetic Reynolds number. D.C.Dalal et.al [8] investigated the problem of free convective heat transfer to a dusty fluid due to differentially heated vertical walls of a rectangular Channel and obtained the solution using a combination of central and second difference scheming.

Jagjit paul Kaur et. al [9] studied the flow of an incompressible viscous electrically conducting dusty fluid in the presence of magnetic field of a uniform intensity in a channel whose cross section is an porous regular hexagonal duct with impermeable boundary under time varying axial pressure gradient. Ali.J.chamkha[10] obtained closed form transient solutions for hydromagnetic two-phase particulate suspension flow in channels and circular pipes and numerical solutions for the thermal problem. Nanigopal Datta and Saroj Kumar Mishra [12], considered the flow valid for any time by employing numerical inverse Laplace Transform. A.K. Ghosh et.al constructed solution of the problem of heat transfer associated with the pulsatile flow of a two-phase fluid particle system in a channel bounded by two infinitely long parallel walls.

At present we know relatively very little about the pulsatile flow of dusty fluids between permeable beds. Because of its intrinsic importance in many industrial problems and its relevance to the general natural phenomena , in this paper we have chosen to study the flow of dusty fluids between permeable beds in the presence of magnetic field.

2 Mathematical Formulation

Consider the pulsatile flow of a viscous incompressible dusty fluid between two permeable beds under the influence of uniform transverse magnetic field. The fluid is injected into the channel from the lower permeable bed with a velocity V and is sucked out into the upper permeable bed with the same velocity. The permeabilities of lower and upper beds are respectively k_1 and k_2 .The flow in the upper and lower beds are governed by the Darcy's law. The flow region between the permeable beds is governed by the Navier Stoke's Equations. The x-axis is taken along the interface and the y-axis perpendicular to it.

The fluid is assumed to be driven by an unsteady pressure gradient

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = A + Be^{i\omega t} \tag{2.1}$$

where A and B are constants and ω is the frequency. The equation governing the flow are given

Fluid Phase

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.2}$$

$$\frac{\partial u}{\partial t}+V\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2 u}{\rho}+\frac{KN}{\rho}\left(u_p-u\right) \tag{2.3}$$

$$\frac{\partial p}{\partial y}=0 \tag{2.4}$$

Particle phase

$$\frac{\partial u_p}{\partial x}+\frac{\partial v_p}{\partial y}=0 \tag{2.5}$$

$$\frac{\partial u_p}{\partial t}+V\frac{\partial u_p}{\partial y}=-\frac{K}{m}\left(u_p-u\right) \tag{2.6}$$

Boundary Conditions

$$y=0: \qquad u=u_{B1} \qquad \frac{du}{dy}=\frac{\alpha}{\sqrt{k_1}}\left(u_{B1}-Q_1\right) \tag{2.7}$$

$$y=h: \qquad u=u_{B2} \qquad \frac{du}{dy}=-\frac{\alpha}{\sqrt{k_2}}\left(u_{B2}-Q_2\right) \tag{2.8}$$

$$u_p\left(0\right)=\xi_s\frac{du_p}{dy}\left(0\right) \tag{2.9}$$

The boundary conditions (7) and (8) are due to Darcy's Law on the boundary of the channel. In the case of flow past a porous medium Beavers and Joseph [13] have shown that the usual no slip condition is no longer valid and they have postulated that slip exists at the porous boundary, because of the transfer of momentum. Equation (9) is taken with reference to [14], as the motion f particle phase resemble that of a rarefied gas and it represents the condition that particles slip at the boundary. ξ_s represents the particle phase dimensional slip coefficient.

Separating Equations (2)-(9) into steady part denoted by (7) and unsteady part denoted by ( ), we obtain ,

$$\frac{\partial \bar{u}}{\partial x}=0 \tag{2.10}$$

$$V\frac{\partial \bar{u}}{\partial y}=-A+\upsilon\frac{\partial^2 \bar{u}}{\partial y^2}-\frac{\sigma B_0^2 \bar{u}}{\rho}+\frac{KN}{\rho}\left(\bar{u}_p-\bar{u}\right) \tag{2.11}$$

$$V\frac{\partial \bar{u}_p}{\partial y}=-\frac{K}{m}\left(\bar{u}_p-\bar{u}\right) \tag{2.12}$$

and the corresponding boundary conditions are

$$y=0: \qquad \bar{u}=\bar{u}_{B1} \qquad \frac{d\bar{u}}{dy}=\frac{\alpha}{\sqrt{k_1}}\left(\bar{u}_{B1}-\bar{Q}_1\right) \tag{2.13}$$

$$y=h: \qquad \bar{u}=\bar{u}_{B2} \qquad \frac{d\bar{u}}{dy}=-\frac{\alpha}{\sqrt{k_2}}\left(\bar{u}_{B2}-\bar{Q}_2\right) \tag{2.14}$$

$$\bar{u}_p\left(0\right)=\xi\frac{d\bar{u}_p}{dy}\left(0\right) \tag{2.15}$$

Unsteady Part

$$\frac{\partial \tilde{u}}{\partial x} = 0 \tag{2.16}$$

$$V\frac{d\tilde{u}}{dy} = -Be^{iw^{*}t} + \upsilon\frac{d^2\tilde{u}}{dy^2} - \frac{\sigma B_0^2\tilde{u}}{\rho} + \frac{KN}{\rho}\left(\tilde{u}_p - \tilde{u}\right) \tag{2.17}$$

$$\frac{d\tilde{u}_p}{dt} + V\frac{d\tilde{u}_p}{dy} = -\frac{K}{m}\left(\tilde{u}_p - \tilde{u}\right) \tag{2.18}$$

and the corresponding boundary conditions are

$$y = 0: \qquad \tilde{u} = \tilde{u}_{B1} \qquad \frac{d\tilde{u}}{dy} = \frac{\alpha}{\sqrt{k_1}}\left(\tilde{u}_{B1} - \tilde{Q}_1\right) \tag{2.19}$$

$$y = h: \qquad \tilde{u} = \tilde{u}_{B2} \qquad \frac{d\tilde{u}}{dy} = -\frac{\alpha}{\sqrt{k_2}}\left(\tilde{u}_{B2} - \tilde{Q}_2\right) \tag{2.20}$$

$$\tilde{u}_p(0) = \xi\frac{d\tilde{u}_p}{dy}(0) \tag{2.21}$$

where ϵ is the non dimensional wall slip coefficient. The coupled ordinary differential equations are solved by usual method of finding complementary functions and particular integral. The solutions are not given due to lack of space.

Non-dimensional quantities for steady part

$y^* = \frac{y}{h}$ $\quad \bar{u}^* = \frac{\bar{u}}{\frac{A_1 h}{V}}$ $\quad \bar{u}_p^* = \frac{\bar{u}_p}{\frac{A_1 h}{V}}$ $\quad \bar{u}_{B1}^* = \frac{\bar{u}_{B1}}{\frac{A_1 h}{V}}$ $\quad \bar{u}_{B2}^* = \frac{\bar{u}_{B2}}{\frac{A_1 h}{V}}$

$Q_1^* = \frac{Q_1}{\frac{A_1 h}{V}}$ $\quad Q_2^* = \frac{Q_2}{\frac{A_1 h}{V}}$ $\quad x^* = \frac{x}{h}$ $\quad A_1 = -A$ $\quad \xi = \frac{\xi_s}{h}$

$M = \frac{\sigma B_0^0 h^2}{\rho\upsilon}$ $\quad f = \frac{Nm}{\rho}$ $\quad \tau = \frac{m}{K}$ $\quad \tau' = \tau\frac{h}{V}$

$$R\frac{d\bar{u}}{dy} = R + \frac{d^2\bar{u}}{dy^2} - M^2\bar{u} + \frac{f}{\tau'}\left(\bar{u}_p - \bar{u}\right) \tag{2.22}$$

$$\frac{d\bar{u}_p}{dy} = -\frac{1}{\tau'}\left(\bar{u}_p - \bar{u}\right) \tag{2.23}$$

and the corresponding boundary conditions are

$$y = 0: \qquad \bar{u} = \bar{u}_{B1} \qquad \frac{d\bar{u}}{dy} = \alpha\sigma_1\left(\bar{u}_{B1} - \frac{R}{\sigma_1^2}\right) \tag{2.24}$$

$$y = 1: \qquad \bar{u} = \bar{u}_{B2} \qquad \frac{d\bar{u}}{dy} = -\alpha\sigma_2\left(\bar{u}_{B2} - \frac{R}{\sigma_2^2}\right) \tag{2.25}$$

$$\bar{u}_p(0) = \xi\frac{d\bar{u}_p}{dy}(0) \tag{2.26}$$

The coupled ordinary differential equations are solved by usual method of finding complementary functions and particular integral. The solutions are not given due to lack of space.

Non-dimensional quantities for unsteady part

$y^* = \frac{y}{h}$ $\quad \tilde{u}^* = \frac{\tilde{u}}{\frac{A_1 h}{V}}$ $\quad \tilde{u}_p^* = \frac{\tilde{u}}{\frac{A_1 h}{V}}$ $\quad \tilde{u}_{B1}^* = \frac{\tilde{u}_{B1}}{\frac{A_1 h}{V}}$ $\quad \tilde{u}_{B2}^* = \frac{\tilde{u}_{B2}}{\frac{A_1 h}{V}}$

$\tilde{Q}_1^* = \frac{\tilde{Q}_1}{\frac{A_1 h}{V}}$ $\quad \tilde{Q}_2^* = \frac{\tilde{Q}_2}{\frac{A_1 h}{V}}$ $\quad x^* = \frac{x}{h}$ $\quad B_1 = -B$ $\quad \xi = \frac{\xi_s}{h}$

$M = \frac{\sigma B_0^2 h^2}{\rho\upsilon}$ $\quad f = \frac{Nm}{\rho}$ $\quad \tau = \frac{m}{K}$ $\quad \tau' = \tau\frac{h}{V}$

The flow equations in steady state without * will become

$$R\frac{d\tilde{u}}{dy} = R + \frac{d^2\tilde{u}}{dy^2} - M^2\tilde{u} + \frac{f}{\tau'}\left(\tilde{u}_p - \tilde{u}\right) \tag{2.27}$$

$$\frac{d\tilde{u}_p}{dy} = -\frac{1}{\tau'}\left(\tilde{u}_p - \tilde{u}\right) \tag{2.28}$$

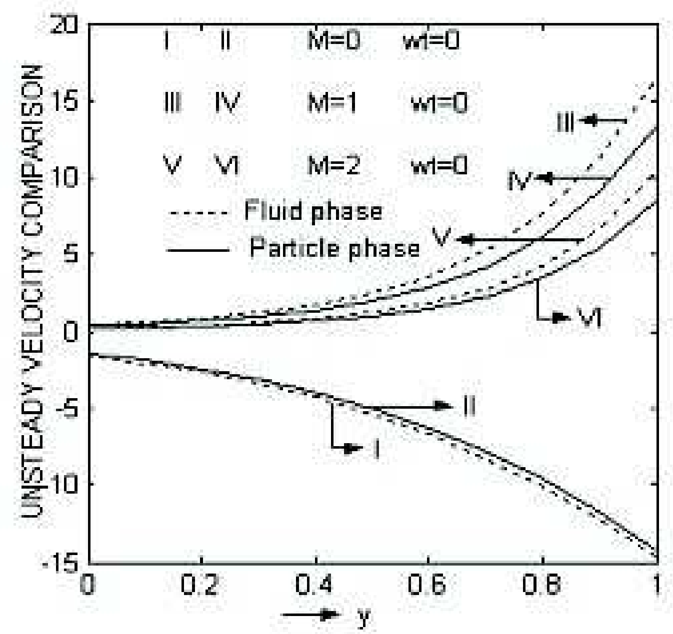
and the corresponding boundary conditions are

$$y = 0: \qquad \tilde{u} = \tilde{u}_{B1} \qquad \frac{d\tilde{u}}{dy} = \alpha\sigma_1\left(\tilde{u}_{B1} - \frac{R}{\sigma_1^2}\right) \tag{2.29}$$

$$y = 1: \qquad \tilde{u} = \tilde{u}_{B2} \qquad \frac{d\tilde{u}}{dy} = -\alpha\sigma_2\left(\tilde{u}_{B2} - \frac{R}{\sigma_2^2}\right) \tag{2.30}$$

$$\tilde{u}_p\left(0\right) = \xi\frac{d\tilde{u}_p}{dy}\left(0\right) \tag{2.31}$$

The coupled ordinary differential equations are solved by usual method of finding complementary functions and particular integral. The solutions are not given due to lack of space.



Fgure 1. COMPARISON OF FLUID AND PARTICLE VELOCITY FOR DIFFERENT VALUES OF M $R = 2; f = 0.2; \tau = 0.025; \sigma_1 = \sigma_2 = 5;$

Results and Discussion

Numerical calculations have been carried out for different values of the parameter entering into the problem and the results are depicted graphically. Due to the presence of complex parameter ω the flow quantities will appear in complex form.

Hence for the discussion the graphs are drawn only for the real part. In figure 1,the fluid and particle velocities are compared for different values of M. In the absence of magnetic field, i.e when M=0, both fluid and particle velocities decrease and particle velocity is slightly less than that of fluid. But as the Hartmann Number increases this trend changes. Both phase velocities increase and also the velocity of particle is greater than that of the fluid. In figure 2 the velocity profile of both phases have been compared for the effect of Reynolds number. It is very obvious that increase in Reynolds number increases the fluid and particle velocity. The flow of fluid is parallel to that of dust.

figure 3 shows that the unsteady velocity of fluid increases with increase in M, for the phases $\omega t = 0\, and\, \pi/4$. But it decreases with increase in M for the phases $\omega t = 3\pi/4\, and\, \pi/2$ which is shown in figure 4, figure 5 and figure 6 show the profile of unsteady velocity for different values of R.

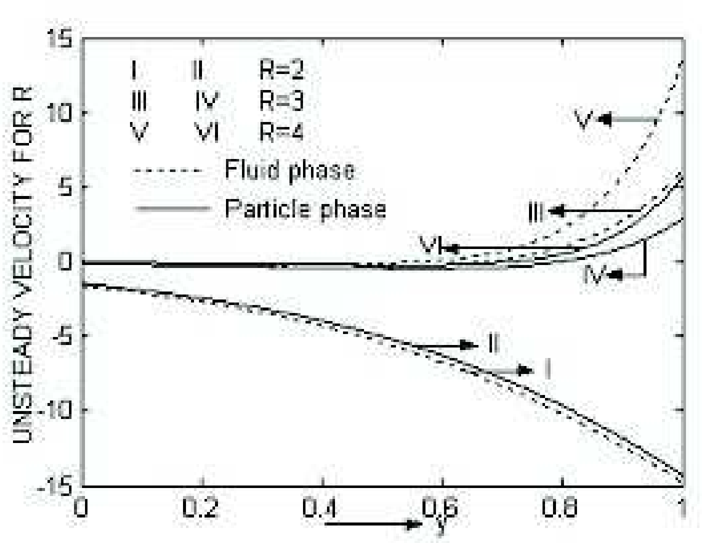


Figure. 2 COMPARISON OF FLUID AND PARTICLE VELOCITY FOR DIFFERENT VALUES OF R $M = 0; f = 0.2; \tau = 0.025; \sigma_1 = \sigma_2 = 5; \omega t = 0$

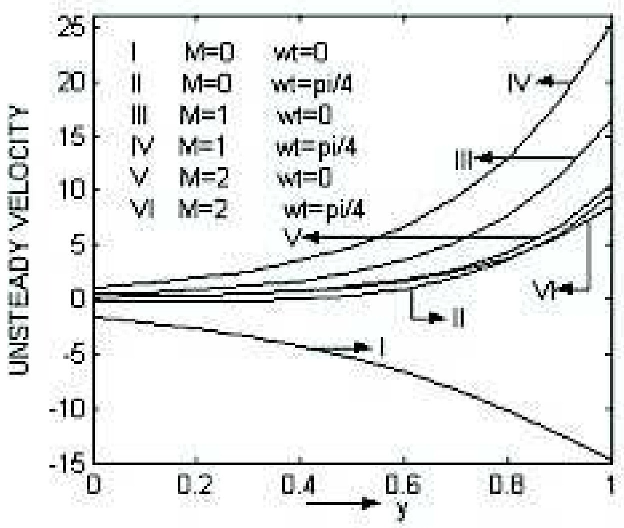


Figure. 3 UNSTEADY VELOCITY OF FLUID FOR DIFFERENT VALUES OF M $R = 2; f = 0.2; \tau = 0.025; \sigma_1 = \sigma_2 = 5$

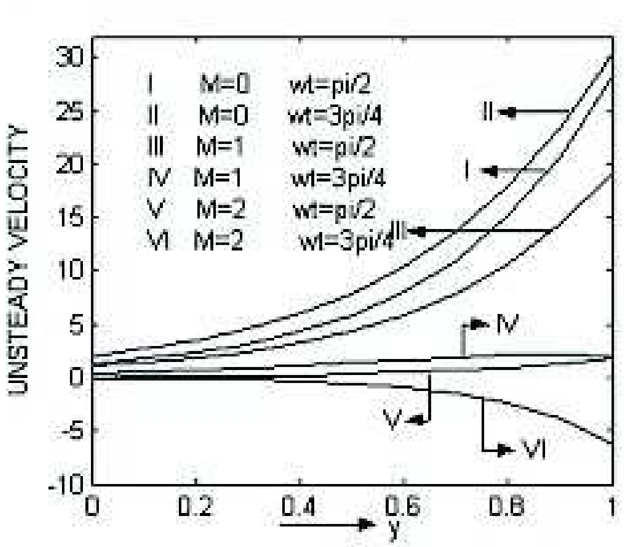


Figure. 4 UNSTEADY VELOCITY OF FLUID FOR DIFFERENT VALUES OF M $R = 2; f = 0.2; \tau = 0.025; \sigma_1 = \sigma_2 = 5;$

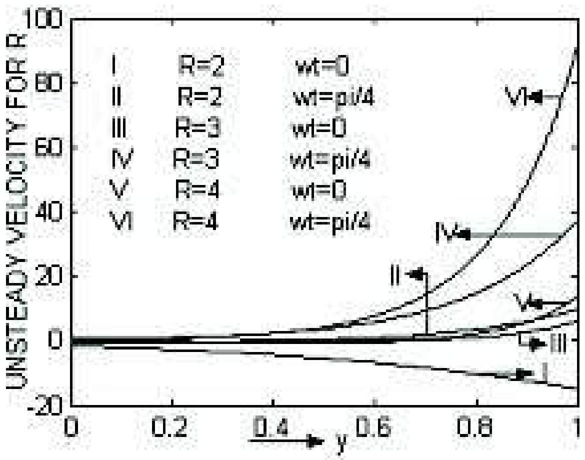


Figure. 5 UNSTEADY VELOCITY OF FLUID FOR DIFFERENT VALUES OF R $M = 0; f = 0.2; \tau = 0.025; \sigma_1 = \sigma_2 = 5; \omega t = 0$

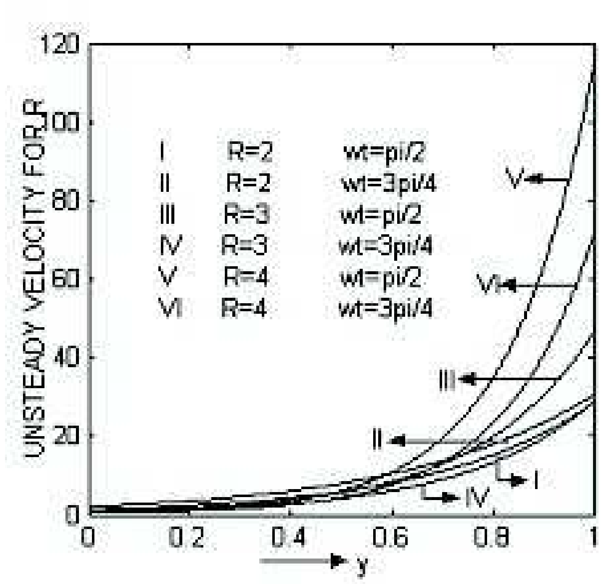


Figure. 6 UNSTEADY VELOCITY OF FLUID FOR DIFFERENT VALUES OF R $M = 0; f = 0.2; \tau = 0.025; \sigma_1 = \sigma_2 = 5; \omega t = 0$

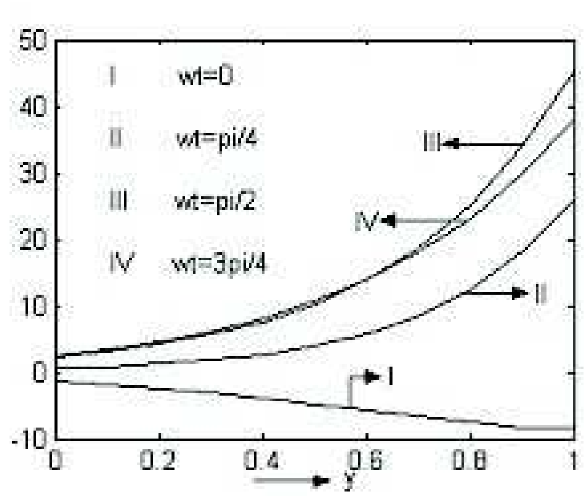


Figure. 7 UNSTEADY VELOCITY OF FLUID FOR DIFFERENT VALUES OF $\sigma M = 0; f = 0.2; \tau = 0.025; \sigma_1 = 5; \sigma_2 = 6; \omega t = 0; R = 6$

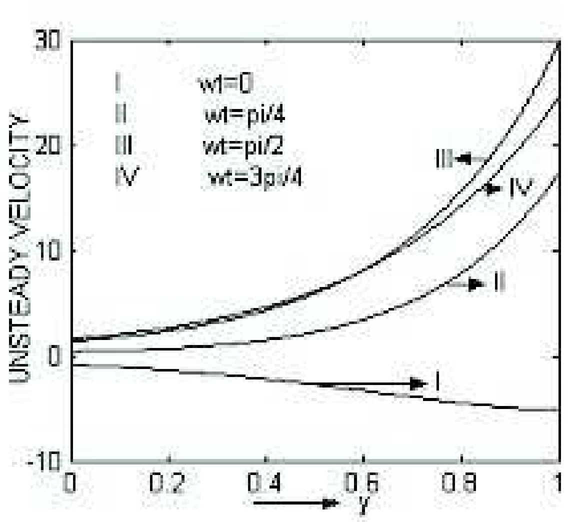


Figure. 8 UNSTEADY VELOCITY OF FLUID FOR DIFFERENT VALUES OF f $M = 0; R = 2; \tau = 0.025; \sigma_1 = 5; \sigma_2 = 5$

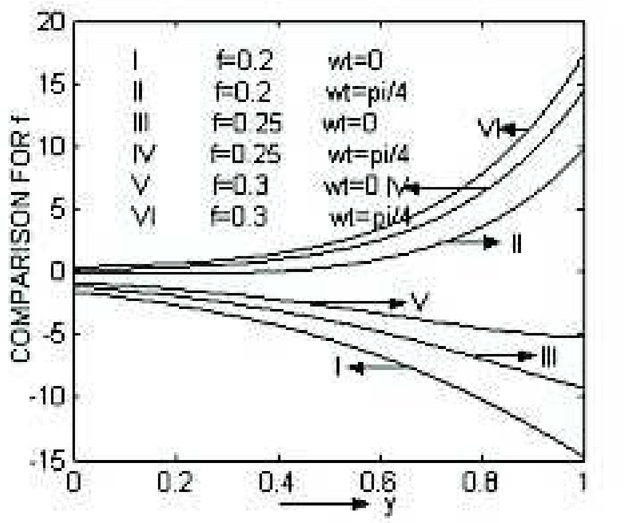


Figure. 9 COMPARISON OF UNSTEADY VELOCITY OF FLUID FOR DIFFERENT VALUES OFf $M = 0; R = 2; \tau = 0.025; \sigma_1 = 5; \sigma_2 = 5$

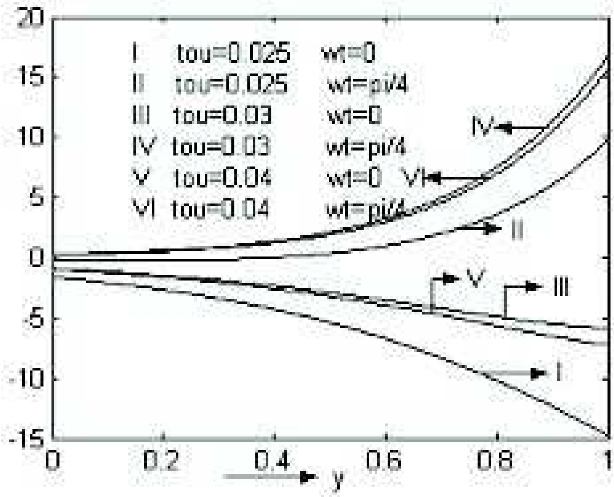


Figure. 10 COMPARISON OF UNSTEADY VELOCITY OF FLUID FOR DIFFERENT VALUES OF τ $M = 0; R = 2; f = 0.2; \sigma_1 = 5; \sigma_2 = 5$

Figure 7 shows the unsteady velocity for different values of σ. The velocity increases when the phase increases from 0 to $\pi/2$. However, when the phase angle increases from $\pi/2$ *to* $3\pi/4$, the velocity increases upto y=0.6, and then cross flow occurs and it starts increasing.

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Understanding the Place Value Concept through the Incorporation of Ideas from Indian History of Mathematics

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Abstract : The decimal place value system with zero that is currently in use originated in India, and it is foundational to progress in mathematics. However, many students, including high school students, experience considerable difficulties with this vital concept. In my PhD thesis, I adopted a multi-dimensional approach to suggesting possible ways to improve junior secondary school students' conceptual understanding of the structure of place value numeration by integrating ideas from Indian history of mathematics and mathematics education research. The first part of the study analysed the history of mathematics, in particular Indian history, for ideas relevant to the place value construct and these ideas were then incorporated into a teaching/learning framework. The second part investigated the effectiveness of the framework developed, on junior secondary school students' understanding of a general place value system. Technology, including CAS technology, and manipulative materials were used in teaching in order to aid conceptual understanding. This paper highlights some of my historical findings and discusses the components of the frameworks or teaching sequences developed out of the historical and psychological research.

1 Introduction

Central to understanding numbers and for developing fluency with numbers is the concept of place value in the Hindu-Arabic decimal system and its application in standard algorithms. The structure of positional notation and arithmetic are the foundations on which advanced mathematical topics are built. However, despite its importance, many researchers (e.g. Fuson, 1990b) have reported students' difficulties in making sense of multidigit numbers, particularly its multiplicative structure. As pointed out by Skemp (1971), this may be due to the fact while simple on the surface, the place value system conceals a complexity of ideas that is too sophisticated for young children to understand. In addition to this, the underlying structure of the numeration system is polynomial and hence implies a need for some knowledge of *exponentiation.* Moreover, as stated by Becker and Varelas (1993), the place value system is a *semiotic/sign system* and understanding it involves the integration of both its *conceptual* and *semiotic* features. That is, that links have to be made between the *written* number symbols and the recursive grouping structure that is implicit within the system. In this context, as proposed by Vygotsky (1962), there is a deeper grasp of the place value idea when it is understood as a specific instance of a general positional notation. Furthermore, generalization of multiple bases to the idea of a general base could provide for a smoother passage to algebra. This suggests that the place value concept needs to be *revisited* when students come across *powers* in upper primary or junior secondary school and the connections between groupings, powers and place value be made clear. However, very few research studies have reported on *secondary* school students' understanding of the above relationships.

In response to students' place value difficulties, a number of research studies (e.g. Association of Independent Schools of South Australia, 2004) have described teaching approaches, however despite the remediation efforts, students' misconceptions continue. Due to students' troubles in mathematics, in recent years, many researchers and educators (Fauvel & van Maanen, 2000; Gupta, 1995b; Katz, 2007) have turned to the history of mathematics in an attempt to understand student difficulties and to inform practice. History may be used either directly or indirectly in the classroom. In the direct approach, teaching and learning is supplemented by historical information, whereas in the indirect approach, the teacher, after undertaking a historical analysis, designs a sequence keeping in mind the historical evolution of the subject and the obstacles encountered in history, so that topics become more accessible to students (Tzanakis & Arcavi, 2000).

While a review of history reveals that every civilization had its own numeration system, the present day decimal system with ten distinct and arbitrary numerals, the place value principle and the use of zero, originated in India and was then transmitted to Europe by the Arab mathematicians (e.g. Datta & Singh, 2001; Ifrah, 1985; Joseph, 2000, 2009, 2011). Hence it is reasonable to suppose that the historical development of this system may hold valuable lessons for today (Katz, 2007) and thus this research focused mainly on the development of the Hindu-Arabic decimal system. Adopting a historical-critical methodology (Piaget & Garcia, 1989), historical ideas were analysed in light of mathematics education research and in turn, integration of ideas of the two domains aided the development of a teaching sequence for improving students' understanding of the structure of place value. The theoretical framework used in this thesis is given in Figure 1.

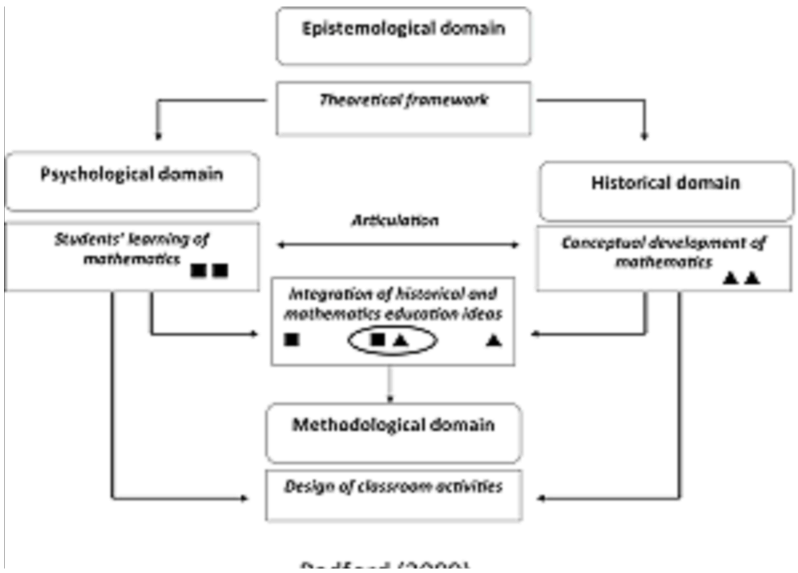


Figure 1. Theoretical framework adapted from Radford (2000) on the use of history in mathematics education.

2 Mathematics Education Research Background

A rich understanding of the base ten system requires an appreciation of its multiplicative structure. This is possible only if teaching strategies include students' experiences in multiplication and exponentiation and their related notation. In this context, Dienes and Golding (1971) state that the ideas of exponents enable one to see the multiplicative structure more clearly. That is, the grouping structure of place values is reflected in the exponential form of number. Related to this, Confrey and Smith (1995), emphasise the importance of repeated multiplication as a beginning strategy for understanding exponentials. In addition to this, exponentiation is crucial to the understanding of algebra and its symbol system. Hence, as suggested by Dienes, in order to learn about powers, students need to have exposure to grouping experiences in different bases formed with different exponents. Moreover, psychologists (e.g. Skemp, 1971; Vygotsky, 1962) believe that teaching multiple bases increases students' awareness of the concept of positional notation. This would imply the provision of opportunities for students to experience multiple representations (M. O. J. Thomas, 2004) of the place value system. These representations could include concrete materials such as blocks and bundles of sticks, pictorial and symbolic representations. More importantly, according to M. O. J. Thomas, clear connections have to be made between representations and across representational systems in order to abstract the concept.

In my PhD thesis (Nataraj, 2012), algebra was employed as a supporting idea to help students understand place value structure. In their first year of high school students in New Zealand are introduced to formal algebra. However, in order to teach multiple bases (using powers) and the idea of a general positional notation, students need some facility with algebraic notation. A review of the documented literature indicates that central to algebraic thinking and underpinning all conceptions in algebra is the concept of variable (Schoenfeld & Arcavi, 1988), and Mason (1996) has stressed the importance of generalization in every mathematics lesson. However, documented research (e.g. Kchemann, 1981, MacGregor & Stacey, 1997) indicates students' difficulties and misconceptions of the variable concept. It appears that letters as specific unknowns in equation solving is more accessible than letters as generalized number and variable. In this context, Srinivasan (1989) states that the use of patterns is worthwhile if it supports students to construct the idea of the variable. Srinivasan developed a terminology for pattern recognition and recommends the use of pre-symbolic language which he calls *pattern language* for number patterns. Srinivasan suggests vocabulary such as *changing, not changing (or repeating), changing in the same way* and *changing in different ways;* all words easily understood by young learners. With this background of mathematics education research, this researcher looked at the history of mathematics, in particular Indian history, to trace the development of the current Hindu-Arabic system and algebraic symbolism. History was reviewed for ideas that would be useful in constructing a didactic framework for a deep understanding of a general place value system.

3 Findings from Indian history related to Hindu-Arabic number system

3.1 Naming of Large Numbers in Indian History

The evidence of the use of mathematics in India appears first in the Indus valley, which dates back to around 3000 BCE (Datta & Singh, 2001; Joseph, 2000). Excavations of the Indus valley reveal an urban civilization and scales and other instruments show accurate gradations and significantly, a *decimal* system was already established then as indicated by an analysis of weights and measures. The period after this is called the 'Vedic period'. The *Vedas* is the sacred literature of the Hindus; it is an enormous body of integrated knowledge with teachings on all aspects of life and these monumental works are possibly the oldest literary documents of mankind (Joseph, 2011). The Vedic literature, in Sanskrit verse is thought to have been composed around 2000 BCE and possibly documented around the start of the first millennium BCE. These Sanskrit texts comprising the Vedic literature were transmitted orally from generation to generation, without being written down. It is in these early Vedic works that one finds *number names* for *powers of ten* in Sanskrit. What is seen in the historical development of the decimal place value system over many centuries is that; i) very large numbers were named and infinity was considered; ii) nomenclature was developed for a system of numeration and ten numerals including zero evolved in a place value system. Some examples of large numbers from Indian history are given below:

(a) A major milestone in the development of the Hindu-Arabic system is a set of number names for powers of ten. In the *Vajasaneyi* (*Sukla Yajurveda*) *Samhita* (17.2) (c. 2000 BC) of the Vedas, the following list of *arbitrary* number names is given in Sanskrit verse: *Eka* (1), *Dasa* (10), *Sata* (10^2), *Sahasra* (10^3), *Ayuta* (10^4), *Niyuta* (10^5), *Prayuta* (10^6), *Arbuda* (10^7), *Nyarbuda* (10^8), *Samudra* (10^9), *Madhya* (10^{10}), *Anta* (10^{11}), *Parardha* (10^{12}) (e.g. Datta & Singh, 2001). They were aptly called the *dasagunottara samjna* (decuple terms), confirming a definite mode of arrangement in naming numbers. The same list was then extended to *loka* (10^{19}) (Gupta, 1987).

(b) The same list up to *parardha* (one trillion) as in the *Vajanaseyi Samhita* is repeated in the *Pancavimsa Brahmana* (17.14.1.2) with further extensions. The following passage from it will give an idea of the context in which large numbers were introduced:

By offering the agnistoma sacrifices, he becomes equal to one who performs a sacrifice of a thousand cows as sacrificial fee. By offering ten of these, he becomes equal to one who performs a sacrifice with ten thousand daksinas (fee). By offering ten of these, he becomes equal to one who sacrifices with a sacrifice of a hundred thousand daksinas... By offering ten of these, he becomes equal to one who sacrifices with a sacrifice of 100 000 million daksinas. By offering ten of these he becomes the cow [one trillion]. (Sen, 1971, p. 141)

(c) In the Buddhist work *Lalitavistara* (c. 100 B.C.E), there are examples of series of number names based on the centesimal scale: Hundred *kotis* are called *ayuta* (10^9), hundred *ayutas* is *niyuta* (10^{11}) and so on to *tallaksana* (10^{53}) giving 23 names. Then follow 8 more series leading to $10^{53+8\times46} = 10^{421}$! (Menninger, 1969).

(d) In the *Paitamaha-siddhanta*, an astronomical treatise, this number is given: "The orbit of heaven is 18 712 069 200 000 000 [*yojanas*]...The diameter of the orbit equals the circumference divided by the square root of ten" (Plofker, 2009, p. 69).

(e) In the Vedic literature, time is reckoned in terms of *yugas* or time cycles. The four *yugas* are *Satya-yuga*, *Treta yuga*, *Dwapara yuga* and *Kali yuga*. According to Hindu cosmology, the time-span of these four yugas is said to be 1728000, 1296000, 864000, and 432000 years, respectively, in the ratio 4:3:2:1. The total is one *Mahayuga* and was thus 4320000 years (Srinivasiengar, 196)

(f) Moreover, it is believed that 1000 such *yuga-cycles* comprise one day in the life of *Brahma,* which is 4,320,000,000 years and one day and night period is 8.64 billion years which was further extended to 311×10^{12}. As pointed out by Plofker (2009) time in the astronomical works is bound by cosmological concepts. In one *kalpa* (4 320 000 000 years), all celestial objects are considered to complete *integer* number of revolutions about the earth.

(g) The Jaina mathematicians' interest in large numbers sometimes resulted in problems that had a potential for calculation, such as: "a *raju* is the distance travelled by a god in six months if he covers a hundred thousand *yojana* (approximately a million kilometres) in each blink of his eye" and "a *palya* is the time it will take to empty a cubic vessel of side one *yojana* filled with the wool of newborn lambs if one strand is removed every century (Joseph, 2011, p. 350).

From the examples given above it is clear that large numbers were not only known in early Indian mathematical contexts but formed an integral part of the knowledge and practice of the times. Still, such numbers were mostly employed in a practical/realistic context such as astronomy (Plofker, 2009) and time measures (for calendar purposes). This suggests that students, many of whom are also fascinated by large numbers may also be more motivated if a meaningful, realistic context is used. In today's world of space travel and research, computer technology and huge government budgets, large numbers are relatively common. Hence there is a need to understand large numbers and students could benefit from *naming,* *reading, writing* and *computing* with them in different contexts. Understanding the *naming convention* is a prerequisite for a grasp of large numbers and their structure. It also increases students' *quantity sense,* which Wagner & Davis (2010) define as a *feel* for amounts and magnitudes. The second historical example from *Pancavimsa Brahmana* indicates repeated multiplication as a strategy, both to understand concept of powers and to develop quantity sense. In this context, a teaching strategy would be to assist students to produce examples of large numbers with a *potential* for calculation which sometimes occurred in Indian history, such as the examples (no: 6) above given in Jaina mathematics. Such problems would involve many skills such as conversion, estimation, calculations and rounding, and working across topics such as measurement. Also, consideration of large numbers and the use of calculator technology (including CAS) may lead to scientific form, notation of powers and related groupings. Thus explicitly focusing on the meaning and notation of powers (see Figure 6), both in base ten and other bases via multiple representations (M. O. J. Thomas, 2004) may help establish a *conceptual base* for a more thorough grasp of the composition of the place value system and also later on for polynomials in algebra.

3.2 Development of Written Numerals, Place Value and Zero in Indian History:

In India, the consistent naming of large numbers in Sanskrit and in the *oral* tradition eventually led to the development of decimal place value system with zero. Consequent to the naming of large numbers, the historical development of the *written* decimal place value numeration system in India suggests the following broad classification of the *four* stages that occurred:

1. Verbal (Initial/Additive) stage in number notation

During this stage, numbers (both small and large) were written down in words without the principle of place value, in the same way as they were spoken, for example:

(*i*). *trini satani trisahasrani trimsa ca nava* (= three thousands + three hundreds + three tens + nine, i.e., 3339) (Datta & Singh, 2001, vol. 1, p. 15).

(*ii*) *ekonna vimsati* (= one less than twenty, 20 ?1, i.e. 19) (Sen, 1971, p. 142).

2. Interim (Multiplicative) stage in number notation

In the early stages of numerical symbolism, according to Datta and Singh (2001), while numbers were written out in full in words, symbols were also used for the smaller units and words for the larger units. In this intermediate stage, before the establishment of the place value principle, numbers were written in symbols with the application of the additive, multiplicative, and a combination of the additive, multiplicative principles. The following are some examples.

a. The examples in Figure 2 are numbers in Kharoshti numerals inscribed in the stone pillars of Asoka (c. 300 BCE). In Kharoshti, script, the numerals are written from right to left.

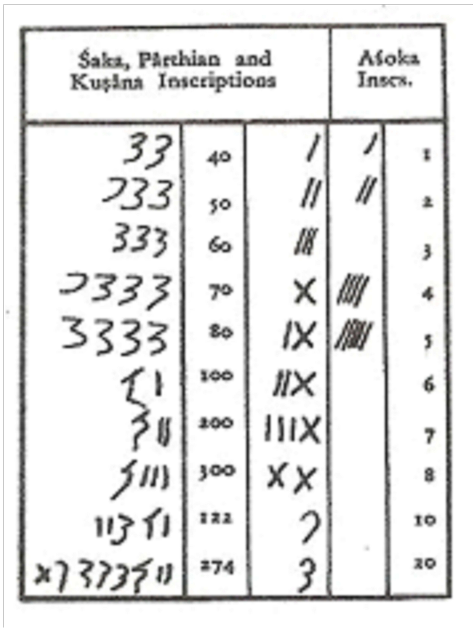


Figure 2. Kharoshti Numerals (Datta & Singh, 2001, vol. 1, p. 105)

Figure 3. Brahmi numerals. (Datta & Singh, 2001, vol. 1, p.26)

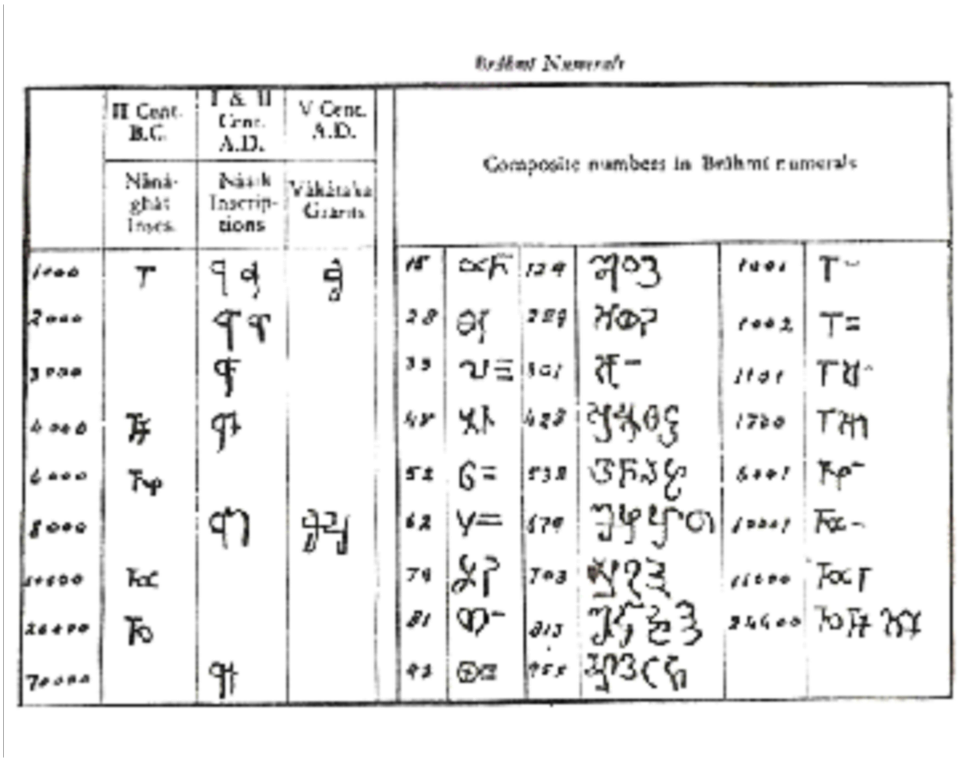
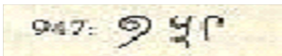


Figure 4. Composite numbers in Brahmi numerals. (Datta & Singh, 2001, vol. 1, p. 117).

b. As can be seen in the Kharoshti numerals, the additive principle alone is applied for 40,50,60,70, 80. Reading 50 from the right we have 20+20+10, and for 60 we have $20 + 20 + 20$.
c. Both multiplicative and additive principles have been applied for the numbers 100, 200, 300, 122 and 274 in Kharoshti numerals. Reading the numeral 274 from the right we have $2 \times 100 + 20 + 20 + 20 + 10 + 4$.
d. Figures 3 and 4 show numbers in Brahmi numerals found in a cave in the Nanaghat hills near Poona in India. It is to be noted that the Brahmi numerals are written from the left to right. The multiplicative principle is applied here. For example, for 6000 the symbols for 6 and 1000 are conjoined. Therefore $6000 = 1000 \times 6$ and not $1000 + 1000 + 1000 \cdots$ six times.
e. Another example of the multiplicative system in writing numbers is the composite number 24400 written in Brahmi numerals (Figure 4). The number given is both multiplicative and *cipherised* similar to the early Chinese and Greek systems. Reading the numeral 24400 from left to right we have, 20000+4000+400 where the numerals for 20000, 4000 and 400 are *cipherised*. Hence $24400 = 2 \times 10000 + 4 \times 1000 + 4 \times 100 = 20000 + 4000 + 400$.
3. Places stage (powers of ten assigned 'places') in number notation

Eventually, these powers of ten were assigned 'places' or *sthanas* and as a consequence, the powers of ten or place values were concealed in the numerical expressions, resulting only in the digits forming the coefficients of powers. This suppression of powers was a natural step given the Indian tendency for conciseness, and perhaps also due to the dearth of writing materials. For example this number is given in Joseph, 2000, p. 241. 

Concept of zero:

In India, *sunya* or zero as a concept(s) probably pre-dated zero as a number by hundreds of years (Joseph, 2002). As indicated by Ifrah (1998), the word *sunya* was not invented specifically for the place value system and the word and its synonyms (with different nuances) possibly existed well before the development of the positional system. It has a very long history and had varied meanings in the different dimensions of philosophy, language, mathematics and science. Zero in the philosophical, linguistic and social contexts appears to have paved the way for the development of the corresponding mathematical concept (Bag & Sarma, 2003).
4. Final abstract stage with full place value and zero

Examples of the final symbolic stage in number notation from the Gwalior inscription (876 CE) are the numbers 50 and 270 

given in Joseph, 2000 p. 241.

What is seen above is the gradual evolution of the *written* numeration system in a broad sense; from the additive form of the symbols to the *multiplicative one* which enabled a *separation* of the face values and the powers of ten (ranks or multi-units of powers of ten) from the total values characterised by powers of ten called as 'places' or 'positions'. Once these powers of ten (which were systematic) were known as 'places' it was not necessary to write or symbolise them, thus leading to an abstract place value structure. Such a structure requires a zero to denote absent powers of ten and since the concept existed from early times in Indian history, its inclusion was a natural progression to a full place value system with zero. A pictorial representation, (bundles of sticks) of the 4 stages of development of written numerals is shown in Fig 8. These representations paralleling the stages of written numerals were used in teaching.

The above set of stages (or representations) offers a route to understanding the place value concept. In the context of base-10 representations, most textbooks use base-10 Dienes' blocks to represent a multidigit number (corresponding to the *first additive* stage above) in order to help students form links between the concrete blocks and the symbolic form of the number such as 2475. 32. Therefore, what is demonstrated in textbooks is the first stage in the form of blocks or its pictorial representation and the final abstract symbol form of the number. However, students' failure to form connections have been reported. Hart (1989) has stated that the gap between the concrete and abstract forms is too big and needs to be bridged by the use of a third transitional form. Hence what is now gained from a study of Indian history is the idea *of two intermediate conceptual stages* reflecting the multiplicative and 'places' stages. These bridging stages can be useful for a deep understanding of the base ten system.

As mentioned earlier, psychologists (e.g. Vygotsky, 1962) have stated that teaching multiple bases and generalizing the place values (powers) could increase students' awareness of the concept of positional notation. Hence, in this research, algebra (in terms of generalization of multiple bases) was an auxiliary idea in understanding the place value concept. The generalization aspect, as well as the idea of general number/variable in the history of Indian algebra was reviewed in order to search ideas for a better understanding of the variable, and which may help students to generalize the place values in different bases.

c) Generalisation and the Variable concept in the History of Indian Algebra

In India, rapid progress was made in algebra after the construction of the decimal number system, and algebraic nomenclature was developed by Brahmagupta's (628 CE) time. The algebraic terminology enabled the Indian mathematicians to write algebraic expressions and equations. As Puig and Rojano (2004) explain, the representation of what is needed for the solution of a problem is done by means of two categories of tools; different names for different unknown quantities and, types or species of numbers (such as square and cube) which were present in Brahmagupta's and Bhaskara's sign systems (Colebrooke, 2005). Some examples:

a) In Pruthudakaswami's commentary on Brahmagupta's *Brahma Sputa Siddhanta* there appears the following representation of an equation:

yava 0 *ya* 10 *ru* 8

yava 1 *ya* 0 *ru* 1

Here *ya* is an abbreviation for *yavat tavat* (the unknown quantity, or x) and *yava* is an abbreviation for *yavat tavat* and *varga* (the square of the unknown quantity or x^2; and *ru* stands for *rupa* (the constant term). In other words this is what we would now write as $10x + 8 = x^2 + 1$. (Joseph, 2000, pp. 272-273))

b) Bhaskara II writes the algebraic expression $x^3 + 3x^2y + 3xy^2 + y^3$ as follows:

ya gh 1 *ya v. ca bh* 3 *ca v. ya bh* 3 *ca gh* 1 (Puig & Rojano, 2004, p. 203).

Here *ya* represents an (first) unknown quantity, *ca* is the abbreviation of *calaca* (black) and represents another unknown quantity, *v* is the beginning of *varga* (square), *gh* is the abbreviation of *ghana* or cube, and *bh* is the beginning of *bhavita* or product. Other unknown quantities were generally represented by colours such as *calaca* or *ca* (black), *nilaka* or *ni* (blue), *pitaca* or *pi* (yellow) and so on. Puig and Rojano (2004) state that Indian mathematicians made a clear distinction between *names of unknown quantities* and *types (powers) of numbers*, which was not present in the Arabic and Latin works. This achievement – different symbols for different unknowns or changing quantities- is significant when it is observed that after Brahmagupta (628 CE), nearly a thousand years elapsed before Viete (1540-1603) in Europe employed different signs for different unknowns. In this thesis, the idea of colours (as signs) to denote different unknowns was combined with Srinivasan's (1989) method of pattern recognition. This method was then employed to help students generalise number patterns (see Figure 7). The idea was that expressing generalities as in Figure 7 would assist students to generalise the place values in multiple bases. From the above historical and psychological analysis pertaining to the development of the Hindu-Arabic numeration system detailed thus far, the following didactic frameworks were constructed and implemented in the classroom in order to help students grasp the compositional structure of the place value system.

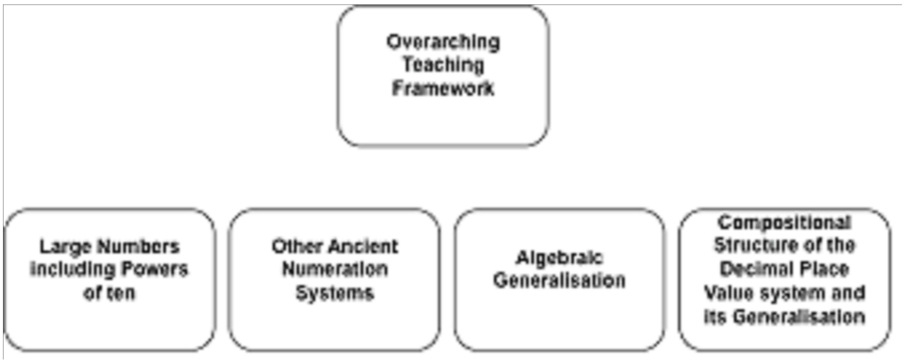


Figure 5. The overarching teaching framework developed and implemented in the study

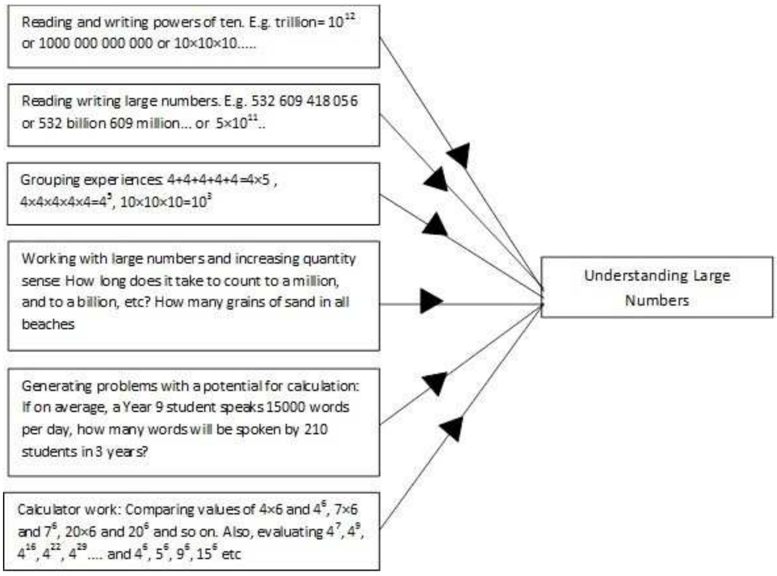


Figure 6. Teaching sequence for understanding large numbers

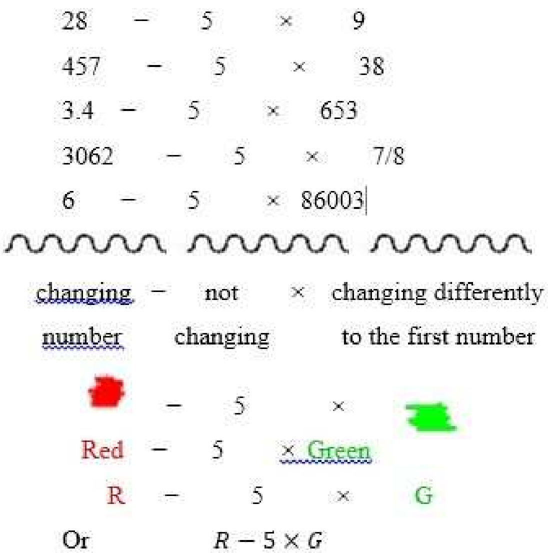
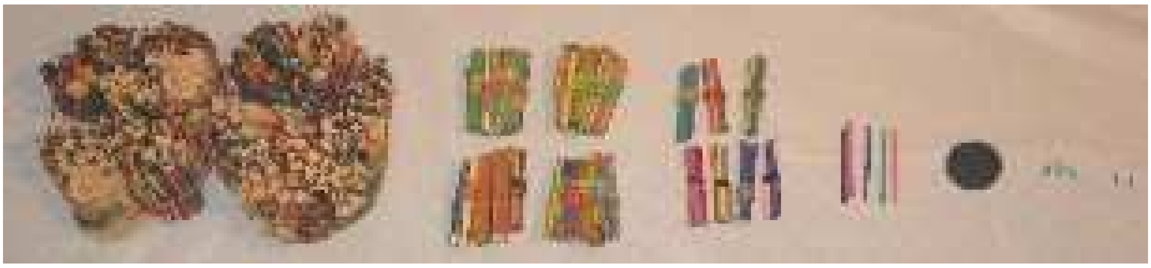
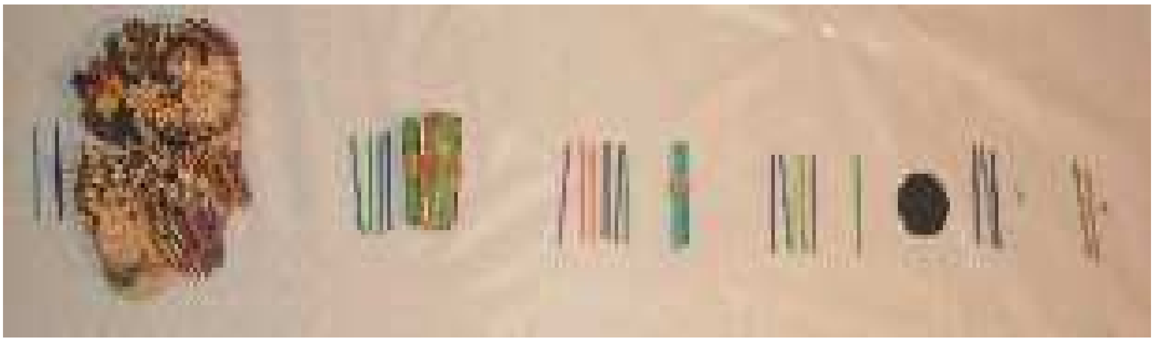


Figure 7. Pattern Recognition using colours for Generalisation and the Variable Concept

Developmental Stages of the Written *Decimal Number System in Indian History*



Verbal/Additive Stage



Interim/Multiplicative Stage

'Places' Stage

Abstract Stage

Final abstract stage using number symbols: 2475.32
Figure 8. Stages in the development of the written Hindu-Arabic numeral system in India

Structure of Numeration – Composition of Number
Review grouping reflecting repeated multiplication and its notation Both decimal and non-decimal bases
Grouping and modelling with sticks as shown in Figure the *four stages* (*additive, multiplicative, places and abstract*) of written numerals thus exposing the implicit multiplicative structure for *base-ten* numbers such as: 4537, 4037, 4537.24, 4537.04, 4037.04 Linking of concrete and symbolic representations
Pictorial representations of the numbers above Linking of pictorial and symbolic representations
Similar process as rows 3 and 4 above in non-decimal bases such as 6,7 or 12

Figure 9. Teaching sequence for the compositional structure of the numeration system

4 Concluding Remarks

This study sought to improve Year 9 students' understanding of a general place value system. To this end, the first part of the study consisted of an *analysis of the historical development* in India of the current Hindu-Arabic decimal system and the evolution of algebraic notation. A historical-critical investigation conducted in the light of mathematics education research revealed, early consideration of *large numbers* including *powers of ten* for many centuries. It also enabled the identification of important *stages* (*additive, multiplicative, places and abstract*) in the development of the *written* decimal place value system with zero. What was also revealed from Indian history is the growth in algebraic symbolism characterised by *different colours* for *different variables* and *names* for *powers of variables.*

The historical findings aided the construction and implementation of an overarching teaching framework and connected teaching sequences (given above) related to a sound grasp of a general place value system. A report on detailed results from the second (classroom) part of the study is beyond the scope of this paper. Briefly, the classroom results suggest that an understanding and use of large numbers, and of exponentiation is reachable to students. The results also showed that students can achieve a certain degree of success in constructing a sense of the structure of the numeration system. In summary, it seems that historical ideas may be useful in developing important new approaches to understanding a general place value system, notation for variables and their powers in algebra.

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Effect of CAI on academic achievement of undergraduates in numerical analysis

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Abstract : This experimental study was aimed to assess the effect of Computer Assisted Instruction (CAI) on academic achievement of undergraduates in Numerical Analysis. For this, Instructional and Measuring tools on Numerical Methods for the solution of Algebraic and Transcendental Equationswere developed. Reliability and validity of these tools were also examined in try-out phases. A sample of ninety (90) female undergraduates from Banasthali Vidyapith, Rajasthan, India were selected after I.Q. test for the division of two equal groups, control group (with conventional method) and experimental group (with CAI). The design used in this study was Pre-test - Post-test Equivalent-Groups Design. SPSS v.16 was used to analyse the data. Results clearly indicated the significant effect of CAI on academic achievement of undergraduates. Findings supported CAI for better understanding of mathematical concepts at undergraduate level.

Keywords : Academic Achievement, CAI, Conventional Method, Numerical Analysis.

1 Introduction

Emphasis on academic achievement of students at college level is constantly growing. Researchers are discovering the need of technological instructions in Mathematics Classrooms. As a result, study on technological instruction CAI andConventional Method aimed to evaluate its effectiveness on academic achievement of undergraduates in mathematics is currently being conducted. Its purpose was to investigate whether CAI is more effective than Conventional Method in increasing undergraduates success in mathematics. This study analysed the use of instructional material for CAI and Conventional Method in Numerical Analysis as predictor of academic achievements of undergraduates in Mathematics. Numerical Analysis is that field of Mathematics which looks for numeric algorithms to solve or approximate problems over a continuous domain. It has many applications in different disciplines at undergraduate level. Keeping in mind to overcome the problem ofundergraduatesperformance in mathematics, the researchers evaluated the instructional material and measuring tools on five sessions of Numerical Methods for the solution of Algebraic and Transcendental Equations in Numerical Analysis. Development of instructional material and measuring tools for CAI and Conventional Method, as prerequisite was also a part of the study.

1.1 Computer Assisted Instruction (CAI)

Computer Assisted Instruction (CAI) is a self - learning technique, which involves interaction of the students with programmed instructional material.It is an interactive instructional technique to present the instructional material and monitor the learning that takes place with help of computer. It uses a combination of text, graphics, sound and video in the learning process.It functions as a tutor with unlimited patience. Immediate feedback to the student, Practice and Self-Evaluation, Self-paced learning opportunities and reinforcement are the main benefits of CAI. In the context of this study, CAI means providing self-learning material to the students through computer. It is an interesting auto mode of instruction. CAI not only decreases the burden of teachers but also provides sufficient time to students for learning abstract concepts. CAI is a student-centred teaching method.

1.2 Conventional Method

Conventional Method is that method of teaching in which the teacher is the centre of class-room activities of teaching learning process. In India, it is a teacher-oriented Lecture Method used for presentation of a textbook material with the help of chalk board. It is also known as Traditional Method, Lecture Method or Expository Method. All these terms convey almost the same meaning. Most of the researchers have taken Conventional Method as it exists in the class-room today. In these class rooms, lessons are not planned, objectives are not stated in behaviour terms and stepwise evaluation of students is not done during teaching. But in this experimental study, Conventional Method means that method of instruction where the teacher plays a major role and the lesson is also planned. Objectives are framed in behavioural terms and stepwise evaluation is done at every stage. Same material meant for CAI is utilized in planning of Conventional Method.

1.3 Academic Achievement

In the present study, academic achievement refers to the scores obtained by students on attainmentand retention of Numerical Analysis at undergraduate level.

1.4 Objectives

The objectives of this study were to

(a) Develop tools on Numerical Methods for the solution of Algebraic and Transcendental Equations in Numerical Analysis for CAI and Conventional Method.

(b) Study the effect of instructional material on academic achievements of undergraduates in Numerical Analysis for CAI Method.

(c) Access the effect of instructional material on academic achievements of undergraduates in Numerical Analysis for Conventional Method.

(d) Compare academic achievements of undergraduates between experimental group using CAI and control group withConventional Method.

1.5 Hypotheses

Hypothesis in this study were framed as follows:

(a) There exists no significant difference in academic achievement of undergraduatesin Numerical Analysis for experimental groupfrom Pre- test to Post - test I and Post-test II.

(b) There exists no significant difference in academic achievement of undergraduates in Numerical Analysis for control group in Numerical Analysis from Pre- test to Post - test I and Post-test II.

(c) There exists no significant difference between mean attainment score of Numerical analysis in experimental group and control group at Pre- test Stage

(d) There exists no significant difference between mean attainment score of Numerical analysis in experimental group and control groupat Post-test I Stage.

(e) There exists no significant difference between mean retention score of Numerical Analysis in experimental group and control group at Post-test II Stage.

2 Rationale of the Study

Many researches have been conducted to evaluate the effect of CAI on theacademic achievement of the students as compared to conventional method of teaching at different levels. Different studies [(see [1])]; [(see [2])]; [(see [5])]; [(see [6])]; [(see [8])]; [(see [11])]; [(see [12])] and [(see [13])])are in favour of CAI for the achievement of students in Mathematics at different levels. But, very few researches on CAI in mathematics at undergraduate level are conducted in Indian situations. Moreover Researchers found no research related to Numerical Analysis with different modes of instruction. Therefore, it was considered worthwhile to develop and compare the effect of CAI and Conventional Method on academic achievement of undergraduates in Numerical Analysis.

3 Method and Procedure of Experiment

3.1 Design of the Study

In this study, Pre-test Post-test Equivalent Groups research design (Fig.1) was used [(see [13])], p.181)

Figure 1: Design of the Study

Groups	Pre-Test (Entry Level)	Treatment	Post-test I (Attainment Level)	Post- test II (Retention Level)
Random	Intelligence Test, NAT (Numerical Analysis Test)	Experimental Group using CAI	NAT (Numerical Analysis Test)	PNAT (Parallel Numerical Analysis Test)
Random	Intelligence Test, NAT (Numerical Analysis Test)	Control Group using Conventional Method	NAT (Numerical Analysis Test)	PNAT(Parallel Numerical Analysis Test)

3.2 Sample and Procedure of the study

The sample for this experimental study was selected from Banasthali University, Rajasthan, India having smart computer lab. Purposive sampling technique was used to select groups by the researchers. In the beginning, 100 under graduates (females), who offered mathematics as elective subject, were selected. Ravens intelligence test was given to students for matching their level of intelligence. Both groups were also pre-tested with Numerical Analysis Test (NAT) for selected topic “Numerical Methods for the solution of Algebraic and Transcendental Equations” of Numerical Analysis in Mathematics to find out whether they have not learnt the unit beforehand. On the basis of the scores obtained in I.Q. Test and Pre-test, the undergraduates were matched and equally distributed in experimental and control group for final experiment. After controlling the effects of confounding Variables, the sample size was reduced to 90 at the time of analysis. Each group had 45 students. The experimental group was exposed to CAI and control group was instructed through Conventional Method. After the treatments, attainment test NAT (Post-test I) was administered to measure their achievements. One week after, retention test PNAT(Post-testII)was conducted to access Retention level. The scores obtained in Pre-test, Post-test I (NAT) and Post-test II (PNAT) were recorded for data analysis.

3.3 Tools for Data Collection

The researchers developed the instructional and measuring tools on five sessions of Numerical Methods for the solution of Algebraic and Transcendental Equations: Introduction to Errors in Numerical Computations:Bisection Method ; Regula Falsi Method ;Newton Raphson Method ;Order of convergence for Numerical Methods in Solving Algebraic and Transcendental Equations (([(see [7])], pp.1-10 and pp.43-64) ; ([10], pp.1-15)).Two types of instructional tools CAIM (Software package for CAIin Microsoft Office word 2007 using hyperlinks) and Lesson plans (for Conventional Method) were developed on five sessions of Numerical Analysis for Conventional Method. The criteria of 85 percentage success were adapted to evaluate the efficiency of CAIM.Validity and suitability of these tools was calculated in terms of gain ratio (.86) for CAIM in pilot study of 33 undergraduates. TwoCriterion - Referenced Tests in parallel forms, NAT (Numerical Analysis

Test) and PNAT (Parallel Numerical Analysis Test) on Numerical Methods for the solution of Algebraic and Transcendental Equations: along with their scoring keys were also developed by the researchers. The content validity of the NAT,PNAT and Scoring keys was determined by the subject experts. Individual and small group try-out was carried for the editing of content material.Reliability co-efficient of measurement tools was Kappa (.87)interms of the consistency of decision making process across NAT and PNAT as alternate forms of attainment test at mastery level. It wascalculatedin terms of mastered and non-mastered group in pilot study for 33 undergraduates([(see [9])] and ([(see [14])], pp.263-267)).

4 Results and Discussions

In order to analyse the data, SPSS v.16 was used. Matched t-test was used to compare the results from Pre-test toPost-test I and Pre-test to Post-test II for each group. Independent t-test was used to compare experimental group and control group at Pre-test Stage, Post-test I Stage and Post-test II Stage ([3], p.268).

Table 1: The results of matched t-tests in experimental group(Pre-test to Post-test I and Pre-test to Post-test II)

Paired Sample Statistics			**Paired Sample Test**			
Stages	Mean	S.D.	Mean Gain Score	t	d. f.	Sig. (2-tailed)
Post I Pre	53.11 13.42	2.673 6.136	39.69	55.19	44	.000
Post II Pre	52.40 13.42	3.227 6.136	38.69	49.84	44	.000

There is a significant difference in the scores for Post-test I ($M = 53.11, SD = 2.673$) and Pre-test ($M = 13.42, SD = 6.136$)conditions; $t(44) = 55.19, p = .000$ in experimental group. Also, there is a significant difference in the scores for Post-test II ($M = 52.40, SD = 3.227$) and Pre-test ($M = 13.42, SD = 6.136$) conditions; $t(4) = 49.84, p = .000$ inexperimental group (Table 1). It also indicates statistically significant mean score gain from the Entry level to Attainment Level and Retention Level in experimental group.

Table 2: The results of matched t-tests in Control group(Pre-test to Post-test I and Pre-test to Post-test II).

Paired Sample Statistics			**Paired Sample Test**			
Stages	Mean	S.D.	Mean Gain Score	t	d. f.	Sig. (2- tailed)
Post I Pre	51.02 13.55	3.299 6.051	37.47	45.198	44	.000
Post II Pre	49.24 13.55	3.227 6.051	35.69	37.342	44	.000

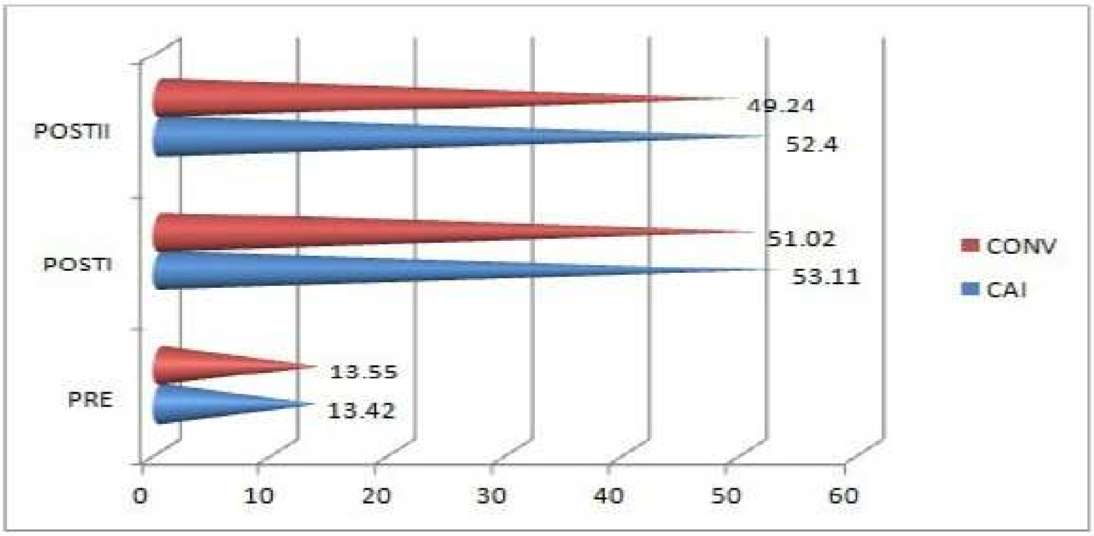
There is asignificant difference in scoresbetween Post-test I ($M = 51.02, SD = 3.299$) and Pre-test ($M = 13.55, SD = 6.051$) conditions; $t(44) = 45.198, p = .000$ in Control group. Also, there is a significant difference in the scores for Post-test II (M = 49.24, SD = 3.227) and Pre-test (M = 13.55, SD = 6.051) conditions; t (44) = 37.342, p = .000 in Control group (Table 2).Hence,It also indicates statistically significant mean score gain from the Entry level to Attainment Level and Retention Level in control group.

Table 3: The results of independent t-test between Experimental Group and Control Group (Pre-test Stage, Post-test I Stage and Post- test II Stage)

Group Statistics					Independent Sample Test t-test for equality of mean		
Stage	Mode	N	Mean	S.D	t	d. f.	Sig.(2-tailed)
Pre-test	CAI	45	13.42	6.136	-.104	88	.918
	CONV	45	13.55	6.051			
Post-test I	CAI	45	53.11	2.673	3.300	88	.000
	CONV	45	51.02	3.299			
Post-test II	CAI	45	52.40	2.396	5.266	88	.000
	CONV	45	49.24	3.227			

ZComparison of Pre-test Score for CAI (M = 13.42, SD = 6.136) and Conventional Method (M = 13.55, SD = 6.051) revealed no significant differences between Groups t (44) = -.104, p= .918 (Pre-test Stage in Table 3).It is also clear from the Table 3(Post-test I Stage) that the experimental group using CAI achieved more score (M = 53.11, SD = 2.673) than the control group (M = 51.02, SD = 3.299). This difference was significant, t (44) = 3.300, p = .000 .The group using CAI recalled significantly more scores (M = 52.40, SD = 2.396) than the group using Conventional Method (M = 49.24, SD = 3.227), t (44) = 5.266, p = .000 (Post-test II Stage in Table 3).

Figure 2: Graphical representation of results



(a) Fig. 2 indicates that Entry Level for both groups were similar.

(b) Mean Gain Scores form Entry Level to Attainment Level and Retention Level (Fig. 2) indicatethe effectiveness of CAIM for CAIand Lesson plans for Conventional Method.

(c) Also, there are clear-cut indications of significant effect of CAI on attainment and retention of Numerical Methods for the solution of Algebraic and Transcendental Equations in Numerical Analysis.

5 Conclusion

This study confirmed the effectiveness of instructional material in academic achievement of undergraduates in Numerical Analysis. Instructional material for both CAI and Conventional Method was effective. But academic achievements in experimental group were better than control group.Hence CAI was found more effective in attaining and retainingNumerical Analysis at undergraduate level. This study will help the mathematicians to choose CAI as mode of instruction in Mathematics Education at undergraduate level. The use of CAI can improve academic achievements of undergraduates in mathematics at their own pace in Indian situations. CAIM for different levels of learners in different subjects can be developed and used for effective teaching.

5.1 Acknowledgement

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"Using technology to make basic mathematical concepts accessible to students"

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Abstract : This presentation will focus on how to help students to make sense of mathematical concepts and increase math awareness so that they begin to view mathematics as part of the real world and not as something alien or extraneous. This presentation will use multi media technologies that do not overwhelm students. Recently I was involved in a decision to choose a book on College Algebra at Ohio University that would best meet the needs of our students. My goal was to choose a book that is readable, affordable, and which would also help my teaching assistants to deliver their lectures effectively. So I ended up choosing, Essential College Algebra by Julie Miller (McGraw-Hill publication) which met my criteria for a 21st century text book for College students.

1 Background

In the 1970s and since 1983, as a teacher and professional, my activities have ranged from teaching mathematics and computer science to promoting multiculturalism in the academic setting. I have taught mathematics and computer science at a wide range of institutions, including the City University of New York (CUNY), Manhattan College (New York City), Middlebury College (Vermont), Fitchburg State University (Massachusetts), Johnson and Wales University (Providence, Rhode Island) and, most recent, Ohio University, Athens, Ohio.

I recount my early experiences with open-admissions students at John Jay College of the City University of New York, SEEK Program (Search for Education, Elevation and Knowledge) in the 1970s as eye opening. It was during this time that I realized, sadly, that students were entering US colleges with minimal mathematical thinking and training. Since then, I have focused my instructional time with my students on filling the gaps they have experienced in their math educationand to inspiring and motivating them to love mathematics. Because of gaps in their education and lack of motivation these students were also dealing with issues related to Math Anxiety.

Students at Fitchburg State University, Johnson and Wales University were not as well prepared academically as those at Manhattan College or Middlebury College. At Ohio University, it is not uncommon to have students with varying abilities in the same classroom, but there is lot of institutional support such as free tutoring, supplemental instruction to help students to succeed in their courses.

2 My Teaching Philosophy

I believe that education is a four-legged stool that rests on four vital legs: educational institutions, parents, teachers, and students. For students to succeed and compete well with the world, each leg has to be working at its best and very closely. You cannot expect any one of these legs alone to achieve the best possible results. My enthusiasm and love for teaching stems from the values of many of my teachers who have had along the way believed in giving, themselves freely to guide help and shape their students lives. My teachers guidance and emphasis on hard work and their reminders, such as - "No Pain, No Gain," and "Mathematics is not a spectators sport, the more you do it, the better you will get at it," has served me well in life.

For decades now, I have been trying to give back and play a role in shaping my students lives by doing my job as a teacher in the best possible way and in helping my students see that Mathematics is sequential and is not a spectators sport. To succeed in Mathematics, one must practice every day to see improvements and I impart this basic tenet to my students.

Rapport between teachers and students is an important part of teaching. Until and unless my students feel comfortable with me, they are not going to seek my help in or outside class, and I wont find out where they are stumbling and might need help. Once they feel comfortable with me, I can challenge their deficiencies and help them beyond their current skills and abilities by convincing them to work hard to learn mathematical reasoning.

Increasingly, we are being called to higher levels of quantitative reasoning. To prepare the workforce for the 21st century, it has become very important that students are able to critically read and process information to solve real world problems using the most efficient process. Without these skills, our students will not be able to contribute meaningfully to society and to participate in the global economy.

Mathematics is considered to be something separate from the rest of our lives, and to some extent, this belief is prevalent even within the academy. I believe my role as teacher is to challenge this myth and convince my students that mathematical reasoning and culture are integral to their lives and future success. With my strong belief in "Mathematics for All," I continue to try to find ways to meet my students varying learning styles and needs. As a teacher one is always under pressure to evolve and grow. Times change, students learning styles and needs change, so has my teaching style evolved from year to year. In order to meet the needs of my 21st -century students, my teaching philosophy and style must evolve constantly. Among other things, I have greatly increased the use of visuals and available technology tools. This approach ensures students ability to see the relevance of Applied Math to their future professions and general life skills. In response to my students varying learning style needs (aural, visual, kinesthetic, etc.), I continue to use a variety of techniques to get their attention and to expand their short attention span for mathematical topics. I will keep on trying any and all approaches that give results.

3 My goals as a mathematics professor

My role as a mathematics teacher is to help my students realize that they can all learn and do well in mathematics. My goal as a mathematics teacher is to equip my students with mathematical tools, sharpen their mathematical skills, as well as to teach them how and where to use these tools effectively. I want them to realize the usefulness and beauty of mathematics. My goal is also to help them to appreciate mathematics and to see the importance and pervasiveness of quantitative reasoning in many academic fields at college as well as in many areas of their future lives.

4 My approach to the teaching-learning process

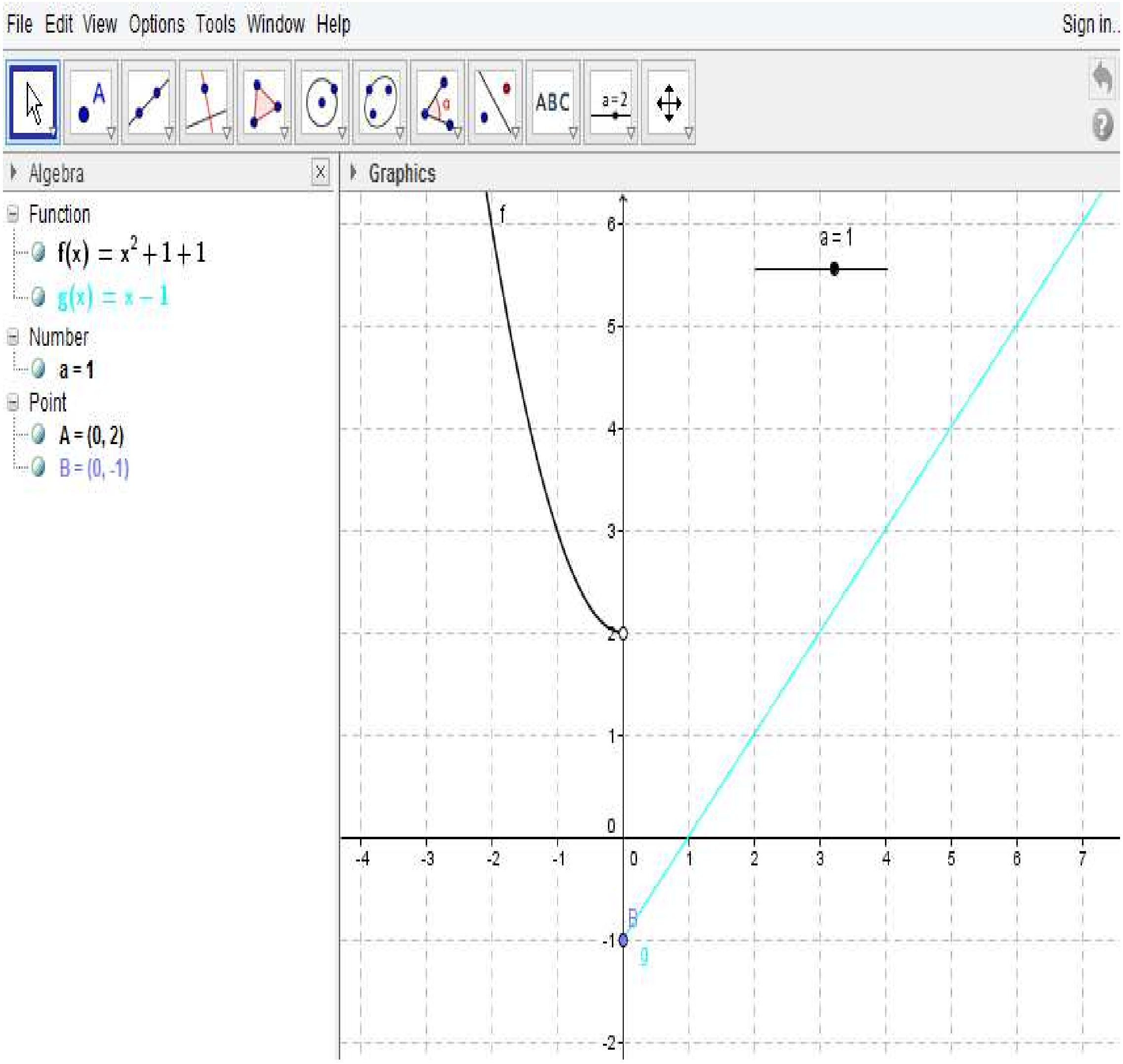
My focus in teaching problem solving to my students is not only to help them find the right answer, but also to help them figure out the best way to solve a problem. To achieve this, I must ensure that students have facility with foundational math skills and concepts. Once that foundation is established, we can pursue the various ways to solving a problem. Definitions play a very important role in the learning and understanding of mathematical concepts/processes. So, to have a clear idea of what the definition means mathematically, I expect my students to express what definitions and/or mathematical symbols for terms such as *Equations, Sets, Functions, Polynomials, Asymptotes, etc.* mean to them in their own words. Since a picture is worth a thousand words, I integrate technologies, more recently, I have been using GeoGebrato help establish a relationship between Algebra and Geometry. Technology can play a role in helping students construct knowledge and build conceptual understanding. I encourage my students to be involved in their learning process and I welcome them to ask me questions during class and make sure that they get all their questions answered before they leave the class. To help mentor and support learning, I make myself easily accessible to my students in office hours and beyond.

My emphasis is also on helping my students to become independent learners. So, sometimes I email them Power Point of the next classs **lecture before I introduce the new topic (using Flip-class model)** so that they can review the material at home and come to the class fully prepared to ask questions during next class.

Teaching has become a very challenging profession. Our classrooms are in- creasingly becoming diverse in age and background, personal interests, physical and mental abilities. Teachers are expected to challenge, and motivate students as well as fill in gaps in students educational background. Our goal is to disseminate information in a way which encourages students to think critically/mathematically and dispel the myth that mathematics is something abstract or unrelated to life. We have to demonstrate that mathematics is pervasive in life so that students dont ask or question: "Why do we have to do mathematics? Where are we going to use it?" Instead, they have to be persuaded that mathematical thinking will serve them in college as well as throughout their entire life and in any career they choose.

In this age of technology, 3Rs (read, write and arithmetic) are not enough to survive global competition, and to be able to succeed it is important to be able to express why and what we are doing as well as how to do it. Technology plays a very important role in helping students visualize what is happening and it gets them motivated and more involved in learning mathematics. Once students have visualized what is happening, they will be able to express these ideas in plain English and figure out how mathematical concepts can help them in succeeding in their studies.

I use GeoGebra to help my students visualize mathematical concepts and to strengthen their critical thinking. Here are some examples how I use GeoGebra to help student understand piecewise function, x-intercept and y-intercept, zeroes of the function, behavior of the function in the long-run, asymptotes, parabola, etc.

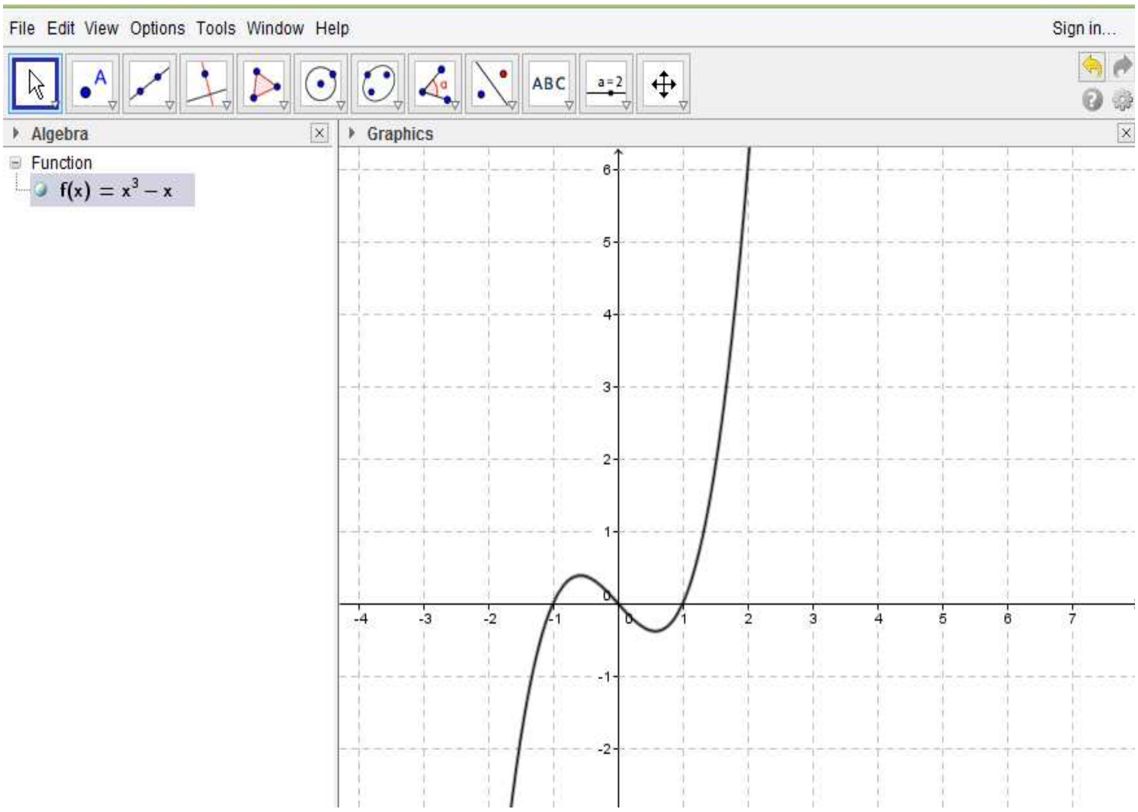


Example 4.1. For $y = x^3 - x$, find

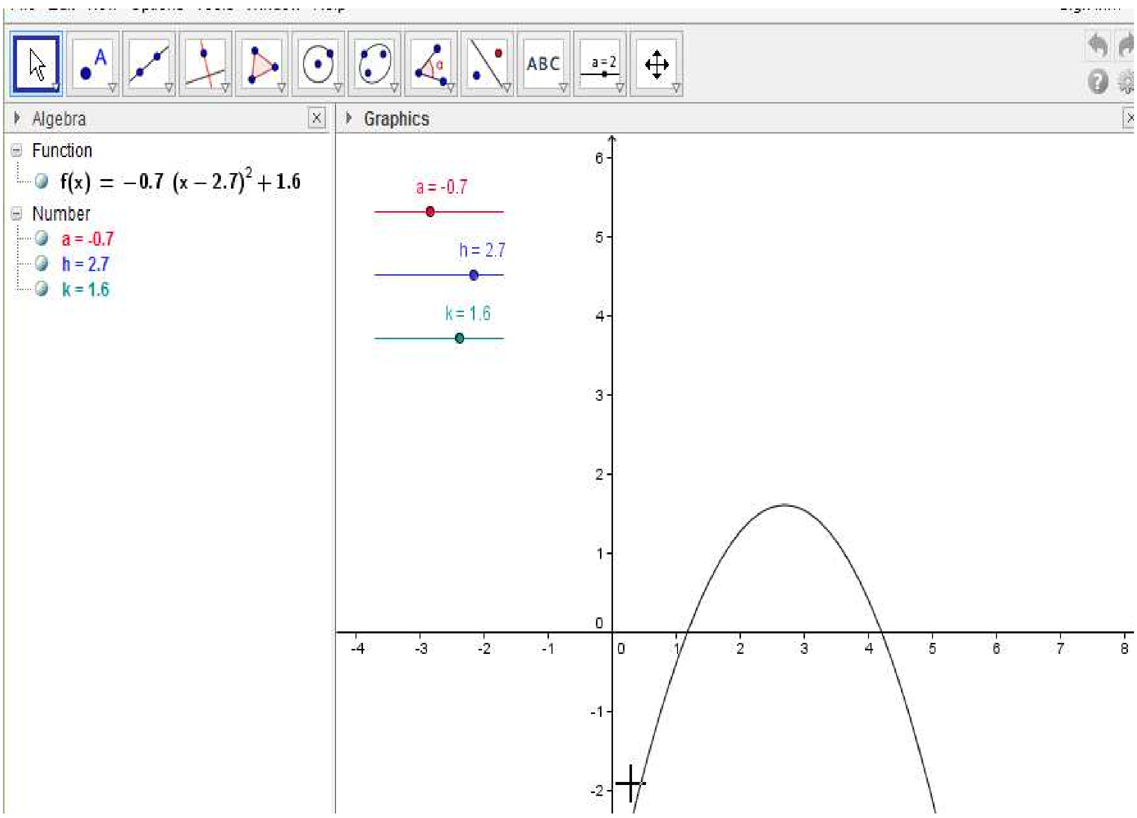
(a) x-intercept and y-intercept.

(b) Zeros of the function.

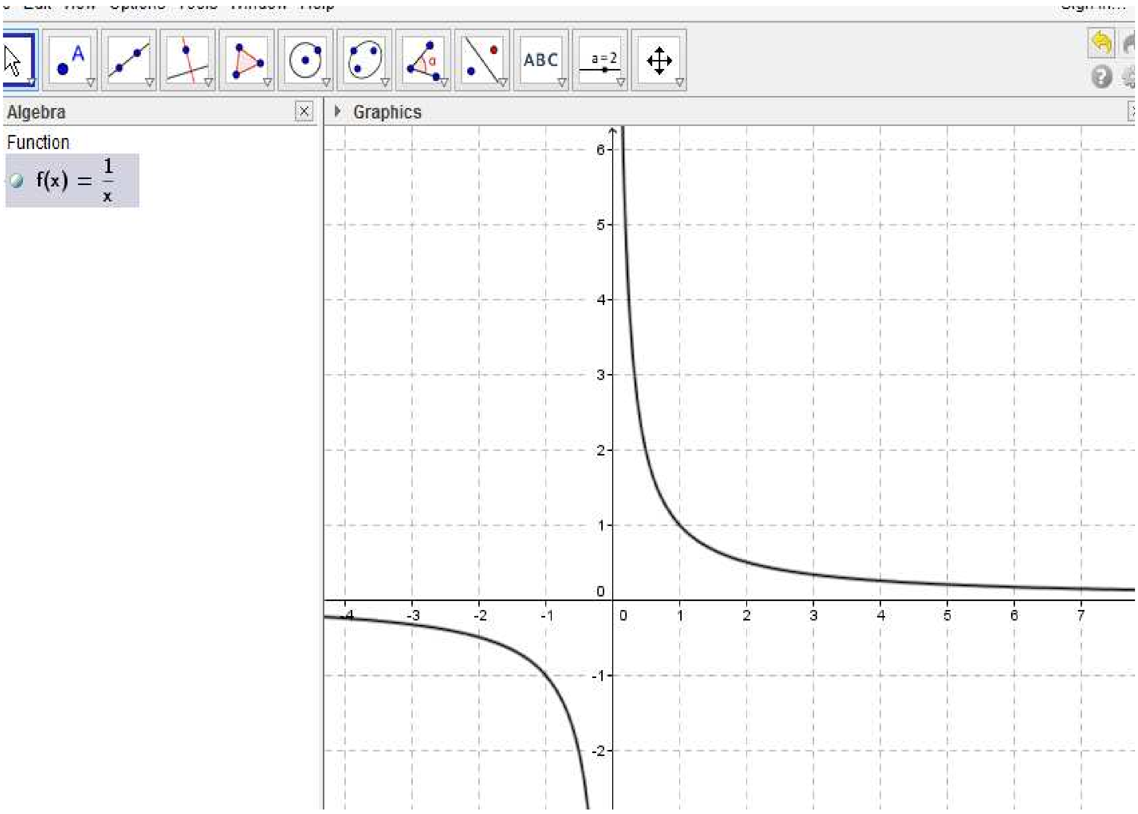
(c) The behavior of the function in long-run



Example 4.2. Visualize parabola with changing values of 'a' using sliders



Example 4.3. Write asymptotes for the graph.



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The onset of ferroconvection in a horizontal porous layer with magnetic field-dependent viscosity

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Abstract : The effect of magnetic field dependent (MFD) viscosity on the onset of ferromagnetic convection in a ferrofluid saturated horizontal Brinkman porous layer heated from below is investigated. The isothermal boundaries of the porous layer are considered to be rigid paramagnetic with low/high magnetic susceptibility. The simultaneous and isolation presence of buoyancy and magnetic forces on the criterion for the onset of ferromagnetic convection is emphasized and the resulting eigenvalue problem is solved numerically using the Galerkin technique. It is observed that the system is found to be more stable when the magnetic force alone is present. The numerical calculations also reveal that increase in the MFD viscosity parameter λ, magnetic susceptibility χ and decrease in the magnetic number $M_1$and nonlinearity of fluid magnetization parameter M_3 is to inhibit the onset of ferromagnetic convection in a porous layer. The critical wave number is found to be independent ofλ, but increasingM_1 and χ as well as decreasing M_3 is to reduce the size of convection cells.

Keywords: Porous medium, Ferromagnetic convection, MFD viscosity, Rigid Paramagnetic boundaries

1 Introduction

Ferrofluids contain single domain nanoparticles of magnetic material (iron, cobalt or magnetite) stably suspended in carrier liquids like water, kerosene or various oils. Each particle is encapsulated by a monolayer of surfactant in order to prevent particle coalescence due to magnetic attraction. The average size of magnetic nanoparticles is about 10 nm. Magnetic colloids have magnetic susceptibility which is thousands times larger than that of natural materials. The study of such fluids became the subject of a special branch of magnetohydrodynamics termed as ferrohydrodynamics and found applications in various areas of science, technology and nanotechnology (Rosensweig [1], Bashtovoy et al. [2], Berkovsky et al. [3],).

The magnetization of ferrofluids depends on the magnetic field, the temperature and the density of the fluid. Any variation of these quantities can induce a change in body force distribution in the fluid. This leads to convection in ferrofluids in the presence of magnetic field gradient, known as ferroconvection, which is similar to buoyancy driven convection. Finlayson [4] was the first to study ferroconvection in a horizontal layer of ferrofluid subject to a vertical temperature gradient and in the presence of a uniform vertical magnetic field. Since then several studies on ferroconvection in a layer of ferrofluid heated uniformly from below in the presence of a uniform magnetic field have been undertaken and copious literature is available on this topic of research (Ganguly et al. [5], Odenbach [6], Sunil and Mahajan [7], Nanjundappa and Shivakumara [8]. Singh and Bajaj [9], Belyaev and Smorodin [10]).

Thermal convection of ferrofluids saturating a porous medium has also attracted considerable attention in the literature owing to its importance in controlled emplacement of liquids or treatment of chemicals, and emplacement of geophysically imageble liquids into particular zones for subsequent imaging (for details see Borglin et al. [11] and references therein). Shivakumara et al. [12, 13] have investigated in detail the onset of thermomagnetic convection in a ferrofluid saturated porous medium for various types of velocity and temperature boundary conditions. Recently, the buoyancy-driven convection in a ferromagnetic fluid saturated porous medium considering the lower boundary to be rigid-paramagnetic while the upper boundary to be either rigid or stress-free has been studied by Nanjundappa et al. [14].

For relatively weak magnetic field, the ferrofluid is not saturated and its properties vary with the magnetic field. One of the well known phenomena generated by the influence of magnetic field on ferrofluids is the change of their viscosity. The first discovery of such changes was made by Rosensweig et al. [15] in concentrated magnetite-ferrofluids followed a few months later by an independent work by McTague [16] using highly diluted Co-ferrofluids. Both the studies contemplated an increase of viscosity of ferrofluids with increasing magnetic field strength. The effect of a homogeneous magnetic field on the viscosity of a ferrofluid with solid particles possessing intrinsic magnetic moments has been investigated by Shliomis [17].

Realizing the fact that the viscosity of the ferrofluid changes with the magnetic field, thermal convective instability in a ferrofluid saturated porous layer subject to MFD viscosity has also been investigated in the recent past. Notably, Vaidyanathan et al. [18] have investigated the effect of MFD viscosity on ferroconvection in a rotating sparsely packed porous medium, while the anisotropy of porous medium on the problem without rotational effects has been analyzed by Ramanathan and Suresh [19] using the Darcy model. Sunil et al. [20] have studied the effect of MFD viscosity on thermal convection in ferromagnetic fluid saturating a porous medium in the presence of dust particles considering the boundaries to be stress-free. Recently, the onset of ferroconvection in a horizontal ferrofluid saturated porous layer subject to MFD viscosity has been investigated theoretically by considering the bounding surfaces of the porous layer are either rigid-ferromagnetic or stress-free with constant heat flux conditions by Nanjundappa et al. [21].

It is a known fact that the nature of bounding surfaces of a porous layer plays a significant role on the onset of convection [22] and it so in the case of ferromagnetic convection in a ferrofluid saturated Brinkman porous layer heated from below as well. Also, probing ferromagnetic convection in a layer of porous medium subject to MFD viscosity and paramagnetic boundary conditions will be quite interesting because of its relevance in many technological problems and in laboratory experiments. To our knowledge, this problem has not received any attention in the literature. The aim of the present paper is, therefore, to study this problem using a non-Darcian model for more realistic magnetic and velocity boundary conditions. In analyzing the problem, the lower and upper rigid isothermal boundaries of the porous layer are considered to be paramagnetic with low/ high magnetic susceptibility. Unlike the free-boundaries case, the solution for the eigenvalue problem in closed form is not possible and hence the critical stability parameters have been extracted numerically by employing a higher-order Galerkin method. The criterion for the onset of ferromagnetic convection has been analyzed when the buoyancy and magnetic forces are acting simultaneously and also in isolation. The existing results in the literature are obtained as limiting cases from the present study.

2 Formulation of the problem

The system considered is an initially quiescent ferrofluid saturated horizontal porous layer of characteristic thickness d in the presence of an applied magnetic field H_0 in the vertical direction. The lower and the upper boundaries of the porous layer are maintained at constant temperature T_0 and $T_1(< T_0)$ respectively, and thus constant temperature difference $\Delta T\,(= T_1 - T_0)$ is maintained between boundaries. A Cartesian co-ordinate system (x, y, z) is used with the origin at the bottom of the porous layer and z-axis is directed vertically upward. Gravity acts in the negative z-direction, $\vec{g} = -g\,\hat{k}$, where $\hat{k}$ is the unit vector in the z-direction. The flow in the porous medium is described by the Brinkman-Lapwood extended Darcy equation. It is observed experimentally that the viscosity of the ferrofluid increases with respect to magnetic field steadily and reaches a saturation value as the strength of the magnetic field is increased [15-17]. As a result, for small field variation, linear variation in viscosity η with respect to magnetic field has been used [18]. We follow this formalism and assume the viscosity of the ferrofluid to vary in the form $\eta = \eta_0(1 + \vec{\delta} \cdot \vec{B})$, where δ is the isotropic variation coefficient of magnetic field dependent viscosity, η_0 is the viscosity of the fluid when the applied magnetic field is absent and $\vec{B} = (B_x, B_y, B_z)$ is the magnetic induction.
The continuity equation for an incompressible ferrofluid is

$$\nabla \cdot \vec{q} = 0 \tag{2.1}$$

where $\vec{q} = (u, v, w)$ is the velocity vector. The momentum equation for an incompressible ferrofluid saturated porous medium with variable viscosity η is given by

$$\rho_0 \left[\frac{1}{\varepsilon}\frac{\partial \vec{q}}{\partial t} + \frac{1}{\varepsilon^2}\left(\vec{q}.\nabla\right)\vec{q}\right] = -\nabla p + \rho\vec{g} - \frac{\eta}{k}\vec{q} + \nabla \cdot \left[\frac{\eta}{\varepsilon}\left(\nabla\vec{q} + \nabla\vec{q}^{\,T}\right)\right] + \mu_0(\vec{M}\cdot\nabla)\vec{H} \tag{2.2}$$

where p is the pressure, $\vec{M}$ is the magnetization, $\vec{H}$ is the magnetic intensity, ρ is the fluid density, ρ_0 is the reference density, μ_0 the magnetic permeability of vacuum, k is the permeability of the porous medium and ε the porosity of the porous medium.
The energy equation for an incompressible ferrofluid saturating a porous medium is

$$\varepsilon\left[\rho_0 C_{V,H} - \mu_0 \vec{H}\cdot\left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\right]\frac{DT}{Dt} + (1-\varepsilon)(\rho_0 C)_S\,\frac{\partial T}{\partial t} + \mu_0 T\left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\cdot\frac{D\vec{H}}{Dt} = k_t\nabla^2 T \tag{2.3}$$

where T is the temperature, k_t is the thermal conductivity of the fluid, C is the specific heat, $C_{V,H}$ is the specific heat at constant volume and magnetic field, the subscript s represents the solid and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator.
The equation of state for the Boussinesq ferrofluid is:

$$\rho = \rho_0\left[1 - \alpha_t\left(T - T_0\right)\right] \tag{2.4}$$

where α_t the thermal expansion coefficient.
Maxwell's equations in the magnetostatic limit are:

$$\nabla\cdot\vec{B} = 0, \nabla\times\vec{H} = 0 \text{ or } \vec{H} = \nabla\varphi \tag{2.5}$$

$$\vec{B} = \mu_0\,(\vec{M} + \vec{H}) \tag{2.6}$$

where ϕ is the magnetic potential.
Since the magnetization depends on the magnitude of magnetic field and temperature, we have

$$\vec{M} = \frac{\vec{H}}{H}M\left(H, T\right) \tag{2.7}$$

The linearized equation of magnetic state about H_0 and T_0 is

$$M = M_0 + \chi(H - H_0) - K(T - T_0) \tag{2.8}$$

where $\chi = \left(\frac{\partial M}{\partial H}\right)_{H_0,T_0}$ is the magnetic susceptibility, $K = -\left(\frac{\partial M}{\partial T}\right)_{H_0,T_0}$ is the pyromagnetic coefficient and $M_0 = M(H_0, T_0)$.
It is clear that there exists the following solution for the basic state:

$$\vec{q}_b = 0, p_b(z) = p_0 - \rho_0 g\, z - \frac{1}{2}\rho_0\alpha_t g\beta\, z^2 - \frac{\mu_0 M_0 \kappa\, \beta}{1+\chi} z - \frac{\mu_0 \kappa^2 \beta^2}{2(1+\chi)^2} z^2$$

$$T_b(z) = T_0 - \beta z\ , \vec{H}_b\,(z) = \left[H_0 - \frac{K\beta\, z}{1+\chi}\right]\hat{k}, \vec{M}_b\,(z) = \left[M_0 + \frac{K\beta\, z}{1+\chi}\right]\hat{k} \tag{2.9}$$

where $\beta = \Delta T/d$ is the temperature gradient and the subscript b denotes the basic state.
To investigate the conditions under which the quiescent basic state is stable against small disturbances, we consider a perturbed state such that

$$\vec{q} = \vec{q}\,', p = p_b(z) + p', \eta = \eta_b(z) + \eta', T = T_b(z) + T'$$

$$\vec{H} = \vec{H}_b(z) + \vec{H}', \vec{M} = \vec{M}_b(z) + \vec{M}' \tag{2.10}$$

where $\vec{q}\,'$, p', η', T', $\vec{H}'$ and $\vec{M}'$ are perturbed variables and are assumed to be small.
Substituting equation (2.10) into equations (2.6) and (2.7), and using Eq.(2.5), we obtain (after dropping the primes)

$$H_x + M_x = (1 + M_0/H_0)\ H_x$$

$$H_y + M_y = (1 + M_0/H_0)\ H_y$$

$$H_z + M_z = (1+\chi)\,H_z - KT. \tag{2.11}$$

Again substituting equation (2.10) into momentum equation (2.2), linearizing, eliminating the pressure term by operating curl twice and using equation (2.11) the z-component of the resulting equation can be obtained as (after dropping the primes)

$$\left(\frac{\rho_0}{\varepsilon}\frac{\partial}{\partial t} + \frac{\eta_b}{k} - \frac{\eta_b}{\varepsilon}\nabla^2\right)\nabla^2 w = -\mu_0 K\beta\frac{\partial}{\partial z}\left(\nabla_h^2\varphi\right) + \frac{\mu_0 K^2\,\beta}{1+\chi}\nabla_h^2 T + \rho_0\alpha_t g\,\nabla_h^2 T \tag{2.12}$$

where $\eta_b = \eta_0[1 + \delta\mu_0(M_0 + H_0)]$ and $\nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the horizontal Laplacian operator. Since the basic state is quiescent, the inertia effect is not appearing in the above equation.
The energy equation (2.3), after using equation (2.10) and linearizing, takes the form (after dropping the primes)

$$(\rho_0 C)_1\frac{\partial T}{\partial t} - \mu_0\, T_0 K\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial z}\right) = k_1\,\nabla^2 T + \left[(\rho_0 C)_2 - \frac{\mu_0\, T_0 K^2}{1+\chi}\right] w\,\beta \tag{2.13}$$

where $(\rho_0 C)_1 = \varepsilon\,\rho_0 C_{V,H} + \varepsilon\,\mu_0 H_0 K + (1-\varepsilon)\,(\rho_0 C)_s$ and $(\rho_0 C)_2 = \varepsilon\,\rho_0 C_{V,H} + \varepsilon\,\mu_0 H_0 K$.
Equations 5(a, b), after substituting equation (2.10) and using equation (2.11), may be written as (after dropping the primes)

$$\left(1 + \frac{M_0}{H_0}\right)\nabla_h^2\varphi + (1+\chi)\,\frac{\partial^2\varphi}{\partial z^2} - K\frac{\partial T}{\partial z} = 0. \tag{2.14}$$

Since the principle of exchange of stability is valid (Finlayson [4]), the normal mode expansion of the dependent variables is assumed in the form

$$\{w, T, \varphi\} = \{W(z),\, \Theta(z),\, \Phi(z)\}\ \exp\left[i\,(\ell x + my)\right] \tag{2.15}$$

where ℓ and m are wave numbers in the x and y directions, respectively.

On substituting equation (2.15) into equations (2.12)-(2.14) and non-dimensionalizing the variables by setting

$$Z^* = \frac{z}{d}, \quad W^* = \frac{d}{\nu A} W, \quad \Theta^* = \frac{\kappa}{\beta v d}\Theta \text{ and } \Phi^* = \frac{(1+\chi)\,\kappa}{\mathrm{K}\,\beta\, v\, d^2}\Phi \tag{2.16}$$

where $v = \eta_0/\rho_0$ is the kinematic viscosity, $\kappa = k_t/(\rho_0 C)_2$ is the effective thermal diffusivity and $A = (\rho_0 C)_1/(\rho_0 C)_2$, we obtain the stability equations in the following form (after dropping the asterisks for simplicity):

$$(1+\lambda)\left[(D^2-a^2) - Da^{-1}\right](D^2-a^2)\, W = -\, a^2 R\left[M_1 D\Phi - (1+M_1)\,\Theta\right] \tag{2.17}$$

$$(D^2-a^2)\,\Theta = -(1-M_2 A)\, W \tag{2.18}$$

$$(D^2 - a^2 M_3)\,\Phi - D\Theta = 0. \tag{2.19}$$

Here $D = d/dz$ is the differential operator, $a = \sqrt{\ell^2+m^2}$ is the overall horizontal wave number, W is the amplitude of vertical component of velocity, Θ is the amplitude of temperature, Φ is the amplitude of magnetic potential, $R = \alpha_t g\beta\, d^4/\nu\,\kappa A$ is the thermal Raleigh number and it is the ratio of buoyant to viscous forces, $\lambda = \delta\,\mu_0(M_0+H_0)/\varepsilon$ is the MFD viscosity parameter, $M_1 = \mu_0 K^2\beta/(1+\chi)\,\alpha_t\rho_0 g$ is the magnetic number and it is the ratio of magnetic to gravitational buoyancy forces, $M_2 = \mu_0 T_0 K^2/(1+\chi)\;(\rho_0 C)_1$ is the magnetic parameter, $M_3 = (1+M_o/H_o)/(1+\chi)$ is the measure of nonlinearity of magnetization, and $Da = k/d^2$ is the Darcy number. The typical value of M_2 for magnetic fluids with different carrier liquids turns out to be of the order of 10^{-6} [4] and hence its effect is neglected as compared to unity.

The above equations are to be solved subject to appropriate boundary conditions. The boundary conditions considered are

$$\left.\begin{aligned} W = 0 = DW, \quad \Theta = 0 \text{ at } z = 0 \\ (1+\chi)\, D\Phi - a\,\Phi = 0 \text{ at } z = 0 \\ (1+\chi)\, D\Phi + a\,\Phi = 0 \text{ at } z = 1 \end{aligned}\right\} \tag{2.20}$$

3 Numerical solution

Equations (2.17)-(2.19) together with boundary conditions (2.20) considered constitute an eigenvalue problem with R as the eigenvalue. For the boundary conditions considered, it is not possible to obtain an analytical solution and we have to resort to numerical methods. The Galerkin method is employed to obtain critical stability parameters. Accordingly, the variables are written in series of basis functions as

$$W = \sum_{i=1}^{n} A_i\, W_i(z),\ \Theta(z) = \sum_{i=1}^{n} B_i \Theta_i(z) \text{a nd } \Phi(z) = \sum_{i=1}^{n} C_i \Phi_i(z) \tag{3.1}$$

where the trial functions $W_i(z)$, $\Theta_i(z)$ and $\Phi_i(z)$ will be generally chosen in such a way that they satisfy the respective boundary conditions, and A_i, B_i and C_i being constants. Substituting equation (3.1) into equations (2.17) – (2.19), multiplying the resulting equation (2.17) by $W_j(z)$, equation (2.18) by $\Theta_j(z)$ and the equation (2.19) by $\Phi_j(z)$; performing the integration by parts with respect to z between z = 0 and z = 1 and using the boundary conditions (2.20) we obtain the following system of linear homogeneous algebraic equations:

$$C_{ji}A_i + D_{ji}B_i + E_{ji}C_i = 0 \tag{3.2}$$

$$F_{ji}A_i + G_{ji}B_i = 0 \tag{3.3}$$

$$H_{ji}B_i + I_{ji}C_i = 0. \tag{3.4}$$

The coefficients $C_{ji} - I_{ji}$ involve the inner products of the basis functions and are given by

$$C_{ji} = (1+\lambda)[< D^2W_j D^2W_i > + (2\,a^2 + Da^{-1}) < DW_j DW_i > + a^2(a^2+Da^{-1}) < W_j W_i >]$$

$$D_{ji} = -a^2 R\,(1+M_1) < W_j\,\Theta_i >$$

$$E_{ji} = a^2 R\,M_1 < W_j D\Phi_i >$$

$$F_{ji} = - < \Theta_j W_i >$$

$$G_{ji} = < D\Theta_j D\Theta_i > + a^2 < \Theta_j \Theta_i >$$

$$H_{ji} = - < D\Phi_j \Theta_i >$$

$$I_{ji} = \frac{a}{1+\chi}\,[\Phi_j(1)\,\Phi_i(1) + \Phi_j(0)\,\Phi_i(0)] + < D\Phi_j D\Phi_i > + a^2 M_3 < \Phi_j \Phi_i > \tag{3.5}$$

where the inner product is defined as $< \cdots\cdots > = \int_0^1 (\cdots)\,dz$.
The above set of homogeneous algebraic equations can have a non-trivial solution if and only if

$$\begin{vmatrix} C_{ji} & D_{ji} & E_{ji} \\ F_{ji} & G_{ji} & 0 \\ 0 & H_{ji} & I_{ji} \end{vmatrix} = 0. \tag{3.6}$$

The eigenvalue has to be extracted from the above characteristic equation. For this, we select the trial functions as

$$W_i = (z^4 - 2\,z^3 + z^2)T^*_{i-1}, \Theta_i = (z^2 - z)T^*_{i-1}, \Phi_i = (z - 1/2)T^*_{i-1}. \tag{3.7}$$

Here, $T_i^{*'}s\ (i \in n)$ are the modified Chebyshev polynomials. It is seen that the velocity trial function satisfies the boundary conditions but the magnetic potential trial functions do not satisfy their respective boundary conditions. However, the residuals from the temperature and magnetic potential conditions are included as residuals from the differential equations. Equation (3.6) leads to a characteristic equation from which the critical Rayleigh number as a function of wave number a is extracted numerically for various values of physical parameters Da^{-1}, λ, M_1, χ and M_3. The inner products involved in equation (3.6) are evaluated analytically rather than numerically in order to avoid errors in the numerical integration. The critical Rayleigh numberR_c is obtained by minimizing R with respect to the wave number a for different fixed values of other parameters. A convergence is obtained for a sixth order expansion ($i = j = 6$) of the trial functions and the results are found to be in good agreement with those of the limiting cases.

4 Results and discussion

The effect of MFD viscosity on the criterion for the onset of convection in a ferrofluid saturated porous layer heated from below is investigated in the presence of a uniform magnetic field using a non-Darcian model. The rigid conducting boundaries are considered to be paramagnetic with low and high magnetic susceptibilities.

The results obtained for paramagnetic boundary conditions covering a wide range of various physical parameters are illustrated in Figs. 1-6. The variation of critical Rayleigh number R_c and the corresponding critical wave number a_c as a function of MFD viscosity parameter λ is shown in Figs.1(a) and 1(b), respectively for different values of Da^{-1} for $M_3 = 1$ and $M_1 = 1$. From Fig. 1(a), it is seen that the effect of increasing λ is to increase the critical Rayleigh number R_c and thus has a stabilizing effect on the system. This is due to the increase in the viscosity of the ferrofluid with an increase in the strength of the magnetic field which in turn retards the flow and hence convection starts at higher values of the Rayleigh number with increasingλ. Moreover, R_c increases monotonically with increasing Da^{-1} indicating its effect is to delay the onset of ferromagnetic convection in a porous medium. Further, it is seen that the paramagnetic boundaries with large magnetic susceptibility ($1 + \chi = 10^4$) are more stable and least stable for paramagnetic boundaries with low magnetic susceptibility ($\chi = 0$).

From Fig. 1(b) it is evident that, an increase in the value of Da^{-1} and magnetic susceptibilityχ is to increase the critical wave number a_c and thus their effect is to decrease the size of the convection cells. However, there is no variation in a_c with λ irrespective of the values of Da^{-1} . Further, a_c values are higher for large magnetic susceptibility paramagnetic boundaries and least for paramagnetic boundaries with low susceptibility.

In Fig. 2(a), plotted the critical Rayleigh number R_c as a function of MFD viscosity parameter λ for different values of magnetic number M_1 with $M_3 = 1$ and$Da^{-1} = 50$. The figure indicates that increasing M_1 is to make the system more unstable due to increase in the destabilizing magnetic force. Besides, the curves of different M_1 become closer as the value of M_1 increases. Figure 2(b) illustrates that increase in M_1 is to increase a_c and hence its effect is to decrease the size of convection cells.

The effect of increase in the measure of non-linearity of ferrofluid magnetization parameter M_3 is to decrease R_c and thus it has a destabilizing effect on the system. This fact is elucidated in Fig. 3(a) by plotting R_c as a function of λ for different values of M_3 and χ for $M_1 = 1$ and $Da^{-1} = 50$. This figure demonstrates that increasing M_3 has a destabilizing effect on the system. Nevertheless, the destabilization due to increase in the nonlinearity of the fluid magnetization is only marginal. This may be attributed to the fact that a higher value of M_3 would arise either due to a larger pyromagnetic coefficient or larger temperature gradient. Both these factors are conducive for generating a larger gradient in the Kelvin body force field, possibly promoting instability. Increase in the value of M_3 is to decrease the value of a_c and hence its effect is to widen the convection cells and this fact is evident from Fig. 3(b). From Figs. 3(a) and 3(b), it is also observed that the critical Rayleigh number R_c and the corresponding critical wave number $a_c$coincide as$M_3 \to \infty$.

The complementary effects of buoyancy and the magnetic forces are made clear in Fig. 4 by displaying the locus of the critical Rayleigh number R_c and the critical magnetic Rayleigh number $N_c(= R_c M_1)$for different values of non-linearity of magnetization parameter M_3 for $\lambda = 0.02$ and $Da^{-1} = 50$. We note that R_c is inversely proportional toN_c. As$M_3 \to \infty$, for paramagnetic boundary conditions considered, the data fit the following relation exactly

$$\frac{R_c}{R_{c0}} + \frac{N_c}{N_{c0}} = 1 \tag{4.1}$$

as observed in the non-porous case (i.e.,$Da^{-1} = 0$). In the above formula, R_{c0} is the critical Rayleigh number in the non-magnetic case ($N = 0$) and N_{c0} is the critical magnetic Rayleigh number in the non-gravitational case($R = 0$).

Figures 5 and 6 exhibit the variation of critical magnetic Rayleigh numberN_c as a function of MFD viscosity parameter λ for different values of Da^{-1} (when $M_3 = 1$) and M_3 (when $Da^{-1} = 50$) respectively in the absence of buoyancy force ($R = 0$). It is seen that increasing Da^{-1} is to delay, while increasing M_3 is to hasten the onset of ferromagnetic convection.

5 Conclusions

The linear stability theory is used to investigate the onset of ferromagnetic convection in a horizontal Brinkman porous layer heated from below in the presence of a uniform magnetic field by considering variation in viscosity with magnetic field for paramagnetic boundary conditions. The resulting eigenvalue problem is solved numerically by employing the Galerkin method.
Some of the important conclusions derived from the study are listed below:

(a) The system is more stabilizing against the ferromagnetic convection if the boundaries are paramagnetic with large magnetic susceptibility and least stable if the paramagnetic with low magnetic susceptibility.

(b) Increase in the value of Da^{-1} , $\chi$and MFD viscosity parameter λ is to delay the onset of ferromagnetic convection in a ferrofluid saturated porous layer. Whereas, the effect of increasing the value of magnetic number $M_1$and the nonlinearity of magnetization parameter M_3 is to hasten the onset of ferromagnetic convection.

(c) The MFD viscosity parameter λ has no influence on the critical wave number, while the effect of increase in χ, Da^{-1} and M_1as well as decrease in M_3 is to increase a_c and hence their effect is to narrow the convection cells. Also, the critical wave numbers are higher for large magnetic susceptibility paramagnetic boundaries and the least for paramagnetic boundaries with low susceptibility.

(d) As$M_3 \to \infty$, the data fit in to a straight line $R_c/R_{c0} + N_c/N_{c0} = 1$ irrespective of the nature of magnetic boundaries.

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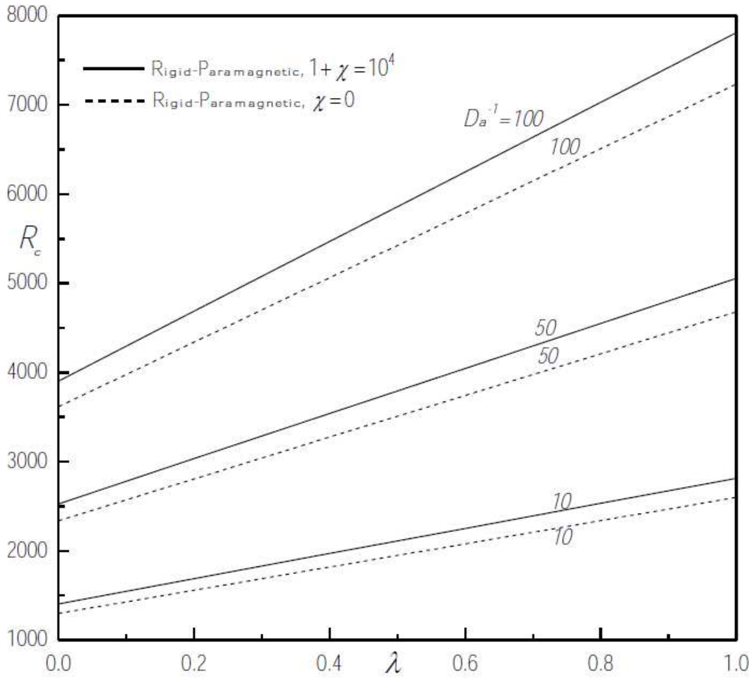

Figure 1(a): Variation of critical Rayleigh number $R_c$as function of λ with three values of Da^{-1} when $M_3 = 1,$ and $M_1 = 2$

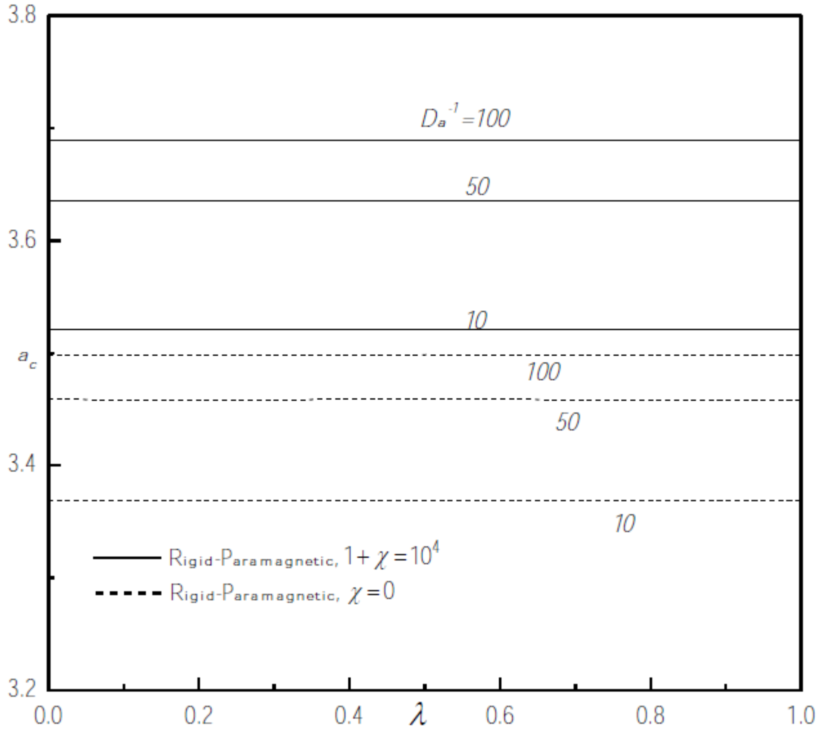

Figure 1(b): Variation of critical wave number $a_c$as function of λ with three values of Da^{-1} when $M_3 = 1,$ and $M_1 = 2$

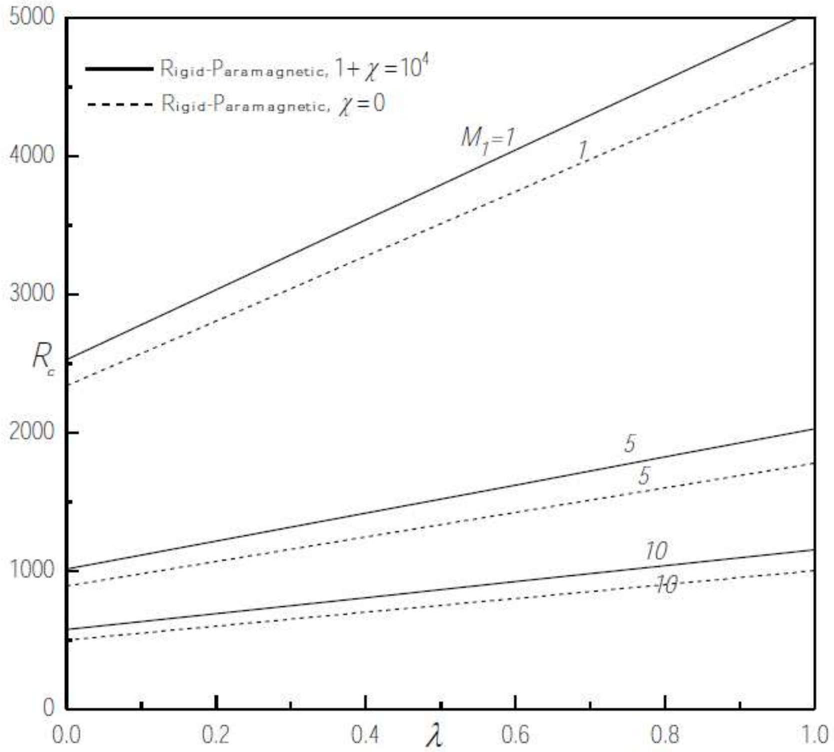

Figure 2(a): Variation of critical Rayleigh number $R_c$as function of λ with three values of M_1 when $M_3 = 1$ and $Da^{-1} = 50$

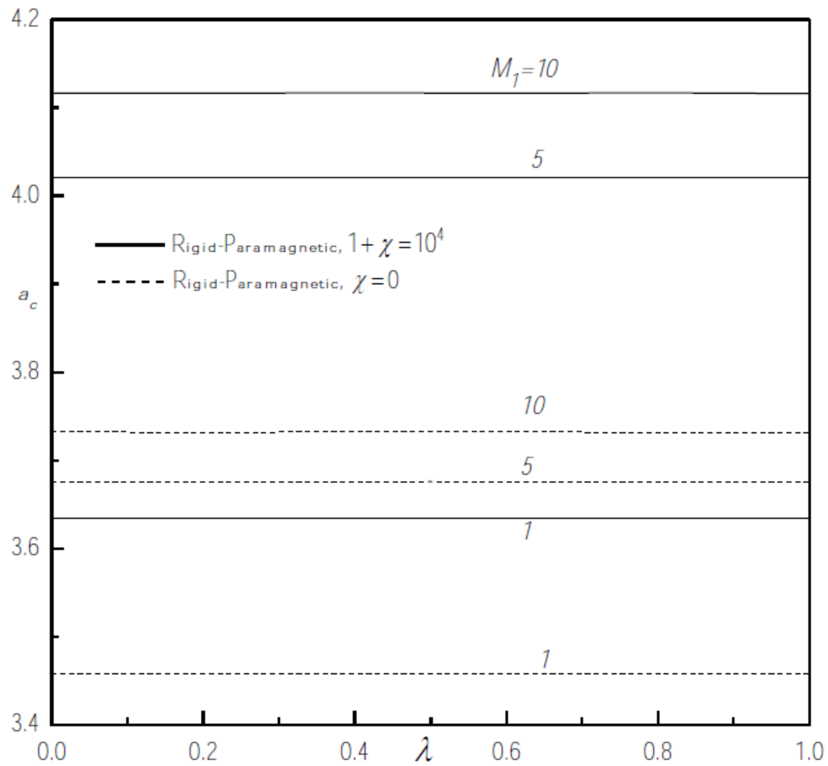

Figure 2(b): Variation of critical wave number $a_c$as function of λ with three values of M_1 when $M_3 = 1$ and $Da^{-1} = 50$

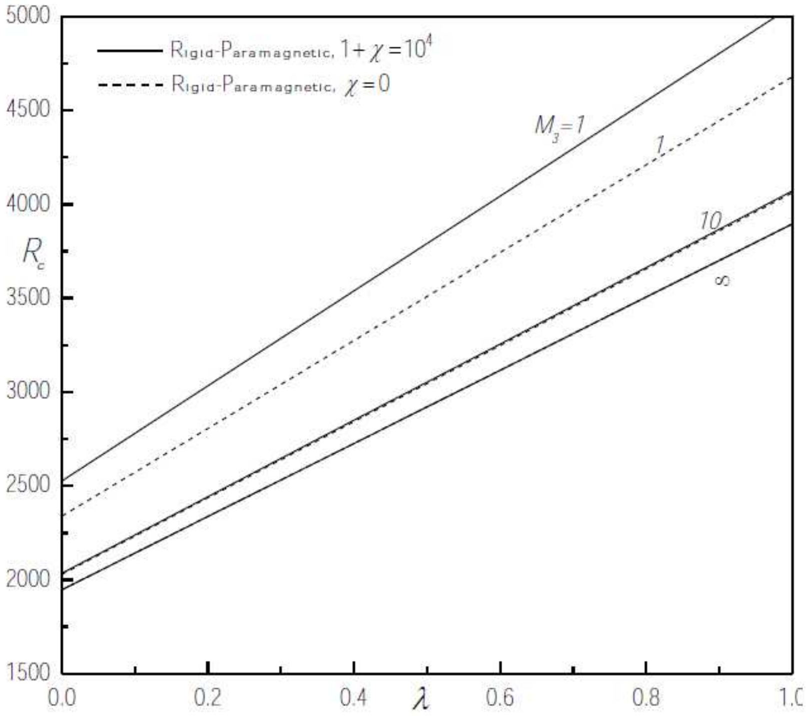

Figure 3(a): Variation of critical Rayleigh number $R_c$as function of λ with three values of M_3 when $M_1 = 1$ and $Da^{-1} = 50$

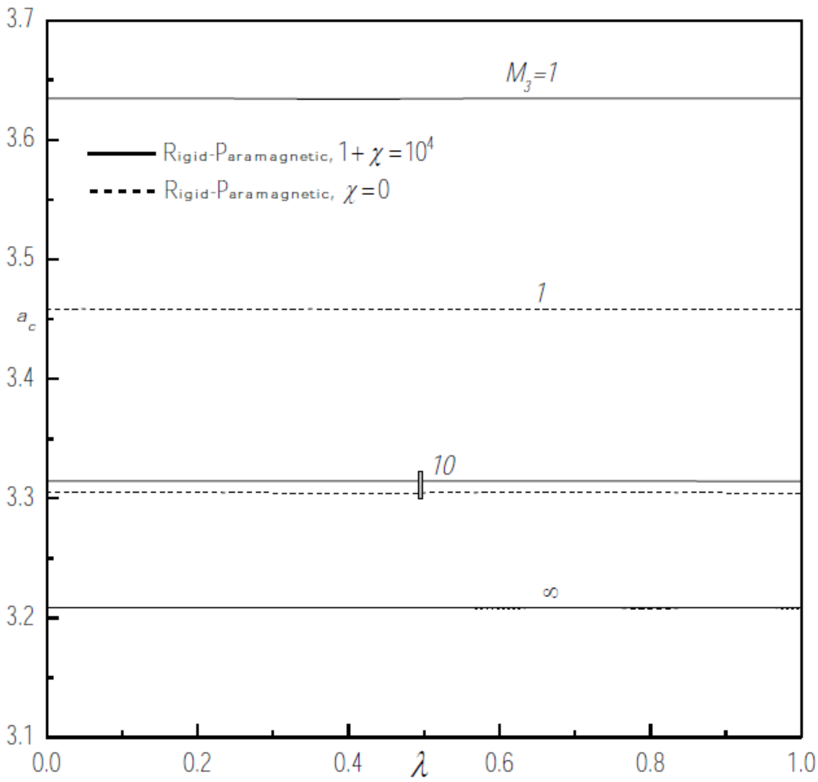

Figure 3(b): Variation of critical wave number $a_c$as function of λ with three values of M_3 when $M_1 = 1$ and $Da^{-1} = 50$

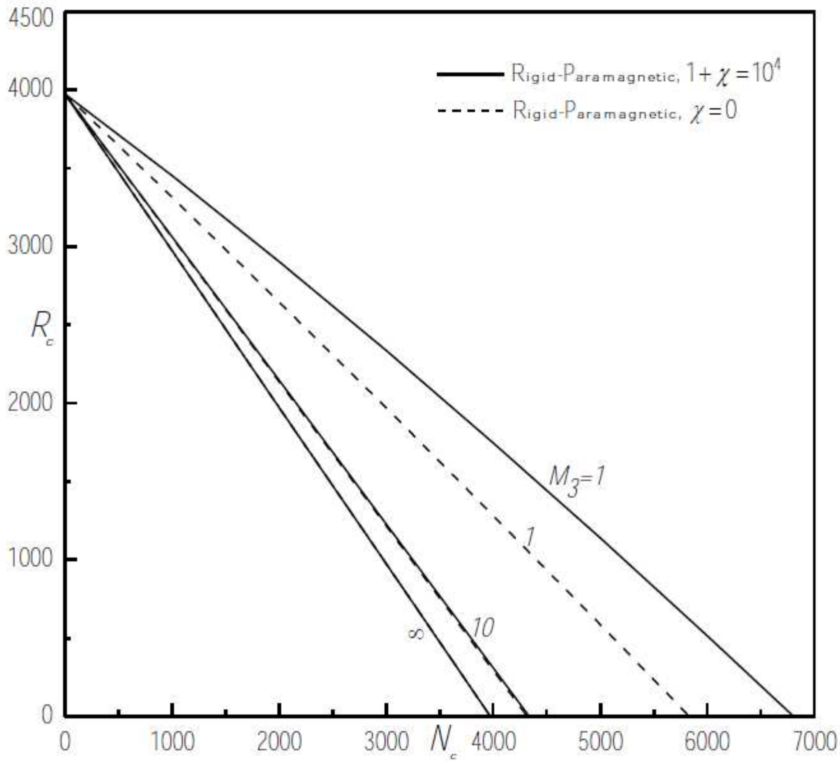

Figure 4: Variation of critical Rayleigh number $R_c$as function of N_c with three values of M_3 when $\lambda = 0.02$ and $Da^{-1} = 50$.

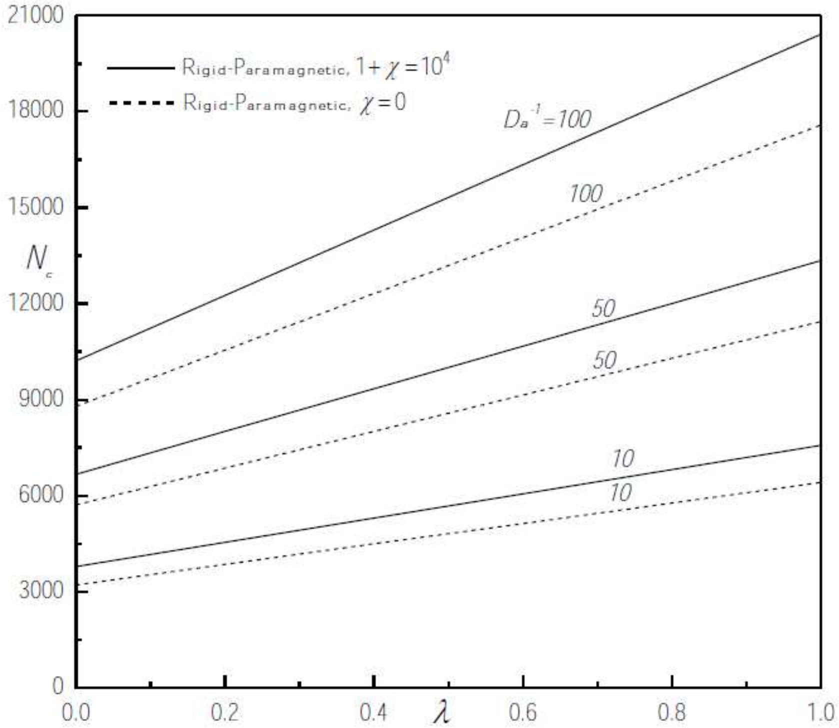

Figure 5: Variation of critical magnetic Rayleigh number $N_c$as function of λ with three values of Da^{-1} when $M_3 = 1$ and $R = 0$(absence of buoyancy)

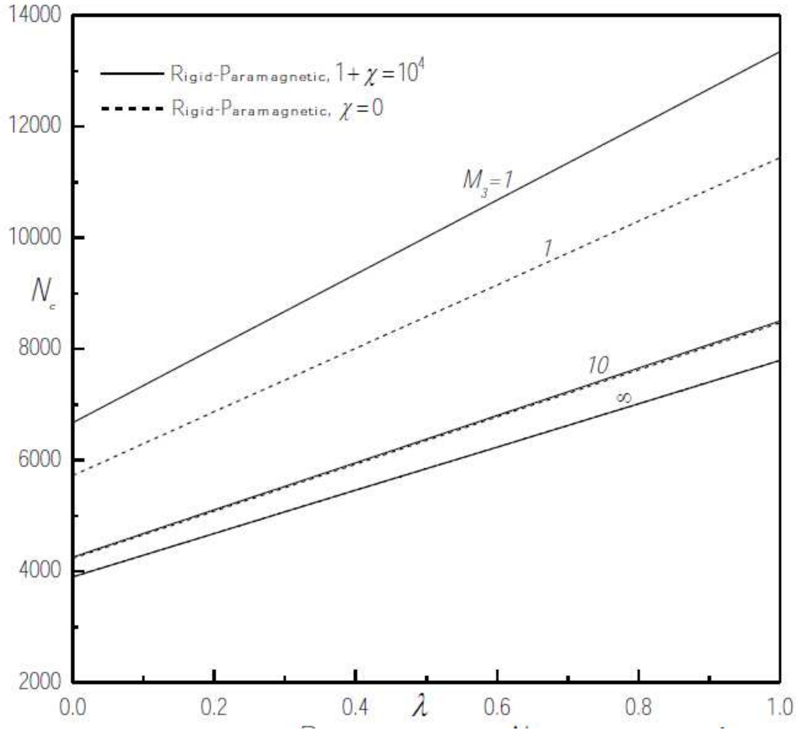

Figure 6: Variation of critical magnetic Rayleigh number $N_c$as function of λ with three values of M_3 when $Da^{-1} = 50$ and $R = 0$(absence of buoyancy)

Mathematical modelling in the academy: what it is and why we face reaction

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Abstract :
Mathematical modelling (MM) is often presented as a unifying perspective on problem solving in 'applied' mathematics, and an essential skill for science, engineering and technology, and even an essential competence in everyday life. As such this perspective can present mathematics as important to education for citizenship and for science, and not just as an academic pursuit: an important factor in overcoming student alienation. It is widely argued that this has significant implications for pedagogy, but also for policy, curriculum and assessment practice. I will present some examples from my own research and experience as to how MM has been implemented (e.g. in schools) and from current work in universities with STEM students. However, I observe that in recent decades there have been limits of the success of the worldwide 'movement' for modelling in mathematics education, with active reactions by 'back to basics' perspectives, and indeed by passive institutional inertia. I argue that to understand this requires a sociological and political perspective on schooling and academe as a cultural field for reproduction of class relations. These underpin certain rightist ideologies and educational positions that have apparently unintended consequences for alienation in schooling and mathematics education.

Introduction

In this paper I aim to

(a) explain what I think 'mathematical modelling' reforms to the curriculum and pedagogy of mathematics are about (what it is), and describe some current work, and

(b) explain where and why the reaction to reform arises (why we face reaction and so often fail).

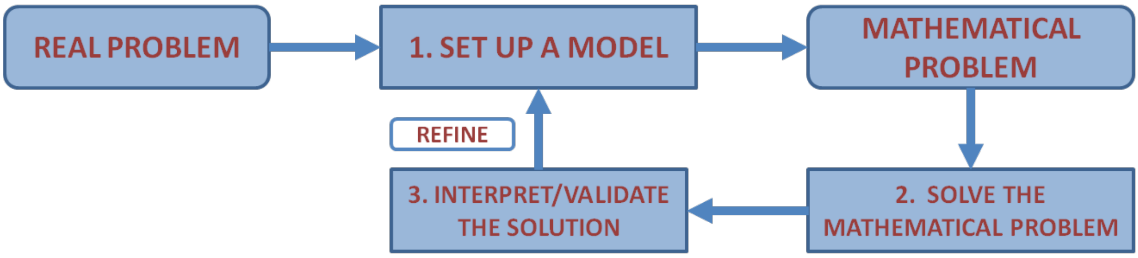
Mathematical modelling as a perspective on unifying mathematics as applied problem solving

There are two dominant perspectives on mathematical modelling in the literature in mathematics education, and they are closely related. The first originated in Higher Education, it is characterised by generalising an approach to applied mathematics, in which the 'mathematics' is seen as providing a set of models of 'real life' and through which mathematical methods solve problems in real situations and tasks arising outside mathematics.Eg presented with a task to find an area of a rectangular-looking field, a 'mathematical model' involving multiplying lengths of sides may solve the problem. The second approach comes from Primary education, where mathematics is seen to be 'modelled' by real objects or representations: here one may solve a problem in mathematics like $5 + 3$ by 'modelling' the numbers with objects, counting 5 and then 3, and then counting the total to get the answer 8. In both cases, one notes that 'modelling' involves some kind of translation between a 'reality' and the abstract mathematics that represents it.

The classic case of Newton's principia might be cited as the quintessence of the first kind of modelling: the problems of explaining planetary motion and tidal effects are brought under one gravitational model using mathematics requiring not only devising the right inverse square model for gravity but also inventing a kind of calculus. On a slightly less illustrious scale, 'everyday folks' and school children required to solve simpler problems may have to choose additive or multiplicative models in the solving of various problems involving household finances. In one sense, mathematics provides many models for problem solving, and selecting the right model may be as important and demanding as actuating and correctly following through on that model.

What then do we mean in this context by 'mathematical modelling'? The art of solving real problems using mathematical models usually requires more than merely selecting an appropriate model and then correct application. In fact the whole process usually is schematised in a diagram such as Figure 1. Given a real problematic situation, the problem solver has to explore and understand the context so as to formulate the problem in an appropriate way. The formulation requires specifying the model: this may involve drawing appropriate diagrams, introducing... variable quantities and parameters/constants, and so writing down equations to be solved. But then the 'mathematics' becomes relevant - a set of equations to be solved say. Then the solution may have to be interpreted in the real context, and there may be processes of validation of the results.

Figure 1



Dan Koko's dive

To take an example from the Higher Education perspective, I have selected the modelling diagram and Dan Koko's dive in the Guinness book of records, which becomes the following task:

> "the greatest height reported for a dive into an air bag is 326feet = 99.36m by Dan Koko from the top of the Vegas World Hotel, on 13th August, 1984... imagine you are the applied mathematician on the technical support: can you set up a model for his likely motion, e.g. one that allows you to predict the time of fall, and the impact speed with the airbag, etc ..." (See Savage & Williams,1990)

Such 'real problems' become particularly interesting when the most elementary models lead to solutions that then have to be tested with real data. In this case there is data on the time of flight, which contradicts the 'free fall under gravity' model which provides an underestimate of the time of flight. In many cases experiments might need to be conducted to test models and assumptions and build in improvements or model refinements.

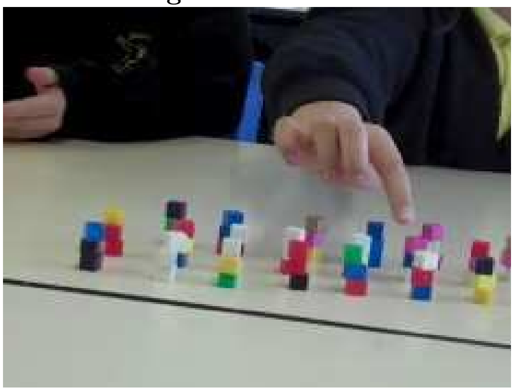
There has been a growing literature on this kind of modelling at the upper phases of mathematics education through the ICTMA conferences and proceedings: and this work continues. Justone remark for now: my own involvement in this work (see for instance Savage & Williams, 1990) led to some major programmes in England (e.g. the SMP 16-19 and MEI A level texts and supporting materials) which came to almost nothing ten years later.

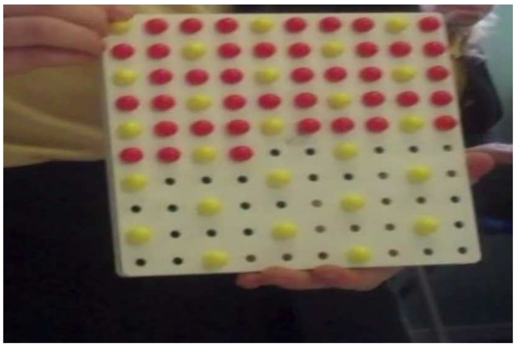
Modelling in early years

On the other hand one sees the second kind of modelling in Primary schools and some lower secondary schooling: I take this following account from a recent study in one Primary schoolby myself and colleagues (reported in Ryan & Williams, 2013).

Using a lesson study approach (see APEC website, `http://hrd.apec.org/index.php/Lesson_Study`) a team of researchers and teachers had developed a lesson asking the children to show using various models that $28 \times 3 = 84$. The problem had originally arisen from calculating the number of drinks required for a Christmas party based on the class size of 28 and three drinks per child. But the transformed problem was designed to provoke explanations using models/equipment and diagrams/reasoning. One group's spontaneous models are shown in the two photos below.

Figure 2 : some of the childrens models used to explain why $28 \times 3 = 84$





Here the children made visible the way they thought about the calculation: the right hand board (incomplete) in particular shows how the pair of children had used yellow beads to represent the 28 children, and then assigned 3 red beads for each drink the child would need. We were able then to tie this spontaneous problem solving of the children to the modelling of multiplication using array models.

The most mature and theoretically supported R& D development of this approach internationally springs from the work of Freudenthal (1991), where mathematics is understood as developing through the combination of spontaneous problem solving in situations that 'beg to be organised' in certain mathematical ways.

But note in this approach that the real situations encountered are there to motivate mathematics, and so the art is to select or design situations that beg to be organised mathematically and so support mathematics education. This is the subject of empirical work, and this is what we pursue in lesson study research (see Williams et al., 2013).

On the other hand the 'applied approach' involves a mixture of motivations to support the solution to problems 'outside mathematics' as well as to motivate the growth of mathematics itself. Yet the applied approach may remain entirely 'uncritical' in the general sense. Lets now explore this possibility of becoming critical.

When modelling gets critical

We now explore how modelling in this sense can become an important and socially significant element in teaching and learning from a critical mathematics education (CME) perspective. Critical Maths involves situating mathematics education and problem solving in ways that involve 'meaningful' problem solving, especially when meaning involves becoming socially critical. Becoming critical can involve at least two distinct processes: on the one hand becoming critical of the automatic (uncritical) acceptance of models that are offered in the curriculum. On the other hand it can involve addressing critical social questions leading to an education in the general sense of expanding consciousness of social, citizenship, and cultural questions/issues.

I cite this example as an example of a non-real problem, or at least a non-critical problem. But I will transform it into a more interesting task below.

Im imagining the former question might be estimated literally allowing each person to lie down in a space 1m by $2m$ . . . so assume Island is 20km by 10km rectangle, that's 200 million square metres, so 100 million people in 1m by 2m 'bunks'. In practice the students tackling this task became involved in all sorts of side issues, e.g. carefully scrutinising maps of the island (off the south coast of England!) and calculating in detail the dimensions involved. The problem here is we have no real context, and no real 'common sense' with which to make estimations that one could judge reasonable . . . lets refine the model and make it more realistic, or lets vary the assumptions in sensible ways and see how robust the answer is to variations in assumptions.

Thus: how many people could you reasonably plan to get into an open air music festival on the Isle of White? How many toilets/doctors/bottles of water etc would be needed to support such a crowd for several days . . . and how much can the organisers invest with an expectation of profit?

A festival cannot use the full 20 Km by 10 Km of the island, as the music wont be heard, and the bands cant be seen. Rather it may be assumed there is a limit beyond which visibility makes it 'not really a festival'. How big? Say 100 to 200m (how far can loud music be heard and seen. anybody? Then maybe assume it's a third of a circle with the stage at the apex (geometrical assumptions) of a circle,. . . so we have a seating/standing area of say 1/3 x pi x (100-200) squared, ie 10 000 40 000 sq metres. Then we might pack them in one to a square metre (say between 1 and 3 to a square metre?) which gives an audience between 10 000 and 120 000 people. Does this sound realistic?

Google "music festival attendances" and you will find headline figures between 10 000 and 177 000 people at the biggest outdoor festivals. . . this suggests a closer look at the assumptions we made. How sensitive is the answer to variation/error in the estimates in the assumptions? Then what about the other parts of the question. . .

Socially critical mathematics: Bayesian models

For my final example I look to an example from a new curriculum development we call "Critical maths". The aim is to present very real problems of social interest where mathematics really makes a difference to one's understanding of the world. For example:

"The chances of DNA being matched between two people is approximately "one in a million": assuming a man in the police data base was matched to a crime scene, and there was no other evidence, what were the chances the matched person was guilty should you convict them?"

Assuming the police database has about a million people on it, one might expect to find a match by chance. But in a total population of say 40 million we might expect to find 40 such matches: thus the chance of the accused person being the guilty person is about one in forty. This is quite counterintuitive for most groups of students and is characteristic of a whole class of such problems, involving screening. It also links to some high profile cases of injustice where expert witnesses misled juries to convict. (Incidentally, a relevant miscarriage of justice of this kind has recently become well publicised in the UK).

Another example being trialled currently involves random screening for 'rare medical conditions' like cancer: suppose rare means one in a thousand of a population will get the condition, but that a test is only 98% accurate: how worried should you be if you are identified as a 'positive' on the test? Thinking critically (imagine 10 000 people in this population, etc.) might lead you to calculate your chances of being a false positive as very good. This shows how important the 'accuracy of the test' is in relation to the rarity of the condition (and hence implicates the ages of the populations being screened). This issue remains one of public, political and media concern in the UK to the present day. Thus this curriculum presents mathematics as an important tool in helping to 'slow down' intuitive decision making with rational science,including social science. The Nobel prize winning theorist of decision-making, Daniel Kahnemann and others work in the area is now becoming well enough known to enter curricula: see "Thinking fast, and slow." (Kahnemann, 2011)

But its hidden curriculum is perhaps more significant: it aims to teach students to think about social situations critically, using data and mathematical reasoning to question intuitive and prejudiced judgments. This is perhaps dangerous thinking.

Implications for pedagogy, teacher education, assessment.

There are well-researched problems with this kind of curriculum work in practice in schools and Colleges/universities: many of these problems are relevant to most curriculum change, especially when they challenge traditional assessment practices, cost money/time, or involve 'old dogs learning new tricks'. Here are some key problems:

(a) the teachers knowledge has been honed to the traditional methods and the new ways demand new knowledge, threatening teachers sense of pedagogic efficacy;

(b) modelling and problem solving may require greater access to information and technology (eg problems/data for models);

(c) the assessment approach that best suits modelling and problem solving (eg using projects and coursework involves a major investment in training for 'fair' assessment, and a trust in the teachers and students (eg corruption) .

All three of these key points have been tackled in major Research and Development projects round the world and they are not impossible to solve. Nevertheless we have seen in a number of cases that these innovations take time and resource over long periods of time. They have also often been set back, overturned or eradicated very quickly in a number of cases, egwhen the politics of mistrust have undermined them (see Williams & Goos (2013) and our appendix in the Royal Society reportto be published online in Howes et al., due on their website in 2014).

Politics of critical mathematics education, and modelling

In this section I set out to understand the education system and its politics in order to explain such reactions to innovation. At first sight it seems perplexing that a hard-won educational innovation that engages learners apparently so effectively can be destroyed so easily, almost at the whim or signature of one key bureaucrat or minister. One surely expects the pressure for the education system to progress to win out over reaction in the long run a modernist perspective?

I look to Bourdieu and his colleagues and followers for a different perspective on the system: influenced by Althusser's argument that capitalism would not survive even one more day of the education (and other cultural elements of the) system did not keep a continuous supply of suitably trained and compliant workers being fed onto the labour market. Thus, the prime function of schooling and education must be seen in relation to the wider system, and this external function must be understood in order to understand the internal structure and functioning of schooling (Bourdieu & Passeron, 1990).

They argue that the key structure of the education system is its capacity to distribute and stratify, and especially to fail students. But the students' failures must be 'misrecognised' as being of their own making, as a natural result of education 'not being for the likes of us', rather than an act of class exclusion. The assessment and examination system is therefore key to the process: it has to be understood as being equitable and fair in order for its work of 'symbolic violence' to be effective. It comes as no surprise then that the assessment systemoften proves the greatest barrier to critical education, and reform in general. It survives in its traditional form almost all curriculum development and innovation, and is the part of the system most vulnerable to reactionary political interference, of the kind currently under way in England in our present ‘back to basics’ reaction.

Conclusions

The purpose of the paper was threefold: first to introduce the ideas of mathematical modelling in mathematics education, to identify some aims and some examples; second to examine the practical obstacles and issues faced in overcoming systemic inertia or opposition to introducing modelling in curriculum and pedagogy; and third to examine the socio-political basis of the opposition to introducing modelling and other critical forms of education.

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Constructivism: a way of knowing and learning in mathematics education.

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Abstract : Many of the current attempts to develop theoreies in Mathematics Education reflect the view that learning is a process of constructing internal relationships. As teachers and school administrators struggle to increase student achievement in mathematics, the recommendatoins on mathematics instruction from National Council of Teachers of Mathematics (2000) require a major shift in Philosophy as well as practice. This shift signifies a move from objectivist based to constructivist practices on student achievement in Mathematics and Traditional method of mathematics. The data indicates that constructivist based instructions is beleived to be an effective means of increasing student understanding of mathematical skills and concepts and therefore should be effective in increasing student achievements in mathematics.

1 Introduction

A constructivist approach in education has been developed on the basis of the paradigm shift from instruction to active, conscious, purposeful learning. Self-constructed acquiring of knowledge, skills and competence is in the center of the approach. The essence of the constructivist approach is reflected in the following features: motivated, meaningful learning based on a previous experience; activity; self-constructed knowledge, skills and competence; a teacher as a facilitator and counselor; positive and supportive learning environment. Therefore the constructivists support a perspective that for each individual the most important is knowledge which is developed meaningfully by the individual based on his/her experience. Constructivists do not support imposition of knowledge when a student is a passive receiver of information. Therefore the aim of the article is to substantiate the principles of the constructivist approach . Cognitive strategies of teaching/learning students learning to control their own learning and enhance student achievement, and develop students ability to learn independently.

Constructivism is a theory of learning which claims that students construct knowledge rather than merely receive and store knowledge transmitted by the teacher. Teaching techniques derived from the theory of constructivism are supposed to be more successful than traditional techniques, because they explicitly address the inevitable process of knowledge construction.

This article is logically divided into two parts. The **first** part-after a motivating example-is a survey of the theory of constructivism in Mathematics Education. The **second** part of the paper contains the investigators attempts to apply this theory in Mathematics Education.

Focus on learning is needed if teachers are to implement a constructive approach to thinking about day-to-day learning by the students. Conventional lesson planning focuses on what the teacher will do. If learning is teacher directed, then the focus of the lesson plan is on what the teacher does. When designing a learning experience for students, teachers focus on what students will do. Our language encourages teachers to focus on thinking about how to organize what learners will do rather than plan their teaching behaviors.

2 Constructivist learning

Constructivist learning has emerged as a prominent approach to teaching during this past decade. The work of Dewey, Montessori, Piaget, Bruner, and Vygotsky among others provide historical precedents for constructivist learning theory. Constructivism represents a paradigm shift from education based on behaviorism to education based on cognitive theory. Behaviorist epistemology focuses on intelligence, domains of objectives, levels of knowledge, and reinforcement. Constructivist epistemology assumes that learners construct their own knowledge on the basis of interaction with their environment. Four epistemological assumptions are at the heart of what we refer to as "constructivist learning."

(a) Knowledge is physically constructed by learners who are involved in active learning.

(b) Knowledge is symbolically constructed by learners who are making their own representations of action;

(c) Knowledge is socially constructed by learners who convey their meaning making to others;

(d) Knowledge is theoretically constructed by learners who try to explain things they don't completely understand.

Most of the time, students explained that physical education, fine arts, or industrial arts were their most interesting classes because they actually got to do something. They were active participants in learning rather than passive recipients of information. This is the primary message of constructivism; students who are engaged in active learning are making their own meaning and constructing their own knowledge in the process.

3 Constructivist Learning Design

The "Constructive Learning Design" emphasizes these six important elements:

(a) **Situation**

(b) **Groupings**

(c) **Bridge**

(d) **Questions**

(e) **Exhibi**

(f) **Reflections**

These elements are designed to provoke teacher planning and reflection about the process of student learning. Teachers develop the **situation** for students to explain, select a process for **groupings** of materials and students, build a **bridge** between what students already know and what they want them to learn, anticipate **questions** to ask and answer without giving away an explanation, encourage students to **exhibit** a record of their thinking by sharing it with others, and solicit students' **reflections** about their learning. We now longer refer to objectives, outcomes, or results since we expect that teachers have that determined by the district curriculum or the textbook they are using in their classroom and need to think more about accomplishing it than about writing it again. This brief overview above indicates how each of these **six** elements integrate and work as a whole, but all need further explanation:

(a) **Situation:** What situation are you going to arrange for students to explain? Give this situation a title and describe a process of solving problems, answering questions, creating metaphors, making decisions, drawing conclusions, or setting goals. This situation should include what you expect the students to do and how students will make their own meaning.

(b) **Groupings:** There are two categories of groupings:

   (a) How are you going to make groupings of students; as a whole class, individuals, in collaborative thinking teams of two, three, four, five, six or more, and what process will you use to group them; counting off, choosing a color or piece of fruit, or similar clothing? This depends upon the situation you design and the materials you have available to you.

   (b) How are you going to arrange groupings of materials that students will use to explain the situation by physical modeling, graphically representing, numerically describing, or individually writing about their collective experience. How many sets of materials you have will often determine the numbers of student groups you will form.

(c) **Bridge:** This is an initial activity intended to determine students' prior knowledge and to build a "bridge" between what they already know and what they might learn by explaining the situation. This might involve such things as giving them a simple problem to solve, having a whole class discussion, playing a game, or making lists. Sometimes this is best done before students are in groups and sometimes after they are grouped. You need to think about what is appropriate.

(d) **Questions:** Questions could take place during each element of the Learning Design. What guiding questions will you use to introduce the situation, to arrange the groupings, to set up the bridge, to keep active learning going, to prompt exhibits, and to encourage reflections? You also need to anticipate questions from students and frame other questions to encourage them to explain their thinking and to support them in continuing to think for themselves.

(e) **Exhibit:** This involves having students make an exhibit for others of whatever record they made to record their thinking as they were explaining the situation. This could include writing a description on cards and giving a verbal presentation, making a graph, chart, or other visual representation, acting out or role playing their impressions, constructing a physical representation with models, and making a video tape, photographs, or audio tape for display.

(f) **Reflections:** These are the students' reflections of what they thought about while explaining the situation and then saw the exhibits from others. They would include what students remember from their thought process about feelings in their spirit, images in their imagination, and languages in their internal dialogue. What attitudes, skills, and concepts will students take out the door? What did students learn today that they won't forget tomorrow? What did they know before; what did they want to know; and what did they learn?

4 Assessment

Assessment becomes an integral part of every step in this learning design. Teachers design the **situation** based on their assessment of students' learning approaches, interests, and needs. Teachers design a process for **groupings** based on their assessment of materials of available and desired mixture of students. Teachers design a simple assessment of what students already know as a **bridge** to what they want students to learn. Teachers design **questions** to assess student understanding of the concepts, skills, or attitudes they are trying to learn. Teachers arrange an **exhibit** for students to record what they thought and submit it to others for assessment. Teachers arrange for **reflections** about what students' have learned and their internal process of representations as a context for self-assessment of individual learning.

There is no doubt that constructivist approach is the right educational tool in Mathematics education for professional practice in the post industrial world. It is likely to re-define professional engineering discourse and the focus on the process leading to the raising of questions rather than convergent problem solving is more likely to trigger critical attitudes.

Traditional Classroom	Constructivist Classroom
Curriculum begins with the parts of the whole. Emphasizes basic skills.	Curriculum emphasizes big concepts, beginning with the whole and expanding to include the parts
Strict adherence to fixed curriculum is highly valued.	Pursuit of student questions and interests is valued.
Materials are primarily textbooks and workbooks.	Materials include primary sources of material and manipulative materials.
Learning is based on repetition.	Learning is interactive, building on what the student already knows.
Teachers disseminate information to students; students are recipients of knowledge.	Teachers have a dialogue with students, helping students construct their own knowledge.
Teacher's role is directive, rooted in authority.	Teacher's role is interactive, rooted in negotiation.
Assessment is through testing, correct answers.	Assessment includes student works, observations, and points of view, as well as tests. Process is as important as product.
Knowledge is seen as inert.	Knowledge is seen as dynamic, ever changing with our experiences.
Students work primarily alone.	Students work primarily in groups.

This study is focused on mathematics education because of its universal importance in Engineering. Therefore the aim of the article is to substantiate the principles of the constructivist approach in Science and Technology.. The investigator find the relationship between the Constructivist approach and the Traditional approach in Mathematics Education. The investigation is to examine Constructivist approach is better than Traditional method .

Often, Twenty five Students are taught in a traditional teacher-centered style. The traditional classroom can sometimes resemble a one-person show with a captive but largely uninvolved audience. Classes are usually dominated by lecture or direct instruction. The idea is that there is a fixed body of knowledge that the student must come to know. Students are expected to blindly accept the information they are given without questioning the instructor . The teacher seeks to transfer thoughts and meanings to the passive student leaving little room for student-initiated questions, independent thought or interaction between students . Even in the class , although done in a group, do not encourage discussion or appreciation . This teacher-centered method of teaching also assumes that all students have the same level of background knowledge in the subject matter and are able to absorb the material at the same pace:

In contrast, constructivist or student-centered learning poses a question to the students, who then work together in small groups to discover one or more solutions . Students play an active role in carrying out experiments and reaching their own conclusions. Teachers assist the students in developing new insights and connecting them with previous knowledge, but leave the discovery and discussion to the student groups . Questions are posed to the class and student teams work together to discuss and reach agreement on their answers, which are then shared with the entire class. Students are able to develop their own understanding of the subject matter based on previous knowledge, and can correct any misconceptions they have.

Descriptive Statistics	Traditional Approach	Constructivist Approach	t- value
Mean	15.1	44.9	
Standard Deviation	1.95	14.3	-3.1356

Test scores were compared using a t-test for comparison of means. Average scores from all students in each class were compared. The results of this test showed a significant difference between the constructivist class and the traditional class, with the constructivist group displaying higher scores .The achievement scores for the constructivist group were higher than those for the traditional group.

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Nonlinear mixed electroconvection in a poorly conducting fluid saturated vertical porous channel

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Abstract : Nonlinear Mixedelectroconvection(MEC) in a poorly conducting fluid saturated vertical porous channel is investigated in the presence of an electric field, Viscous and Ohmic dissipations. The nonlinear coupled Darcy-Lapwood-Brinkman and energy equations governing the flow are solved numerically using finite difference technique and analytically using regular perturbation method with B_r as the perturbation parameter using different boundary conditions on temperature. The velocity and temperature fields are obtained for various values of electric number, Brinkman number and GR which is the ratio of Grashof number to the Reynolds number. We note that, if GR=0, the problem reduces to the forced convection. We have computed the local Nusselt number at the walls for three types of thermal boundary conditions. The analytical solutions are compared with those obtained from the numerical solution and good agreement is found. The results obtained for velocity, temperature, skin friction, heat transfer and mass flow rate are graphically represented and found the effect of increase in the temperature difference between the plates is to increase the velocity and temperature distributions due to increase in convection. The effect of perturbation parameter(B_r) is shown to decrease the skin friction and heat transfer and increase the mass flow rate. These results are useful in the effective control of heat transfer in many industrial problems.

Keywords: Electrohydrodynamics, Heat transfer, Mixed-Oberbeck Convection, Mass flow rate, Nonlinear, Porous channel, Skin friction, Vertical plates.

1 Introduction

We know that the mixed convection in a vertical channel in the absence of porous medium have been extensively investigated in the literature. We note that convective heat transfer in porous media has attracted a great deal of attention during the last five decades in view of its several distinct advantageous in transport process in modern technologies. Its occurs, for example, in geothermal systems, underground heat exchangers for energy storage, recovery and temperature controlled reactors, cooling of electronic systems, oil recovery, processes in petroleum reservoirs, ground water hydrology, coal combustors, grain storage, fiber and granular insulation, thermal insulation engineering, food processing, pollutant dispersion in aquifers, fibrous insulation to name just a few applications of the topic of convective flow in porous media. Several excellent monographs summarizing the state-of-the –art are available in the literature which testify the utility of this area (See Nield and Bejan(1984), Ingham and Pop (1998, 2002, 2005), Vafai (2000, 2005), Pop and Ingham (2001), Ingham et al (2004), Bejan et al (2004) and Kohr and Pop (2004)).Free convective boundary layer over a vertical cylinder embedded in a porous medium has also been extensively studied in the literature during the recent years (See Rudraiah et al 1982, 1977). This is the principal mode of heat transfer in numerous applications such as in oil/gas lines, insulation of horizontal pipes, cryogenics as well as in the context of water distribution lines, underground electrical power transmission lines and burial of nuclear waste and so on. Aung and Worku (1987) have discussed the theory of combined free and forced convection in a vertical channel with flow reversal conditions for both developing and fully developed flows.

Many analysis of combined forced and free convection flow in a parallel plate vertical channel are available in the literature, but mixed convection(MC) in vertical porous channel in the presence of an applied electric field considering the dissipations has not been given any attention. The study of it is the main objective of this paper.

To achieve the objective of this paper it is planned as follows. Mathematical formulation of the problem together with the required boundary conditions are given in section 2.In section 3, the solutions for electrical potential is obtained when the potential difference applied either in the same direction of opposing the temperature difference. Analytical solutions for velocity and temperature distributions are obtained in section 4.
Numerical solutions are obtained in section 5. Section 6 is derived to skin friction, rate of heat transfer and mass flow rate. Results and discussions are given in section 7.

2 Formulation of the problem

We consider a steady, laminar, fully developed mixed convection in a poorly conducting flow in an open-ended vertical parallel plate channel filled with a porous medium. The poorly conducting fluid is assumed to be Newtonian and the porous medium is homogeneous and isotropic. The Oberbeck-Boussinesq approximation is assumed to hold. In other words, we consider Oberbeck-Boussinesq, homogeneous, poorly conducting fluid, together with electrohydrodynamic (EHD) approximations (See Rudraiah et al, 1996), namely σ is very small and hence induced magnetic field is negligible and there is no applied magnetic field. It is assumed that the thermal conductivity, the dynamic viscosity and the thermal expansion coefficient are all constant. The X-axis is chosen parallel to the gravitational field, but with opposite direction and Y-axis is transverse to the plates. The origin is such that the channel walls are at positions $Y = 0$ and $Y = 2L$, respectively as shown in figure1.

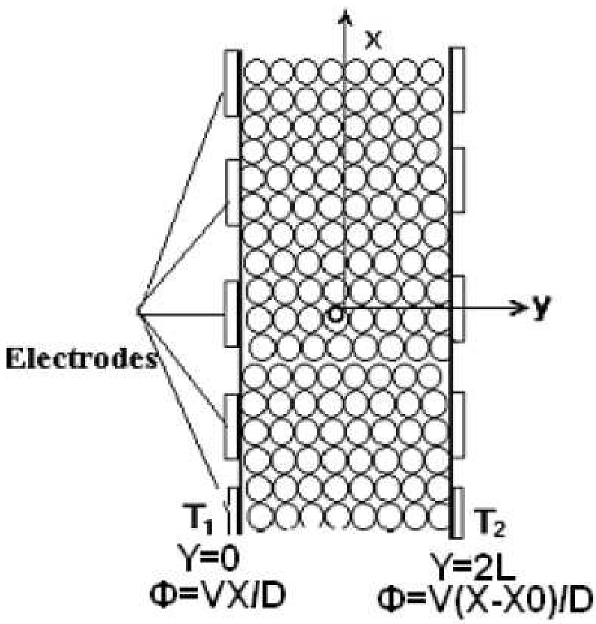


Figure 1 : Physical Configuration

The basic equations for two-dimensional incompressible poorly conducting Boussinesq viscous fluid are

Conservation of Mass

$$\nabla.\vec{q} = 0 \tag{2.1}$$

Equation of State

$$\rho = \rho_0 \left[1 - \beta\left(T - T_0\right)\right] \tag{2.2}$$

Conservation on Momentum

$$\rho_0 \left[\frac{\partial \vec{q}}{\partial t} + \left(\vec{q}.\nabla\right)\vec{q}\right] = -\nabla p + \rho\vec{g} + \bar{\mu}\nabla^2\vec{q} - \frac{\mu}{k}\vec{q} + \rho_e\vec{E} \tag{2.3}$$

Conservation of Energy

$$M\frac{\partial T}{\partial t} + \left(\vec{q}.\nabla\right)T = \kappa\nabla^2 T + \frac{\Phi}{\left(\rho C_p\right)_f} + \frac{\mu}{k(\rho C_p)_f}\left|\vec{q}\right|^2 \tag{2.4}$$

Where $\Phi = \bar{\mu}\left|\nabla\vec{q}\right|^2 + \frac{J^2}{\sigma}$

These equations are supplement with the continuity of charge

$$\frac{\partial \rho_e}{\partial t} + \left(\vec{q}.\nabla\right)\rho_e + \nabla.\vec{J} = 0 \tag{2.5}$$

togerher with Maxwells equations

$$\nabla.\vec{E} = \frac{\rho_e}{\varepsilon_0}, \quad \vec{E} = -\nabla\phi, \quad \vec{J} = \sigma\vec{E}, \sigma = \sigma_0\left[1 + \alpha_b\left(T - T_0\right)\right] \tag{2.6}$$

We also assume a fully developed flow for which, the mass balance equation(2.1) will becomes $\frac{\partial U}{\partial X} = 0$, so that U depends only on Y, where U is the velocity along the X-axis.

Then the above basic equations (2.3) to (2.5) for steady flow using Boussinesq approximation reduce to the required basic equations for steady flow are

$$g\beta\left(T - T_0\right) - \frac{1}{\rho_0}\frac{\partial P}{\partial X} + \bar{\nu}\frac{\mathrm{d}^2 U}{\mathrm{d}Y^2} - \frac{\nu}{k}U + \frac{\rho_e E_x}{\rho_0} = 0 \tag{2.7}$$

$$\chi\frac{\mathrm{d}^2T}{\mathrm{d}Y^2}+\frac{\bar{\mu}}{(\rho c_p)_f}\left(\frac{\mathrm{d}U}{\mathrm{d}Y}\right)^2+\frac{\sigma\left|\vec{E}\right|^2}{(\rho c_p)_f}+\frac{\mu}{k\,(\rho c_p)_f}U^2=0 \tag{2.8}$$

$$\sigma\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}X^2}+\frac{\mathrm{d}^2\varphi}{\mathrm{d}Y^2}\right)+\frac{\mathrm{d}\varphi}{\mathrm{d}Y}.\frac{\mathrm{d}\sigma}{\mathrm{d}Y}=0 \tag{2.9}$$

The physical quantities are defined in the nomenclature and $P=[p+\rho_0 gx]$ is the difference between the pressure and the hydrostatic pressure.

We assume that the temperature of the boundary at $Y=0$ is T_1 while that at $Y=2L$ is T_2 with $T_2\geq T_1$ and $\frac{\mathrm{d}P}{\mathrm{d}X}$ is independent of X and hence we assume that

$$\frac{\mathrm{d}P}{\mathrm{d}X}=A \tag{2.10}$$

where A is a constant.
Taking the derivative of equation (2.7) with respect to X and using (2.10), and the boundary conditions on E are such that $\rho_e E_x$ is independent of X, we get $\frac{\mathrm{d}T}{\mathrm{d}X}=0$.

This implies that the temperature is also depends only on the variable Y. The energy eqn (2.8) balances the effect of conduction with those of viscous and ohmic dissipations.
Equation (2.7), using (2.8) and (2.10), becomes

$$\frac{\mathrm{d}^4U}{\mathrm{d}Y^4}-\frac{g\beta}{\chi\,(c_p)_f}\left(\frac{\mathrm{d}U}{\mathrm{d}Y}\right)^2-\frac{g\beta}{\bar{\nu}K}\sigma\left|\vec{E}\right|^2-\frac{g\beta\mu}{\bar{\nu}Kk}U^2-\frac{\nu}{\bar{\nu}k}\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}+\frac{1}{\bar{\mu}}\frac{\mathrm{d}^2}{\mathrm{d}Y^2}(\rho_e E_x) \tag{2.11}$$

2.1 Boundary conditions

The fluid is bounded by rigid impermeable boundaries and hence the following no-slip boundary conditions are valid

$$U(0)=U(2L)=0 \tag{2.12}$$

Using (2.7) and (2.10), together with the boundary conditions on T, we obtain

$$\begin{aligned}\left(\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right)_{y=0}&=\frac{A}{\bar{\mu}}-\frac{g\beta\,(T_1-T_0)}{\bar{\nu}}-\frac{\rho_e E_x}{\bar{\mu}}\text{ at }Y=0\\ \left(\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right)_{y=2L}&=\frac{A}{\bar{\mu}}-\frac{g\beta\,(T_2-T_0)}{\bar{\nu}}-\frac{\rho_e E_x}{\bar{\mu}}\text{ at }Y=2L\end{aligned} \tag{2.13}$$

and isothermal conditions on T are

$$\begin{aligned}T&=T_1\text{ at }Y=0\\ T&=T_2\text{ at }Y=2L\end{aligned} \tag{2.14}$$

$$\begin{aligned}\phi&=VX/D\text{ at }Y=0\\ \phi&=V\,(X-X_0)/D\text{ at }Y=2L\end{aligned} \tag{2.15}$$

Equations (2.7) to (2.9) and (2.12) to (2.15) are made dimensionless using the quantities

$$u^*=\frac{U}{U_0},\ \theta=\frac{T-T_0}{\Delta T},\ y^*=\frac{Y}{\mathrm{d}},\ E^*=\frac{E}{V/D},\ x^*=\frac{X}{\mathrm{d}},\ \rho_e^*=\frac{\rho_e}{\varepsilon_0 V/D^2},\ \phi^*=\frac{\phi}{V} \tag{2.16}$$

where $D=2L$ is the hydraulic diameter. The reference velocity U_0 and the reference temperature T_0 are given by

$$U_0=-\frac{AD^2}{48\mu};\ T_0=\frac{T_1+T_2}{2} \tag{2.17}$$

The physical meanings of the quantities in (2.16) are defined in the nomenclature. Moreover, the reference temperature difference ΔT is given by

$$\Delta T = T_2 - T_1 \text{ if } (T_1 < T_2) \tag{2.18}$$

Then, (2.7) to (2.15), using (2.16) take the form,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \lambda B_r \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + B_r\sigma_p^2 u^2 + T_e\sigma\left[E_x^2 + E_y^2\right] = 0 \tag{2.19}$$

$$\begin{aligned}\frac{\mathrm{d}^4u}{\mathrm{d}y^4} - GR.B_r\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 - \frac{GR.B_r.\sigma_p^2}{\lambda}u^2 - \frac{\sigma_p^2}{\lambda}\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right) - \frac{GR.T_e}{\lambda}\sigma\left[E_x^2 + E_y^2\right]\\ +W_e\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\rho_e E_x\right) = 0\end{aligned} \tag{2.20}$$

$$\sigma\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\varphi}{\mathrm{d}y^2}\right) + \frac{\mathrm{d}\varphi}{\mathrm{d}y}.\frac{\mathrm{d}\sigma}{\mathrm{d}y} = 0 \tag{2.21}$$

$$\nabla.\vec{E} = \rho_e,\ \sigma = 1 + \alpha\theta_B \text{ where } \alpha = \alpha_b\Delta T, \tag{2.22}$$

$$u\left(0\right) = u\left(1\right) = 0 \tag{2.23}$$

Also using

$$\begin{aligned}\theta &= -\frac{1}{2} \text{ if } y = 0\\ \theta &= -\frac{1}{2} \text{ if } y = 1\end{aligned} \tag{2.24}$$

$$\begin{aligned}\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)_{y=0} &= -\frac{48}{\lambda} + \frac{GR}{2\lambda} - \frac{W_1}{\lambda}\\ \left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)_{y=1} &= -\frac{48}{\lambda} - \frac{GR}{2\lambda} - \frac{W_1}{\lambda}\mathrm{e}^{-\alpha}\end{aligned} \tag{2.25}$$

$$\begin{aligned}&\phi = x \text{ at } y = 0\\ &\phi = \left(x - x_0\right) \text{ at } y = 1\end{aligned} \tag{2.26}$$

Further, (2.7), using (2.17) and (2.18), can be written as

$$\frac{\mathrm{d}^2u}{\mathrm{d}y^2} - \delta^2 u + \frac{GR}{\lambda}\theta + \frac{W_e}{\lambda}\left(\rho_e E_x\right) + \frac{48}{\lambda} = 0 \tag{2.27}$$

Rearrange (2.27), we get

$$\theta = -\frac{\lambda}{GR}\left(\frac{48}{\lambda} - \frac{\sigma_p^2}{\lambda}u + \frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right) - \frac{W_e}{\lambda GR}\left(\rho_e E_x\right) \tag{2.28}$$

where $Gr = \frac{g\beta\Delta TD^3}{\nu^2}$ is the Grashof number,
$R_e = \frac{U_0 D}{\nu}$the Reynolds number,
$B_r = \frac{\mu {U_0}^2}{\kappa\Delta t}$, the Brinkman number, physically represents the measure of viscous heating relative to the heat flow
$W_e = \frac{\varepsilon_0 V^2}{\rho_0 g\beta\Delta TD^3}$ the electric number,
$T_e = \frac{\sigma_0 V^2}{K\Delta T}$is the thermal electric number.
$\delta^2 = \frac{\sigma_p^2}{\lambda}$, $GR = \frac{Gr}{Re}$ are constants.

3 Solution for ϕ

The solution for ϕ, according to (2.21) depends on σ which in turn depends on the temperature θ as in (2.22) . In a poorly conducting fluid (i.e., $\sigma \ll 1$), the dissipations in (2.8) are negligible and hence σ will depend on the conduction temperature, θ_b, satisfying

$$\frac{\mathrm{d}^2\theta_B}{\mathrm{d}y^2} = 0 \tag{3.1}$$

The solution of this satisfying the boundary conditions

$$\theta_B = -1/2 \text{ and } y = 0 \text{ and } \theta_B = 1/2 \ \text{ at } y = 1 \tag{3.2}$$

is

$$\theta_B = y - 1/2 \tag{3.3}$$

Equation (2.22), using (3.3) becomes

$$\sigma = 1 + \alpha\,(y - 1/2) \approx \mathrm{e}^{\alpha(y-1/2)}\,(\because \alpha << 1) \tag{3.4}$$

Then (2.21) using (3.4), becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \alpha\frac{\partial\phi}{\partial y} = 0 \tag{3.5}$$

To find the solution of, (3.5), we consider the following two cases:

<u>Case A:</u> The Potential Difference Applied Opposing the Temperature Difference
The solution of (3.5), satisfying (2.26), is

$$\phi = x - \frac{x_0}{(1 - e^{-\alpha})}\left(1 - e^{-\alpha y}\right) \tag{3.6}$$

From (2.6), after making them dimensionless using (2.16) and using (3.6), we get

$$\rho_e = \nabla.\vec{E} = -\nabla^2\phi = -\frac{\alpha^2 x_0}{(1 - \mathrm{e}^{-\alpha})}\mathrm{e}^{-\alpha y}, E_x = -1, E_y = \frac{\alpha x_0}{(1 - \mathrm{e}^{-\alpha})}\mathrm{e}^{-\alpha y} \tag{3.7}$$

Also

$$\rho_e E_x = \frac{\alpha^2 x_0}{(1 - e^{-\alpha})}e^{-\alpha y}, \quad E_x^2 + E_y^2 = 1 + \frac{x_0^2\alpha^2 e^{-2\alpha y}}{(1 - e^{-\alpha})^2} \tag{3.8}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\rho_e E_x\right) = \frac{\alpha^4 x_0}{(1 - \mathrm{e}^{-\alpha})}\mathrm{e}^{-\alpha y}$$

<u>Case B</u> : The Potential Difference Applied in the Same Direction of Temperature Difference

In this case the boundary conditions on ϕ, in dimensionless form, are opposite to those specified in, (2.26) and they are

$$\phi = x \text{ and } y = 1 \quad \textit{and} \quad \phi = x - x_0 \text{ and } y = 0 \tag{3.9}$$

In this case the solution of (3.5), satisfying (3.9), is

$$\phi = x + \frac{x_0}{(1 - \mathrm{e}^{-\alpha})}\left(\mathrm{e}^{-\alpha} - \mathrm{e}^{-\alpha y}\right) \tag{3.10}$$

In this case ρ_e, E_y and $\rho_e E_x$ are opposite to those obtained in case A, where as E_x and $E_x^2 + E_y^2$ remain the same as in case A.

4 Analytical Solution

Since (2.19) and (2.27) are coupled equations, we find the solutions of these equations analytically using a regular perturbation technique with B_r as the perturbation parameter and using the boundary conditions given in (2.23) to (2.25) and temperature for different cases mentioned below. Another way of finding the solution of the problem is using (2.20) together with the boundary conditions (2.23) to (2.25) . Since these two cases leads to the same values. We proceed this problem by using the first method. This will be done below. We note that when the buoyancy forces are dominating i.e., when $GR \to \pm\infty$, in the case of asymmetric heating, it is purely natural convection. Suppose when the buoyancy forces are negligible and viscous dissipation is dominating, i.e., $GR = 0$, so that a purely forced convection occurs.

The solution of (2.19) and (2.27) for a fixed value of $GR(\neq 0)$ can be expressed for small values of B_r in the form of series

$$u\left(y\right) = u_0\left(y\right) + B_r u_1\left(y\right) + B_r^2 u_2\left(y\right) + \cdots = \sum_{n=0}^{\infty} B_r^n u_n\left(y\right) \tag{4.1}$$

$$\theta\left(y\right) = \theta_0\left(y\right) + B_r \theta_1\left(y\right) + B_r^2 \theta_2\left(y\right) + \cdots = \sum_{n=0}^{\infty} B_r^n \theta_n\left(y\right) \tag{4.2}$$

Case 1: Isothermal-Isothermal Walls

Substitute (3.7) and (3.8) in (2.19) and (2.27), we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \delta^2 u + \frac{GR}{\lambda}\theta + \frac{W_1}{\lambda}\mathrm{e}^{-\alpha y} + \frac{48}{\lambda} = 0 \tag{4.3}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \lambda B_r\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + B_r\sigma_p^2 u^2 + T_e\ \mathrm{e}^{-\alpha/2}\mathrm{e}^{\alpha y} + W_2\mathrm{e}^{-\alpha y} = 0 \tag{4.4}$$

Substituting series (4.1) and (4.2) in to (4.3) and (4.4), and equating coefficients of like powers of B_r to zero, one obtains the boundary value problem for $n = 0$ and 1 as

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}y^2} - \delta^2 u_0 + \frac{GR}{\lambda}\theta_0 + \frac{W_1}{\lambda}\mathrm{e}^{-\alpha y} + \frac{48}{\lambda} = 0 \tag{4.5}$$

$$\frac{\mathrm{d}^2\theta_0}{\mathrm{d}y^2} + T_e\mathrm{e}^{-\alpha/2}\mathrm{e}^{\alpha y} + W_2\mathrm{e}^{-\alpha y} = 0 \tag{4.6}$$

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}y^2} - \delta^2 u_1 + \frac{GR}{\lambda}\theta_1 = 0 \tag{4.7}$$

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}y^2} + \lambda\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)^2 + \sigma_p^2 u_0^2 = 0 \tag{4.8}$$

$$u_0\left(0\right) = u_0\left(1\right) = 0 \tag{4.9}$$

$$u_1\left(0\right) = u_1\left(1\right) = 0 \tag{4.10}$$

$$\theta_0 = -1/2 \text{ and } y = 0 \ \text{ and } \ \theta_0 = 1/2 \text{ at } y = 1 \tag{4.11}$$

$$\theta_1 = 0 \text{ and } y = 0 \text{ and } 1 \tag{4.12}$$

Solving (4.5) to (4.8) using (4.9) to (4.12), we get

$$\theta_0 = -W_2\left(\mathrm{e}^{-\alpha} - \mathrm{e}^{-\alpha y} - 1\right) + T_e\mathrm{e}^{-\alpha/2}\left(\mathrm{e}^{\alpha} - \mathrm{e}^{-\alpha y} - 1\right) + \alpha^2 y \tag{4.13}$$

$$u_0 = C_4\left(\mathrm{e}^{-\delta y} - 1\right) + a_1\left(e^{\alpha y} - \mathrm{e}^{\delta y}\right) + a_2\left(\mathrm{e}^{-\alpha y} - \mathrm{e}^{\delta y}\right) + b_0\left(1 - \mathrm{e}^{\delta y}\right) + a_3 y \tag{4.14}$$

$$
\begin{aligned}
\theta_1 = & \left(\frac{a_4 \mathrm{e}^{2\delta y} + a_5 \mathrm{e}^{-2\delta y}}{4\delta^2}\right) + \left(\frac{a_6 \mathrm{e}^{2\alpha y} + a_7 \mathrm{e}^{-2\alpha y}}{4\alpha^2}\right) + \left(\frac{a_8 \mathrm{e}^{(\alpha-\delta)y} + a_{10} \mathrm{e}^{-(\alpha-\delta)y}}{(\alpha-\delta)^2}\right) \\
& + \left(\frac{a_9 \mathrm{e}^{(\alpha+\delta)y} + a_{11} \mathrm{e}^{-(\alpha+\delta)y}}{(\alpha+\delta)^2}\right) + y\left(\frac{a_{16} \mathrm{e}^{\alpha y} + a_{17} \mathrm{e}^{-\alpha y}}{\alpha^2} + \frac{a_{16} \mathrm{e}^{\delta y} + a_{17} \mathrm{e}^{-\delta y}}{\delta^2}\right) \\
& + \left(a_{26} \mathrm{e}^{\alpha y} + a_{27} \mathrm{e}^{-\alpha y} + a_{28} \mathrm{e}^{\delta y} + a_{29} \mathrm{e}^{-\delta y} + C_6\right) \\
& + y\left(\frac{a_{20}}{6} y^2 + \frac{a_{21}}{2} y - \frac{\sigma_p^2 a_3^2}{12} y^3 + C_5\right)
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
u_1 = & \left(a_{30} \mathrm{e}^{2\delta y} + a_{31} \mathrm{e}^{-2\delta y}\right) + \left(a_{32} \mathrm{e}^{2\alpha y} + a_{33} \mathrm{e}^{-2\alpha y}\right) + y^2\left(a_{38} \mathrm{e}^{-\delta y} + a_{39} \mathrm{e}^{\delta y}\right) \\
& + y\left(a_{40} \mathrm{e}^{\alpha y} + a_{41} \mathrm{e}^{-\alpha y} + a_{42} \mathrm{e}^{\delta y} + a_{43} \mathrm{e}^{-\delta y} - \frac{\sigma_p^2 a_3^2 GR}{12\lambda\delta^2} y^3 + \frac{a_{20} GR}{6\lambda\delta^2} y^2 + a_{46} y + a_{47}\right) \\
& + \left(a_{34} \mathrm{e}^{(\alpha-\delta)y} + a_{35} \mathrm{e}^{(\alpha+\delta)y} + a_{36} \mathrm{e}^{-(\alpha-\delta)y} + a_{37} \mathrm{e}^{-(\alpha+\delta)y}\right) + a_{48} \\
& + \left(a_{44} \mathrm{e}^{\alpha y} + C_7 \mathrm{e}^{\delta y} + C_8 \mathrm{e}^{-\delta y} + a_{45} \mathrm{e}^{-\alpha y}\right)
\end{aligned} \tag{4.16}
$$

Where $W_i(i = 1 \text{ to } 2),\ C_i(i = 1 \text{ to } 8),\ a_i(i = 1 \text{ to } 48)$ are all constants and

$$
W_1 = \frac{W_e x_0 \alpha^2}{(1 - \mathrm{e}^{-\alpha})}, \quad W_2 = \frac{T_e x_0^2 \alpha^2 \mathrm{e}^{-\alpha/2}}{4(1 - \mathrm{e}^{-\alpha})^2}
$$

Case2: Isothermal-Isoflux Walls

The Isoflux and Isothermal boundary conditions for the channel walls are given by

$$
q_1 = K\left(\frac{\mathrm{d}T}{\mathrm{d}Y}\right)_{Y=0} \qquad T(2L) = T_2 \tag{4.17}
$$

In the non-dimensional form (4.17), using (2.16) with $\Delta T = \frac{q_1 D}{K}$, becomes

$$
\left(\frac{\mathrm{d}\theta}{\mathrm{d}Y}\right)_{y=0} = 1; \quad \theta(1) = \theta_2 \tag{4.18}
$$

where $\theta_2 = \frac{T_2 - T_0}{\Delta T}$ is the thermal ratio parameter. Solving (3.1) using (4.18) and substituting the resulting solution in to (2.22), we get

$$
\sigma = 1 + \alpha(y + \Omega) \approx \mathrm{e}^{\alpha(y+\Omega)} \tag{4.19}
$$

where $\Omega = \theta_2 - 1$
From this, we get

$$
\phi = x - \frac{x_0}{(1 - \mathrm{e}^{-\alpha})}\left(1 - \mathrm{e}^{-\alpha y}\right) \tag{4.20}
$$

From (4.20) we obtain $\rho_e E_x,\ E_x,\ E_y$.
Following the above procedure, we have

$$
\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \delta^2 u + \frac{GR}{\lambda}\theta + \frac{W_1}{\lambda}\mathrm{e}^{-\alpha y} + \frac{48}{\lambda} = 0 \tag{4.21}
$$

$$
\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \lambda B_r\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + B_r\sigma_p^2 u^2 + T_e\,\mathrm{e}^{\alpha\Omega}\mathrm{e}^{\alpha y} + W_4\mathrm{e}^{-\alpha y} = 0 \tag{4.22}
$$

Substituting series(4.1) and (4.2) in to (4.21) and(4.22) and equating coefficients of like powers of B_r to zero, we obtain the boundary value problem for $n = 0$ and 1 as

$$
\frac{\mathrm{d}^2 u_0}{\mathrm{d}y^2} - \delta^2 u_0 + \frac{GR}{\lambda}\theta_0 + \frac{W_1}{\lambda}\mathrm{e}^{-\alpha y} + \frac{48}{\lambda} = 0 \tag{4.23}
$$

$$\frac{\mathrm{d}^2\theta_0}{\mathrm{d}y^2} + T_e\,\mathrm{e}^{\alpha\Omega}\mathrm{e}^{\alpha y} + W_4\mathrm{e}^{-\alpha y} = 0 \tag{4.24}$$

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}y^2} - \delta^2 u_1 + \frac{GR}{\lambda}\theta_1 = 0 \tag{4.25}$$

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}y^2} + \lambda\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)^2 + \sigma_p^2 u_0^2 = 0 \tag{4.26}$$

$$u_0(0) = u_0(1) = 0 \tag{4.27}$$

$$u_1(0) = u_1(1) = 0 \tag{4.28}$$

$$\left(\frac{\mathrm{d}\theta_0}{\mathrm{d}y}\right)_{y=0} = 1 \text{ and } \theta_0 = 1/2 \text{ at } y = 1 \tag{4.29}$$

$$\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}y}\right)_{y=0} = 0 \text{ and } \theta_1 = 0 \text{ at } y = 1 \tag{4.30}$$

Solving (4.23) to (4.26) using (4.27) to (4.30), we get

$$\theta_0 = 2T_e\mathrm{e}^{\alpha\Omega}\left(\mathrm{e}^{\alpha} - \mathrm{e}^{\alpha y} + \alpha(y-1)\right) + 2W_4\left(\mathrm{e}^{-\alpha} - \mathrm{e}^{-\alpha y} + \alpha(1-y)\right) + \alpha^2(2y-1) \tag{4.31}$$

$$u_0 = C_4\left(\mathrm{e}^{-\delta y} - 1\right) + a_1\left(\mathrm{e}^{\alpha y} - \mathrm{e}^{\delta y}\right) + a_2\left(\mathrm{e}^{-\alpha y} - \mathrm{e}^{\delta y}\right) + b_0\left(1 - \mathrm{e}^{\delta y}\right) + a_3 y \tag{4.32}$$

$$\begin{aligned}
\theta_1 = &\frac{a_4}{4\delta^2}\mathrm{e}^{2\delta y} + \frac{a_5}{4\delta^2}\mathrm{e}^{2\delta y} + \frac{a_6}{4\alpha^2}\mathrm{e}^{2\alpha y} + \frac{a_7}{4\alpha^2}\mathrm{e}^{-2\alpha y} + \frac{a_8}{(\alpha-\delta)^2}\mathrm{e}^{(\alpha-\delta)y} \\
&+ \frac{a_9}{(\alpha+\delta)^2}\mathrm{e}^{(\alpha+\delta)y} + \frac{a_{10}}{(\alpha-\delta)^2}\mathrm{e}^{-(\alpha-\delta)y} + \frac{a_{11}}{(\alpha+\delta)^2}\mathrm{e}^{-(\alpha+\delta)y} + \frac{a_{16}}{\alpha^2}y\mathrm{e}^{\alpha y} \\
&+ \frac{a_{17}}{\alpha^2}y\mathrm{e}^{-\alpha y} + \frac{a_{18}}{\delta^2}y\mathrm{e}^{\delta y} + \frac{a_{19}}{\delta^2}y\mathrm{e}^{-\delta y} + a_{26}\mathrm{e}^{\alpha y} + a_{27}\mathrm{e}^{-\alpha y} + a_{28}\mathrm{e}^{\delta y} \\
&+ a_{29}\mathrm{e}^{-\delta y} + \frac{a_{20}}{6}y^3 + \frac{a_{21}}{2}y^2 - \frac{\sigma_p^2 a_3^2}{12}y^4 + c_5 y + c_6
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
u_1 = &C_7\mathrm{e}^{\delta y} + C_8\mathrm{e}^{-\delta y} + a_{30}\mathrm{e}^{2\delta y} + a_{31}\mathrm{e}^{-2\delta y} + a_{32}\mathrm{e}^{2\alpha y} + a_{33}\mathrm{e}^{-2\alpha y} \\
&+ a_{34}\mathrm{e}^{(\alpha-\delta)y} + a_{35}\mathrm{e}^{(\alpha+\delta)y} + a_{36}\mathrm{e}^{-(\alpha-\delta)y} + a_{37}\mathrm{e}^{-(\alpha+\delta)y} + a_{38}y^2\mathrm{e}^{-\delta y} \\
&+ a_{39}y^2\mathrm{e}^{\delta y} + a_{40}y\mathrm{e}^{\alpha y} + a_{41}y\mathrm{e}^{-\alpha y} + a_{42}y\mathrm{e}^{\delta y} + a_{43}y\mathrm{e}^{-\delta y} + a_{44}\mathrm{e}^{\alpha y} \\
&+ a_{45}\mathrm{e}^{-\alpha y} - \frac{\sigma_p^2 a_3^2 GR}{12\lambda\delta^2}y^4 + \frac{a_{20}GR}{6\lambda\delta^2}y^3 + a_{46}y^2 + a_{47}y + a_{48}
\end{aligned} \tag{4.34}$$

where $W_i\,(i = 1, 4)\,, C_i\,(i = 1 \text{ at } 8)\,, a_i\,(i = 1 \text{ at } 48)$ are constants with $W_4 = \frac{T_e x_0^2 \alpha^2 \mathrm{e}^{\alpha\Omega}}{4(1-\mathrm{e}^{-\alpha})^2}$.

Case3: Isoflux-Isoflux Walls

The Isoflux and Isoflux boundary conditions for the channel walls are given by

$$q_1 = K\left(\frac{\mathrm{d}T}{\mathrm{d}Y}\right)_{Y=0}; \quad q_2 = K\left(\frac{\mathrm{d}T}{\mathrm{d}Y}\right)_{Y=1} \tag{4.35}$$

In the non-dimensional form (4.35) using (2.16) with $\Delta T = \frac{q_1 D}{K}$, becomes

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}Y}\right)_{y=0} = 1; \quad \left(\frac{\mathrm{d}\theta}{\mathrm{d}Y}\right)_{y=1} = \lambda_1 \tag{4.36}$$

in addition to this we use the boundary condition

$$\int_0^1 \theta\,\mathrm{d}y = Q \tag{4.37}$$

where $\lambda_1 = \frac{q_2}{q_1}$ is the ratio of heat fluxes.
Solving (3.1), using (4.36) and (4.37) and the resulting solution of (2.22) is,

$$\sigma = 1 + \alpha\left(\lambda_1 y + \Omega_1\right) \approx \mathrm{e}^{\alpha(\lambda_1 y + \Omega_1)} \tag{4.38}$$

from this, we get

$$\phi = x - \frac{x_0}{\left(1 - \mathrm{e}^{-\alpha_1}\right)}\left(1 - \mathrm{e}^{-\alpha_1 y}\right) \tag{4.39}$$

where $\Omega_1 = Q - \frac{\lambda_1}{2}, \alpha_1 = \alpha\lambda_1$ is a constant.
From (4.39) we obtain $\rho_e E_x,\ E_x,\ E_y$.
Substitute the above results in (2.19) and (2.27), we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \delta^2 u + \frac{GR}{\lambda}\theta + \frac{W_1}{\lambda}\mathrm{e}^{-\alpha_1 y} + \frac{48}{\lambda} = 0 \tag{4.40}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \lambda B_r\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + B_r\sigma_p^2 u^2 + T_e\ \mathrm{e}^{\alpha\Omega_1}\mathrm{e}^{\alpha_1 y} + W_3\mathrm{e}^{-\alpha_1 y} = 0 \tag{4.41}$$

Substituting series (4.1) and (4.2) in to equations (4.40) and (4.41) and equating coefficients of like powers of B_r to zero, we obtains the boundary value problem for $n = 0$ and 1 as

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}y^2} - \delta^2 u_0 + \frac{GR}{\lambda}\theta_0 + \frac{W_1}{\lambda}\mathrm{e}^{-\alpha_1 y} + \frac{48}{\lambda} = 0 \tag{4.42}$$

$$\begin{gathered}\frac{\mathrm{d}^2\theta_0}{\mathrm{d}y^2} + T_e\ \mathrm{e}^{\alpha\Omega_1}\mathrm{e}^{\alpha_1 y} + W_3\mathrm{e}^{-\alpha_1 y} = 0 \\ \frac{\mathrm{d}^2 u_1}{\mathrm{d}y^2} - \delta^2 u_1 + \frac{GR}{\lambda}\theta_1 = 0\end{gathered} \tag{4.43}$$

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}y^2} + \lambda\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)^2 + \sigma_p^2 u_0^2 = 0 \tag{4.44}$$

$$u_0(0) = u_0(1) = 0 \tag{4.45}$$

$$u_1(0) = u_1(1) = 0 \tag{4.46}$$

$$\left(\frac{\mathrm{d}\theta_0}{\mathrm{d}Y}\right)_{y=1} = \lambda_1 \text{ and } \int_0^1 \theta_0 dy = Q \tag{4.47}$$

$$\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}Y}\right)_{y=1} = 0 \text{ and } \int_0^1 \theta_1 dy = 0 \tag{4.48}$$

Solving (4.42) to (4.44) using (4.45) to (4.48), we get

$$\begin{aligned}\theta_0 = &\frac{T_e\mathrm{e}^{\alpha\Omega_1}}{2\alpha_1^3}\left(2\mathrm{e}^{\alpha_1}y\alpha_1 - \mathrm{e}^{\alpha_1}\alpha_1^2 - 2\mathrm{e}^{\alpha_1 y}\alpha_1 + 2\left(\mathrm{e}^{\alpha_1} - 1\right)\right) \\ &+ \frac{W_3}{2\alpha_1^3}\left(\mathrm{e}^{\alpha_1}\alpha_1^2 - 2\mathrm{e}^{\alpha_1 y}\alpha_1 - 2\mathrm{e}^{\alpha_1}y\alpha_1 + 2W_3\left(1 - \mathrm{e}^{-\alpha_1}\right)\right) + \frac{\lambda_1}{2}(2y - 1) + Q\end{aligned} \tag{4.49}$$

$$u_0 = C_4\left(\mathrm{e}^{-\delta y} - 1\right) + a_1\left(\mathrm{e}^{\alpha y} - \mathrm{e}^{\delta y}\right) + a_2\left(\mathrm{e}^{-\alpha y} - \mathrm{e}^{\delta y}\right) + b_0\left(1 - \mathrm{e}^{\delta y}\right) + a_3 y \tag{4.50}$$

$$\begin{aligned}\theta_1 = &\left(\frac{a_4\mathrm{e}^{2\delta y} + a_5\mathrm{e}^{-2\delta y}}{4\delta^2}\right) + \left(\frac{a_6\mathrm{e}^{2\alpha_1 y} + a_7\mathrm{e}^{-2\alpha_1 y}}{4\alpha_1^2}\right) + \left(\frac{a_8\mathrm{e}^{(\alpha_1-\delta)y} + a_{10}\mathrm{e}^{-(\alpha_1-\delta)y}}{(\alpha_1-\delta)^2}\right) \\ &+ \left(\frac{a_9\mathrm{e}^{(\alpha_1+\delta)y} + a_{11}\mathrm{e}^{-(\alpha_1+\delta)y}}{(\alpha_1+\delta)^2}\right) + y\left(\frac{a_{16}\mathrm{e}^{\alpha_1 y} + a_{17}\mathrm{e}^{-\alpha_1 y} + a_{26}\alpha_1^2\mathrm{e}^{\alpha_1 y} + \alpha_1^2 a_{27}\mathrm{e}^{-\alpha_1 y}}{\alpha_1^2}\right) \\ &+ y\left(\frac{a_{18}\mathrm{e}^{\delta y} + a_{17}\mathrm{e}^{-\delta y} + a_{28}\mathrm{e}^{\delta y}\delta^2 + a_{29}\mathrm{e}^{-\delta y}\delta^2}{\delta^2}\right) + \left(\frac{a_{21}}{2} + \frac{a_{20}}{6}y - \frac{\sigma_p^2 a_3^2}{12}y^2\right)y^2 \\ &- b_1 + C_5(1 - y)\end{aligned} \tag{4.51}$$

$$
\begin{aligned}
u_1 = & C_7 e^{\delta y} + C_8 e^{-\delta y} + a_{30} e^{2\delta y} + a_{31} e^{-2\delta y} + a_{32} e^{2\alpha_1 y} + a_{33} e^{-2\alpha_1 y} \\
& + a_{34} e^{(\alpha_1-\delta)y} + a_{35} e^{(\alpha_1+\delta)y} + a_{36} e^{-(\alpha_1-\delta)y} + a_{37} e^{-(\alpha_1+\delta)y} + a_{38} y^2 e^{-\delta y} \\
& + a_{39} y^2 e^{\delta y} + a_{40} y e^{\alpha_1 y} + a_{41} y e^{-\alpha_1 y} + a_{42} y e^{\delta y} + a_{43} y e^{-\delta y} + a_{44} e^{\alpha_1 y} \\
& + a_{45} e^{-\alpha_1 y} - \frac{\sigma_p^2 a_3^2 GR}{12\lambda\delta^2} y^4 + \frac{a_{20} GR}{6\lambda\delta^2} y^3 + a_{46} y^2 + a_{47} y + a_{48}
\end{aligned}
\tag{4.52}
$$

$W_3 = \frac{T_e x_0^2 \alpha_1^2 e^{\alpha\Omega_1}}{4(1-e^{-\alpha_1})^2}$, where $W_i\,(i=1,3)\,, C_i\,(i=1 \text{ at } 8)\,, l_i\,(i=1 \text{ at } 48)$ are constants.

5 Numerical Method

The analytical solutions obtained in Section 4, using regular perturbation technique are valid only for small values of the parameter B_r. In numerical method, we solve the velocity and energy equations using the central difference method. The use of central difference scheme replaces the derivative with the corresponding central difference approximations leading to set of linear algebraic equations. The solutions of the reduced algebraic equations are obtained by the method of SOR. The relaxation parameter ω is fixed by comparing the numerical results with those obtained by analytical method.

5.1 Isothermal-Isothermal Walls

The finite difference equations of (2.19) and (2.27) using central difference scheme with 21 mesh point with the step size (h) 0.05 are:

$$
\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} - \delta^2 u_j + \frac{W_1}{\lambda} e^{-\alpha y_j} + \frac{GR}{\lambda}\theta_j + \frac{48}{\lambda} = 0, \tag{5.1.1}
$$

Solving for u_j,

$$
L_1 = u_j = \frac{1}{(2+\delta^2 h^2)}\left[u_{j+1} + u_{j-1} + \frac{W_1}{\lambda} e^{-\alpha y_j} h^2 + \frac{GR}{\lambda}\theta_j h^2 + \frac{48}{\lambda} h^2\right] \tag{5.1.2}
$$

Similarly,

$$
\frac{\theta_{j+1} - 2\theta_j + \theta_{j-1}}{h^2} + \lambda Br\left(\frac{u_{j+1} - u_{j-1}}{2h}\right)^2 + Br\sigma_p^2 u_j^2 + T_e e^{-\alpha/2} e^{\alpha y_j} + W_2 e^{-\alpha y_j} = 0 \tag{5.1.3}
$$

Solving for θ_j

$$
\begin{aligned}
L_2 &= \theta_j \\
&= \frac{1}{2}\left[\theta_{j+1} + \theta_{j-1} + \frac{\lambda Br}{4}(u_{j+1} - u_{j-1})^2 + Br\sigma_p^2 h^2 u_j^2 + T_e h^2 e^{-\alpha/2} e^{\alpha y_j} + W_2 h^2 e^{-\alpha y_j}\right]
\end{aligned}
\tag{5.1.4}
$$

where $h = (\Delta y) = 0.05$ and j represents the mesh point and it varies from 0 to 20.
Equations (5.1.2) and (5.1.4) give two implicit equations in $u_j$and θ_j denoted by L $_1$ and L $_2$ respectively. They separately generate systems of linear equations in u_j, u_{j+1}, u_{j-1} and $\theta_j, \theta_{j+1}, \theta_{j-1}$, resulting in tri-diagonal co-efficient matrices. These equations are solved using the Method of Relaxation.

The Algorithm

Initially we choose all u_j and θ_j to be zero. This assumption is automatically corrected as the iteration proceeds. The Scheme of the Relaxation Method for the k^{th} iteration and j^{th} mesh point is:

$u_j^k = \omega[u_j^k]_{L_1} + (1-\omega)u_j^{k-1}$ and $\theta_j^k = \omega[\theta_j^k]_{L_2} + (1-\omega)\theta_j^{k-1}$
Here ω is the relaxation parameter. This translates into:

$$u_j^k = \omega\left[\frac{1}{(2+\delta^2h^2)}\left(u_{j+1}^k + u_{j-1}^k + \frac{W_1}{\lambda}\mathrm{e}^{-\alpha y_j}h^2 + \frac{GR}{\lambda}\theta_j h^2 + \frac{48}{\lambda}h^2\right)\right] + (1-\omega)\,u_j^{k-1} \tag{5.1.5}$$

and similarly for θ we get:

$$\theta_j^k = \omega\left[\frac{1}{2}\left(\theta_{j+1}^k + \theta_{j-1}^k + \frac{Br\lambda}{4}\left(u_{j+1}-u_{j-1}\right)^2 + Br\sigma_p^2h^2u_j^2 + h^2T_e\mathrm{e}^{-\alpha/2}\mathrm{e}^{\alpha y_j} + W_2h^2\mathrm{e}^{-\alpha y_j}\right)\right] + (1-\omega)\,\theta_j^{k-1} \tag{5.1.6}$$

where $y_j = y_0 + jh$, with an initial guess $y_0 = 0$.
The boundary conditions for u, given by $u_{y=0} = 0$ and $u_{y=1} = 0$ are incorporated as:
$u_0 = 0$ and $u_{20} = 0$. Similarly, we use $\theta_{y=0} = -0.5$ and $\theta_{y=1} = 0.5$. That is
$\theta_0 = -0.5$ and $\theta_{20} = 0.5$.Then (5.1.5) and (5.1.6) are solved simultaneously from k=0 at each mesh j point till the required order of iterative difference is achieved after comparison with the analytical results.

5.2 Case2: Isothermal-Isoflux Walls

Even in this case, applying the same procedures, as described above, to (4.21) and (4.22) we get:

$$u_j^k = \omega\left[\frac{1}{(2+\delta^2h^2)}\left(u_{j+1}^k + u_{j-1}^k + \frac{W_1}{\lambda}\mathrm{e}^{-\alpha y_j}h^2 + \frac{GR}{\lambda}\theta_j h^2 + \frac{48}{\lambda}h^2\right)\right] + (1-\omega)\,u_j^{k-1} \tag{5.2.1}$$

The boundary conditions for u given by $u_{y=0} = 0$ and $u_{y=1} = 0$ are incorporated as:
$u_0 = 0$ and $u_{20} = 0$
Since we encounter a Neumann Boundary Condition at one wall for θ, we have employed the Shooting Method to determine θ_j from j=0 to 20. The boundary conditions for θ are given by

$$\frac{\mathrm{d}\theta}{\mathrm{d}y}_{y=0} = 1 \text{ and } \theta_{y=1} = 0.5. \tag{5.2.2}$$

We define:

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)_j = t_j,\ \text{ and make an assumption } \theta_0 = 1. \tag{5.2.3}$$

Therefore (4.22) becomes

$$\left(\frac{\mathrm{d}t}{\mathrm{d}y}\right)_j + \lambda Br\left(\frac{u_{j+1}-u_{j-1}}{2h}\right)^2 + Br\sigma_p^2u_j^2 + T_e\mathrm{e}^{\alpha\Omega}\mathrm{e}^{\alpha y_j} + W_4\mathrm{e}^{-\alpha y_j} = 0, t_0 = 0 \tag{5.2.4}$$

Equations (5.2.3) and (5.2.4) convert the boundary value problem (4.22) into two initial value problems in θ and t which are solved using the Euler's Method as given below:

$$\theta_j^k = \theta_{j-1}^k + h\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)_{j-1} \tag{5.2.5}$$

$$t_j^k = t_{j-1}^k + h\left(\frac{\mathrm{d}t}{\mathrm{d}y}\right)_{j-1} \tag{5.2.6}$$

Where $\frac{\mathrm{d}\theta}{\mathrm{d}y}$ and $\frac{\mathrm{d}t}{\mathrm{d}y}$ are given by (5.2.3) and (5.2.4) respectively. The equation (5.2.1) is separately solved at each mesh point j for $k = 0$. Then (5.2.4) to (5.2.6) are solved simultaneously at each mesh point j for $k = 0$. This process is then iterated and in the end θ_0 is varied so that we get $\theta_{y=1} = 0.5$.

5.3 Case3: Isoflux-Isoflux Walls

Applying the same procedure described above to (4.40) we get:

$$u_j^k = \omega \left[\frac{1}{(2+\delta^2 h^2)} \left(u_{j+1}^k + u_{j-1}^k + \frac{W_1}{\lambda} \mathrm{e}^{-\alpha y_j} h^2 + \frac{GR}{\lambda} \theta_j h^2 + \frac{48}{\lambda} h^2 \right) \right] + (1-\omega)\, u_j^{k-1} \qquad (5.3.1)$$

The boundary conditions for u given by $u_{y=0} = 0$ and $u_{y=1} = 0$ are incorporated as:
$u_0 = 0$ and $u_{20} = 0$. Since we encounter Neumann Boundary Condition at both the walls for θ, we have employed the Shooting Method to determine θ_j for $j = 0$ to 20. The boundary conditions for θ given are given by $\frac{\mathrm{d}\theta}{\mathrm{d}y}_{y=0} = 1$ and $\frac{\mathrm{d}\theta}{\mathrm{d}y}_{y=1} = \lambda_1$. Proceeding in the same way as described in Section 5.2, we get :

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}y} \right)_j = t_j, \text{ assuming } \theta_0 = 1 \qquad (5.3.2)$$

$$\left(\frac{\mathrm{d}t}{\mathrm{d}y} \right)_j + \lambda Br \left(\frac{u_{j+1} - u_{j-1}}{2h} \right)^2 + Br\sigma_p^2 u^2 + T_e \mathrm{e}^{\alpha\Omega_1} \mathrm{e}^{\alpha_1 y_j} + W_3 \mathrm{e}^{-\alpha_1 y_j} = 0, t_0 = 0 \qquad (5.3.3)$$

$$\theta_j^k = \theta_{j-1}^k + h \left(\frac{\mathrm{d}\theta}{\mathrm{d}y} \right)_{j-1} \qquad (5.3.4)$$

$$t_j^k = t_{j-1}^k + h \left(\frac{\mathrm{d}t}{\mathrm{d}y} \right)_{j-1} \qquad (5.3.5)$$

The above equations are solved as described in Section 5.2 and the value of θ_0 is varied so to get $\left(\frac{\mathrm{d}\theta}{\mathrm{d}y} \right)_{y=1} = \lambda_1$.
These solutions of u and θ in each section are computed for different values of the parameters and the results are depicted graphically in figures (2) to (16) and in tables (1) and (2) along with the analytical results and conclusions are drawn in the final section.

6 Skin friction, Rate of heat transfer and Mass flow rate:

In many practical applications involving separation of flow it is advantage us to know the skin friction and the rate of heat transfer at the boundaries. These can be determined once we know the velocity and temperature distributions. The skin friction τ at the walls is defined as

$$\tau' = \mu \left(\partial U / \partial Y \right)_{Y=0,2L} \qquad (6.1)$$

Making this dimensionless, using the $\mu U_0/D$ for τ and using the scales for U and Y used earlier, we get

$$\tau = \left(\frac{\mathrm{d}u}{\mathrm{d}y} \right)_{y=0,1} \qquad (6.2)$$

This, using (4.1), takes the form

$$\tau = \left(\frac{\mathrm{d}u_0}{\mathrm{d}y} \right)_{y=0,1} + B_r \left(\frac{\mathrm{d}u_1}{\mathrm{d}y} \right)_{y=0,1} \qquad (6.3)$$

Here $\left(\frac{\mathrm{d}u_0}{\mathrm{d}y} \right)$ and $\left(\frac{\mathrm{d}u_1}{\mathrm{d}y} \right)$ can be obtained using the analytical solutions for case 1, 2 and 3 is also computed numerically using the finite difference scheme as explained in section 5. Similarly, the rate of heat transfer between the fluid and the plate is given by the heat flux

$$q' = -K \left(\frac{\partial T}{\partial y} \right)_{y=0,2L} \qquad (6.4)$$

This, using the scale $\Delta T = qD/K$ for the heat flux, is expressed in terms of the Nusselt number, Nu, given by,

$$Nu = \left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)_{y=0,1} \tag{6.5}$$

The Nusselt number is calculated using the analytical solutions for all the three cases. This Nu, is given by (6.5), is also numerically evaluated using the finite difference scheme as explained in section 5.

The skin friction and the rate of heat transfer given above obtained from analytical and numerical techniques are compared graphically and suitable conclusions are drawn in the final section.

If m_E denotes the mass flow rate per unit channel width in the presence of dissipation, then

$$m_E = \int_{-b}^{b} \rho_0 u \,\mathrm{d}y \tag{6.6}$$

It is of practical interest to find the mass flow rate given by (6.6) using both analytical and numerical techniques explained in sections 4 and 5 respectively. The results so obtained are discussed in the final section.

7 Results and discussions:

Table.1: Temperature profiles for different values of W_e (Isotheraml-Isoflux)

Y	θ	θ	θ	θ	θ	θ
0	−0.40458	−0.40329	−0.38883	−0.40458	−0.40329	−0.38883
0.05	−0.35484	−0.35356	−0.33916	−0.35458	−0.35329	−0.33883
0.1	−0.3056	−0.30433	−0.29006	−0.30458	−0.30329	−0.28883
0.15	−0.25681	−0.25555	−0.24149	−0.25507	−0.25379	−0.23941
0.2	−0.20846	−0.20724	−0.19344	−0.20602	−0.20475	−0.19053
0.25	−0.16057	−0.15937	−0.1459	−0.15741	−0.15616	−0.14216
0.3	−0.11314	−0.11197	−0.09889	−0.10926	−0.10804	−0.0943
0.35	−0.06617	−0.06504	−0.05242	−0.06157	−0.06037	−0.04698
0.4	−0.01968	−0.0186	−0.0065	−0.01434	−0.01318	$-1.96E-004$
0.45	0.02632	0.02735	0.03884	0.03241	0.03352	0.04603
0.5	0.07183	0.0728	0.08361	0.07866	0.07973	0.09167
0.55	0.11685	0.11774	0.12779	0.12442	0.12543	0.13674
0.6	0.16136	0.16218	0.17139	0.16969	0.17064	0.18123
0.65	0.20538	0.20612	0.21442	0.21445	0.21533	0.22512
0.7	0.24891	0.24956	0.25687	0.25872	0.25952	0.26844
0.75	0.29196	0.29252	0.29877	0.3025	0.30321	0.31118
0.8	0.33453	0.33499	0.34012	0.34579	0.34641	0.35336
0.85	0.37663	0.37698	0.38093	0.38861	0.38913	0.395
0.9	0.41826	0.4185	0.42121	0.43095	0.43138	0.4361
0.95	0.4594	0.45952	0.46092	0.47282	0.47314	0.47666
1	0.5	0.5	0.5	0.5	0.5	0.5
$GR = 10$, $B_r = 0.1$, $T_e = 0.1$	$W_e = 1$ Analytical	$W_e = 10$ $\lambda = 12$	$W_e = 100$ $\sigma_p = 15$	$W_e = 1$ Numerical	$W_e = 10$	$W_e = 100$

Table 2 : Temperature profiles for different values of W_e (Isoflux-Isoflux)

Y	θ	θ	θ	θ	θ	θ
0	-0.12763	-0.12964	-0.15348	-0.12763	-0.12964	-0.15348
0.05	-0.01155	-0.01327	-0.03366	-0.01263	-0.01364	-0.03403
0.1	0.10397	0.10253	0.08538	0.10237	0.10236	0.08542
0.15	0.21899	0.2178	0.20374	0.21681	0.21778	0.20408
0.2	0.33351	0.33257	0.32143	0.33072	0.33266	0.32204
0.25	0.44753	0.44682	0.43847	0.44411	0.44702	0.43933
0.3	0.56102	0.56053	0.55481	0.55699	0.56083	0.55593
0.35	0.67398	0.6737	0.67044	0.66933	0.6741	0.67183
0.4	0.78639	0.78631	0.78533	0.78112	0.78681	0.787
0.45	0.89824	0.89833	0.89944	0.89233	0.89894	0.90141
0.5	1.0095	1.00975	1.01277	1.00297	1.01046	1.01503
0.55	1.12018	1.12058	1.12531	1.11302	1.12138	1.12785
0.6	1.23027	1.2308	1.23704	1.22246	1.23167	1.23986
0.65	1.33978	1.34041	1.34799	1.33129	1.34135	1.35105
0.7	1.44869	1.44943	1.45815	1.43952	1.4504	1.46143
0.75	1.55704	1.55786	1.56756	1.54716	1.55884	1.57101
0.8	1.66482	1.66571	1.67623	1.6542	1.66668	1.67982
0.85	1.77203	1.77298	1.78417	1.76065	1.77391	1.78788
0.9	1.87868	1.87968	1.89139	1.86654	1.88056	1.89519
0.95	1.98474	1.98576	1.99782	1.97184	1.98662	2.00176
1	2.09012	2.09115	2.10335	2.07652	2.09205	2.10754
$GR = 10$, $B_r = 0.01$,	$W_e = 1$ Analytical	$W_e = 10$ $\sigma_p = 15$, $\lambda = 12$	$W_e = 100$ Numerical	$W_e = 1$	$W_e = 10$	$W_e = 100$

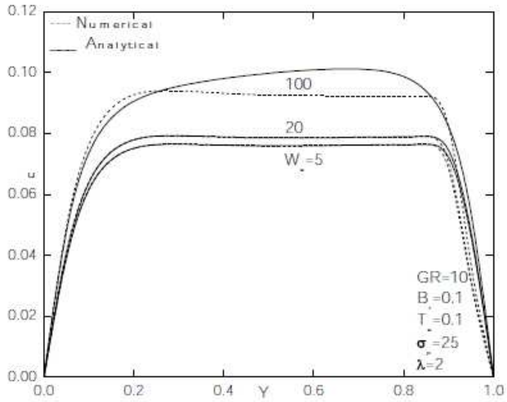


Figure 2a: Velocity vs Y for different values of W_e (Opposite direction) (Isothermal-Isothermal)

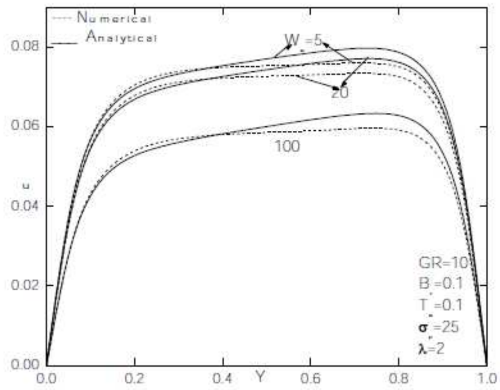


Figure 2b: Velocity vs Y for different values of W_e (Same direction) (Isothermal-Isothermal)

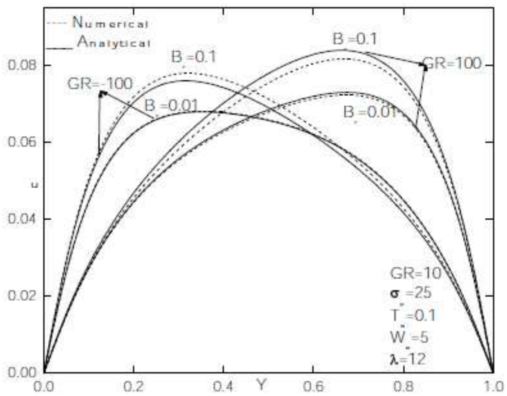


Figure 2c: Velocity vs Y for different values of B_r (Isothermal-Isothermal)

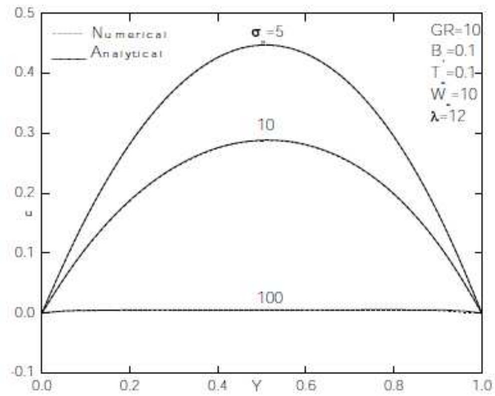


Figure 2d: Velocity vs Y for different values of σ_p (Isothermal-Isothermal)

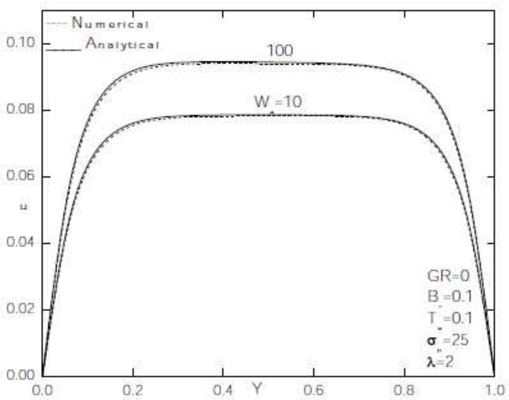


Figure 2e: Velocity vs Y for different values of W_e for $GR = 0$ (Isothermal-Isothermal)

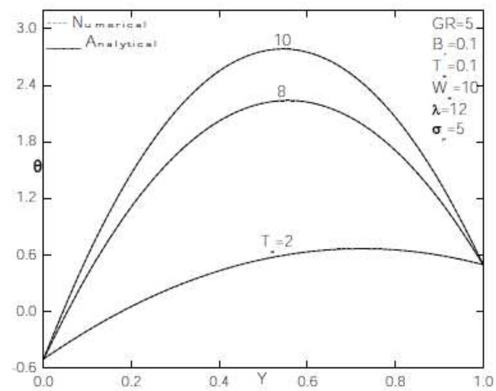


Figure 3a: Temperature vs Y for different values of T_e (Isothermal-Isothermal)

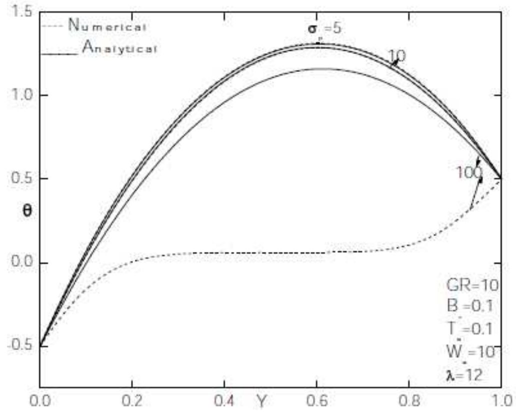


Figure 3b: Temperature vs Y for different values of σ_p (Isothermal-Isothermal)

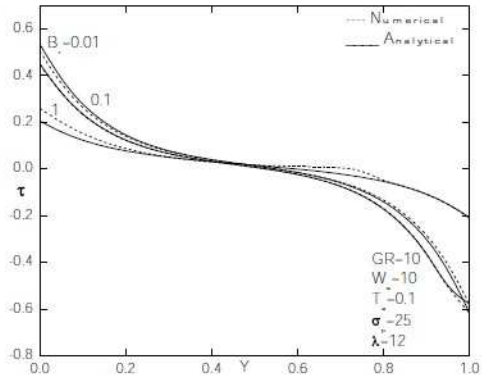


Figure 4a: Skin friction vs Y for different values of B_r (Isothermal-Isothermal)

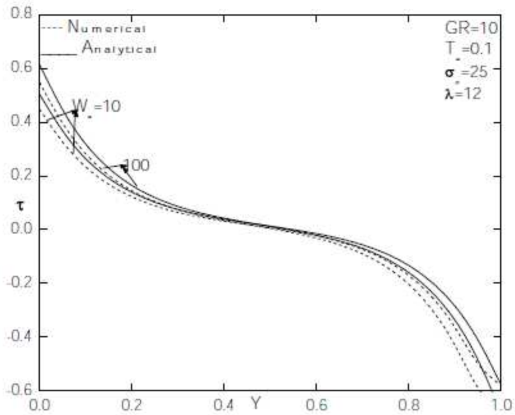


Figure 4b: Skin friction vs Y for different values of W_e (Isothermal-Isothermal)

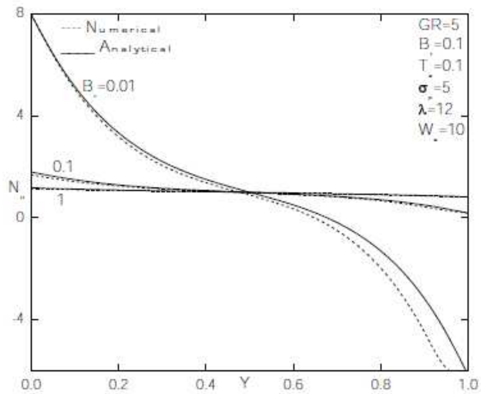


Figure 5a: Rate of heat transfer vs Y for different values of B_r (Isothermal-Isothermal)

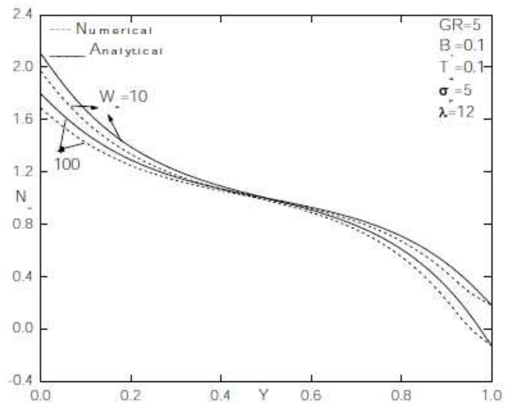


Figure 5b: Rate of heat transfer vs Y for different values of W_e (Isothermal-Isothermal)

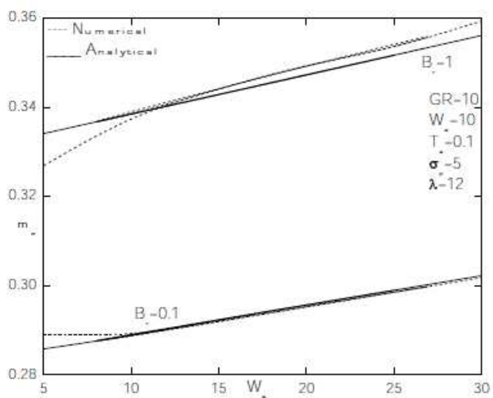
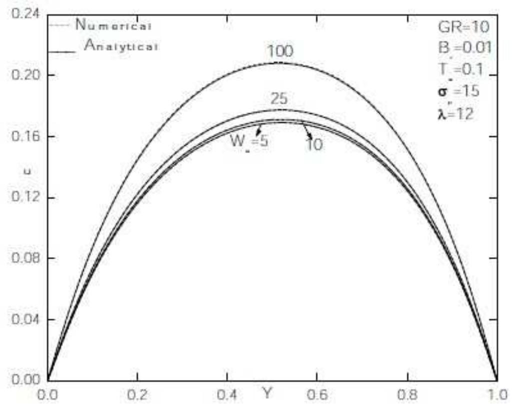


Figure 6: Mass fow rate vs W_e for different values of B_r (Isothermal-Isothermal)



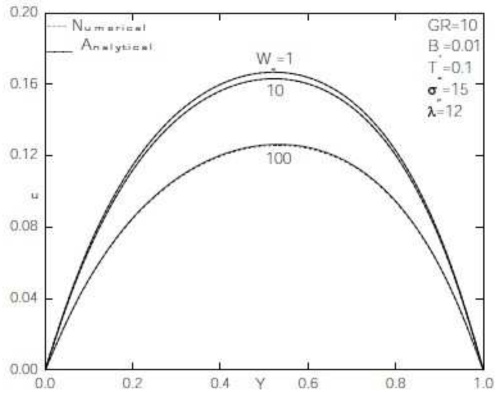
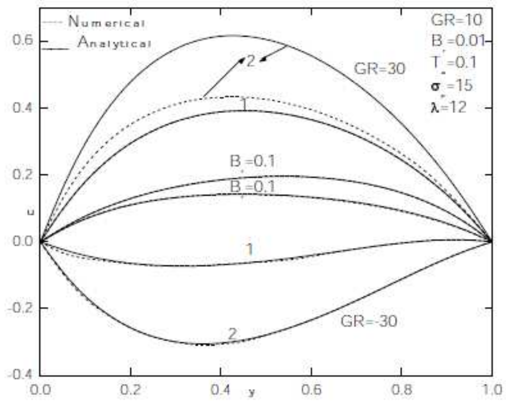


Figure 7a: Velocity vs Y for different values of W_e (Opposite direction) (Isothermal-Isoflux)

Figure 7b: Velocity vs Y for different values of W_e (Same direction) (Isothermal-Isoflux)



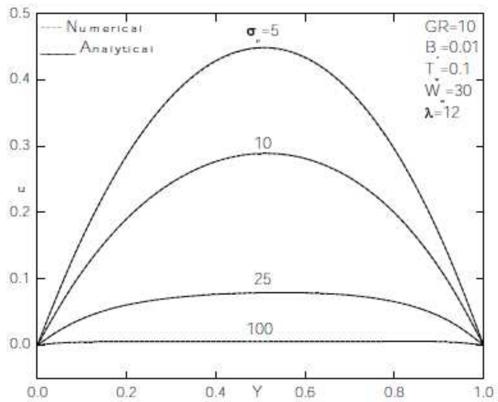


Figure 7c: Velocity vs Y for different values of B_r (Isothermal-Isoflux)

Figure 7d: Velocity vs Y for different values of σ_p (Isothermal-Isoflux)

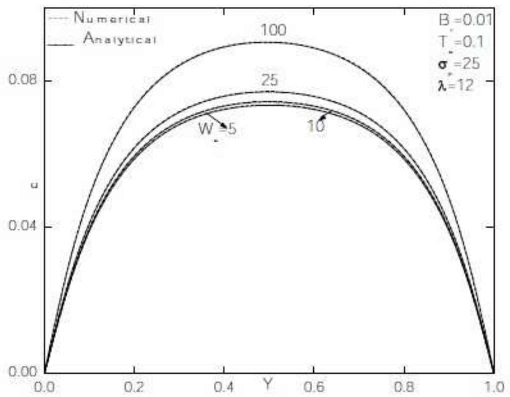


Figure 7e: Velocity vs Y for different values of W_e for $GR = 0$ (Isothermal-Isoflux)

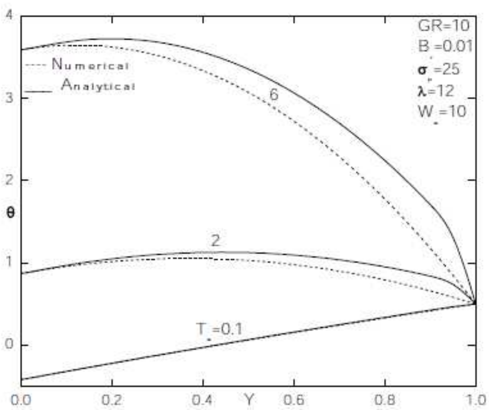


Figure 8: Temperature vs Y for different values of T_e (Isothermal-Isoflux)

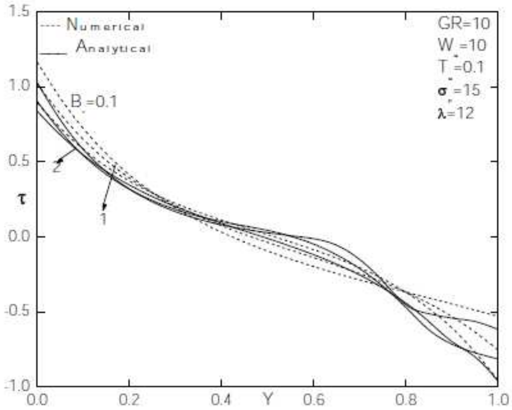


Figure 9: Skinfriction vs Y for different values of B_r (Isothermal-Isoflux)

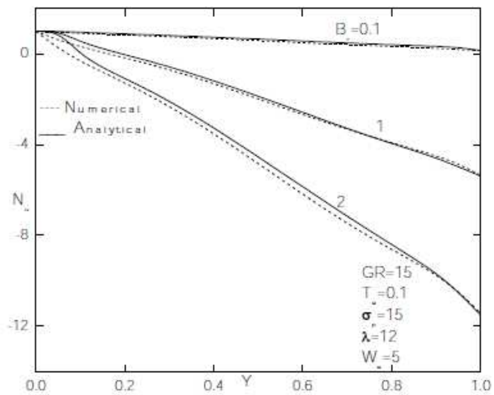


Figure 10: Rate of heat transfer vs Y for different values of B_r (Isothermal-Isoflux)

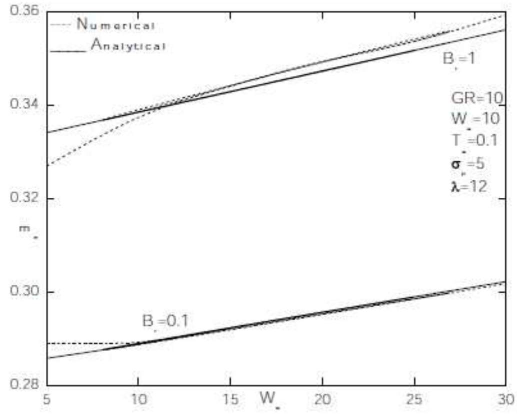


Figure 11: Mass fow rate vs W_e for different values of B_r (Isothermal-Isoflux)

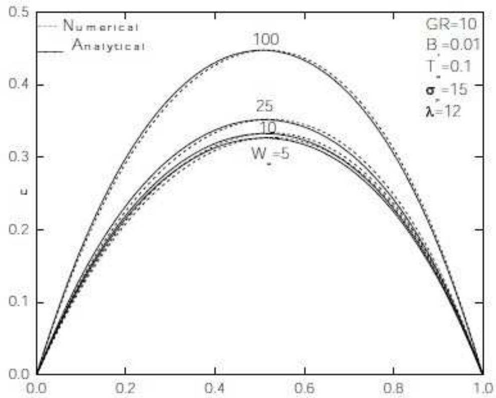


Figure 12a: Velocity vs Y for different values W_e (Opposite direction) (Isoflux-Isoflux)

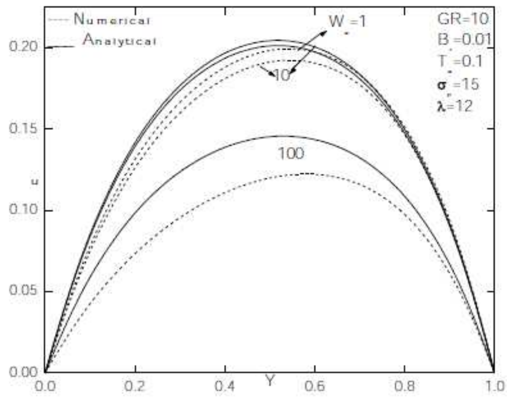


Figure 12b: Velocity vs Y for different values of W_e (Same direction) (Isoflux-Isoflux)

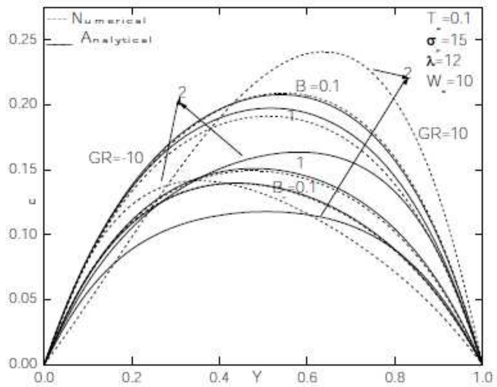


Figure 12c: Velocity vs Y for different values of B_r (Isoflux-Isoflux)

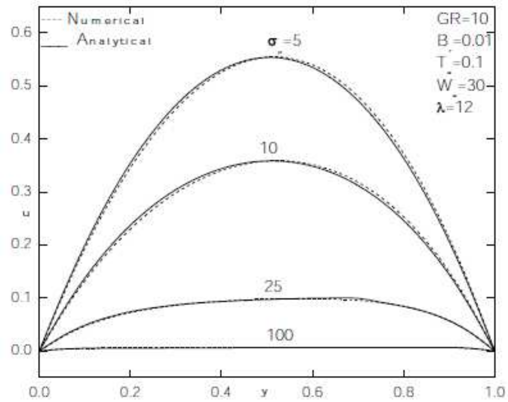


Figure 12d: Velocity vs Y for different values of σ_p (Isoflux-Isoflux)

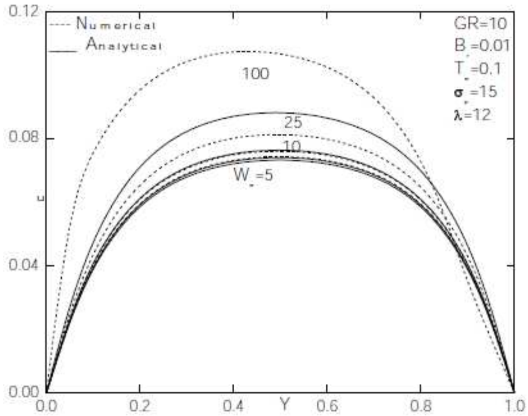


Figure 12e: Velocity vs Y for different values of W_e for $GR = 0$ (Isoflux-Isoflux)

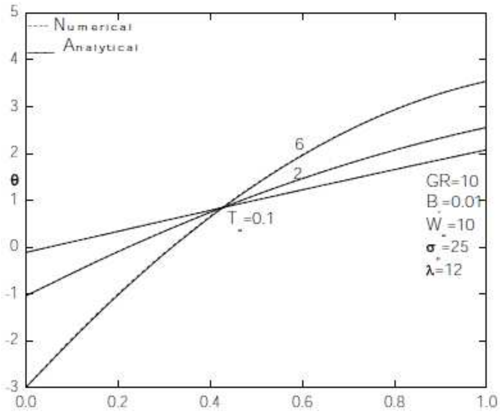


Figure 13: Temperature vs Y for different values of T_e (Isoflux-Isoflux)

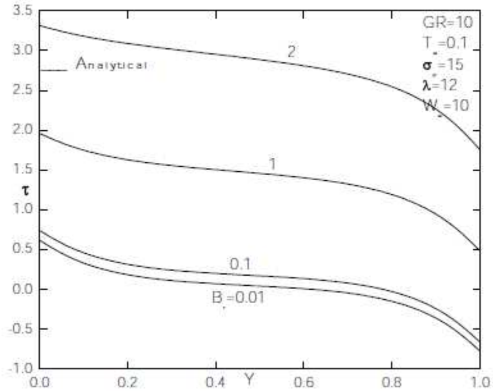


Figure 14: Skin friction vs Y for different values of B_r (Isoflux-Isoflux)

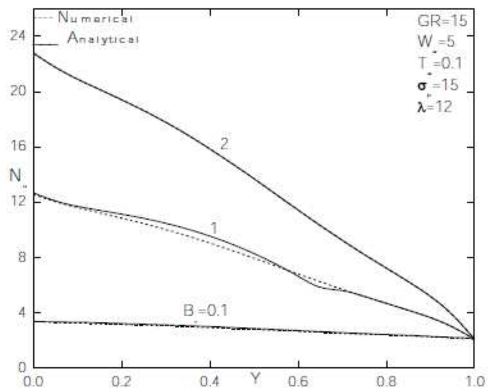


Figure 15: Rate of heat transfer vs Y for different values of B_r (Isoflux-Isoflux)

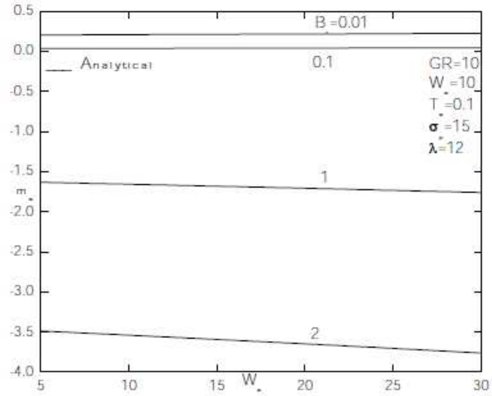


Figure 16: Mass flow rate vs W_e for different values of B_r (Isoflux-Isoflux)

In this paper, the numerical solution of the velocity and temperature fields in the case of asymmetric heating are obtained and the results are depicted in Figures (2a) to (2e), (3), (7a) to (7e), (8) and (12a) to (12e), (13). For asymmetric heating the temperature at the boundaries are different and hence velocity and temperature fields depend on the perturbation parameter B_r. Figure 7c, shows that the flow is upward when GR is positive. On the other hand flow is downward when GR is negative. We also observe that the results obtained form analytical and numerical solutions agree very well. This is due to the fact that for $GR > 0$ there is a buoyancy assisted flow and hence flow is upward and when $GR < 0$ there is buoyancy opposed flow and hence flow is downwards. A greater energy generated by dissipation forces yields greater fluid temperature which in turn increases the buoyancy force.

Figures (2b), (7b), (12b) represents velocity drawn for different values of electric number W_e and for all the three types of thermal boundary conditions, considering the temperature difference (ΔT) being in the same direction of potential difference ($\Delta\varphi$).In this case we find that the velocity decreases with an increase in $W_{e.}$ This implies that the electric field suppress the convection .On the other hand, if $\Delta\varphi$ and ΔT are in the opposite directions, they augment convection.

Figures (2c), (7c), (12c) give the velocity for different values of B_r. An increase in B_r increases the velocity and also flow reversal occurs for 2$^{\text{nd}}$ and 3$^{\text{rd}}$ cases.

Figures (3a,b), (8), (13) and Table 1 and Table 2 give the temperature obtained for different values of W_e and T_e. We see good agreement between numerical and analytical results. We find that temperature increase with an increase in W_e and W_e.

Figures (4a, b), (9a), (14), represent the skin friction for different values of B_r. Skin friction decreases linearly through the distance y and also it decreases as B_r increases in cases 1 and 2. However the skin friction increases as B_r increases in case 3.

Similarly, Figures, (5a, b), (10), (15) show that the rate of heat transfer decreases as B_r increases. Figures (6), (11), (16), show that the mass flow rate increases with an increase in B_r for cases 1 and 2, but decreases for case 3.

8 Conclusions:

Finally, we conclude that the problem of steady, laminar, electrohydrodynamics flow and heat transfer in a vertical channel with asymmetric wall temperatures is studied in this paper using both analytical and numerical methods. The nonlinear dimensionless equations are solved analytically using regular perturbation technique with B_r as the perturbation parameter. In the analytical method only two terms of the perturbation series are considered for evaluation. The governing equations are also solved numerically using finite-difference technique. The numerical results are successfully validated by the analytical solutions. We find that the dissipations enhance the effect of flow reversal in the case of downward flow while it encounters this effect in the case of upward flow. We also find that the suitable value of electric number suppresses the convection.

Nomenclature

The symbols in the above equations have the following meaning

$\vec{q} = (U, V, W)$, velocity vector
p = pressure
μ = viscosity of the fluid
$\bar{\mu}$ = effective viscosity
ν = kinematic viscosity
ρ = density of the fluid
ρ_0 = reference density
ε = porosity
$\vec{E}$ = electric field
$\vec{g} = (-g, 0, 0)$ acceleration due to gravity
$\vec{J}$ = electric current density
σ = electric conductivity
α_b = Temperature coeffecient of electriconductivity
φ = electric potential
k = Permeability
$M = \frac{(\rho c_p)_m}{(\rho c_p)_f}$ Specific heat ratio
$(\rho c_p)_m = (1-\varepsilon)(\rho c_p)_s + \varepsilon(\rho c_p)_f$ ρ_e = electric charge density
$\vec{D}$ = electric displacement or dielectric field
T = temperature
$\sigma_p = \frac{h}{\sqrt{k}}$, porous parameter
A = constant pressure gradient
B_r = Brinkman number

R_e = Reynolds number
L = channel width
λ = ratio of viscosity
$D = 2L$, hydraulic diameter
X, Y = space coordinates
W_e = electricnumber
T_e = thermal electric number
N_u = local nusselt number
K = thermal conductivity
k = permeablity of a porous medium

Greek symbols

β = thermal expansion coefficient
ε_0 = dielectric constant (electric permittivity)
$\chi = \frac{K}{(\rho c_p)_f}$ heat diffusivity (Thermal diffusivity)

Subscripts

0 reference
e charge density
p porous
f fluid
s solid

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Using technology to improve the conceptual understanding of three dimensional geometry

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Abstract : The use of manipulatives as part of mathematics lessons has long been advocated as part of a comprehensive mathematics learning experience. Recent developments such as virtual manipulatives, along with research have caused some to question the role that manipulatives play in learning mathematics. In this paper I re-examine the use of manipulatives within the constructivist paradigm. A software which is called POLY and a CD from National Library of Virtual Manipulatives are used to illustrate how mathematical thinking may be developed.

Key-Words: constructivist, manipulatives, Poly, virtual manipulatives

1 Introduction

Experiential education is based on the idea that active involvement enhances learners learning. Applying this idea to mathematics is difficult, in part, because mathematics is so abstract. One way of bringing experience to bear on learners mathematical understanding, however, is the use of manipulatives. Manipulatives are small, usually very ordinary objects that can be touched and moved by learners to introduce or reinforce a mathematical concept. Manipulatives come in a variety of forms, from inexpensive, simple buttons or empty spools of thread to tangrams and pattern blocks. Typically, it has been the primary grades educators who have generally accepted the importance of manipulatives. "Both Pestalozzi, in the 19th century, and Montessori, in the early 20th century, advocated the active involvement of children in the learning process. Computers can be used to enhance a learners knowledge of mathematics, focusing on what can be done above and beyond with pencil and paper alone [(see [14])]. Using computers as cognitive tools to assist learners in learning powerful mathematics that they could have approached without the technology should be a key goal for research and developmentnot only learning the same mathematics better, stronger, faster, but also learning fundamentally different mathematics in the along the edges, and finally taping together neighbouring faces.

2 THEORETICAL FRAMEWORK THE VAN HIELE THEORY

The van Hiele theory is arguably the best-known framework presently available for studying teaching and learning processes in Geometry. They also investigated the role of instruction in assisting learners to acquire geometric knowledge and raise their thought levels. Resulting from his research van Hiele developed the notion of developmental levels of thinking in geometry. Van Hieles ideas have, according to Pegg [(see [14])], a lot in common with Piagets model of cognitive development in that both models ascribe the development of learner under standing to a process[(see [10])].

Poly is a program for investigating polyhedral shapes. Poly can display polyhedral shapes in three main ways:
as a three-dimensional image,
as a flattened, two-dimensional net, and
as a topological embedding in the plane.

The three-dimensional images may be interactively rotated and Folded/unfolded. Physical models may be produced by printing out the flattened two-dimensional net, cutting around its perimeter, folding Level 0: (Recognition/Visualization).
Learners identify, name, compare and operate on geometric figures on the basis of their appearance in a holistic manner.
Level 1: (Analysis). Learners analyze figures in terms of their components and discover the relationships among those components as well as derive the properties/rules of a class of shapes empirically.
Level 2: (Informal Deduction). Learners logically interrelate previously discovered properties/rules by giving or following informal arguments.
Level 3: (Deduction). Learners prove theorems deductively and establish interrelationships among networks of theorems.
Level 4:(Rigor). Learners establish theorems in different postulation systems and analyze/compare these systems.
Van Hiele has, in addition, suggested that progress from one level to the next could be facilitated by the utilization/ application of five phases of learning which he called information, guided orientation, explicitation, free orientation and integration[(see [20])].

This research project and intervention programme has, to a large extent, been inspired by the following factors:

Fuys et al [(see [7])] indicated in their research findings that attainment of Level 1 (Analysis) is a reasonable series of levels or stages. Pegg[(see [13])] mentions the following differences as important features that make the van Hiele Theory stand apart from other similar models:

(a) It places great importance on the role of language to facilitate the passage through the levels;

(b) It emphasizes learning rather than development, thus the focus is on how to help develop learner understanding, and

(c) It postulates that ideas at a higher level result from the study of the structure at a lower level.

From his research van Hiele has identified and proposed five levels of understanding through which learners must progress in their geometric thought development. Fuys et al [(see [7])] characterised the levels as follows:

(a) goal for children whilst at primary school.

(b) Two previous studies exploring the van Hiele Theory were done in South Africa. The first [(see [17])] used Grades 9 and 10 learners as sample whilst the second by McAuliffe[(see [11])]concentrated on the geometric understanding of pre-service educators. There seems to be a lack of evidence of research in this field at primary school level.

(c) The apparent neglect of geometry teaching in the primary school and the subsequent poor performance at high school level.

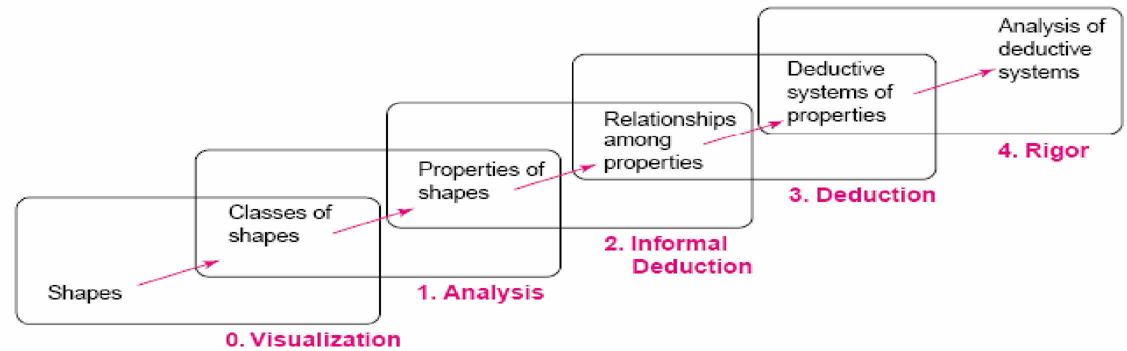


Figure 1 Van Hieles Model [21]

3 METHODOLOGY

3.1 The Sample

A Pretest-Posttest model, was used to measure the effect of the intervention programme on the geometric performance. The programme was implemented at a South African urban primary school with the sample consisting of 40 English speaking learners from a Grade 7 class.

3.2 The Instruments

The instruments used for the research consisted of a test that was administered as pretest and posttest. The test consisted of :

(a) Matching 2 dimensional shapes in different orientation.

(b) Identifying nets of cubes, tetrahedra and octahedron

(c) Making solid platonic shapes using newspaper

Figure 2 learners making solid shapes

The pretest was administered at the commencement of the research project. This was then followed by an intervention programme. The learners were now exposed to POLY and National Library of Virtual Manipulatives where they were able to manipulate 3 dimensional shapes. At the same time learners were asked to make platonic solids using paper.

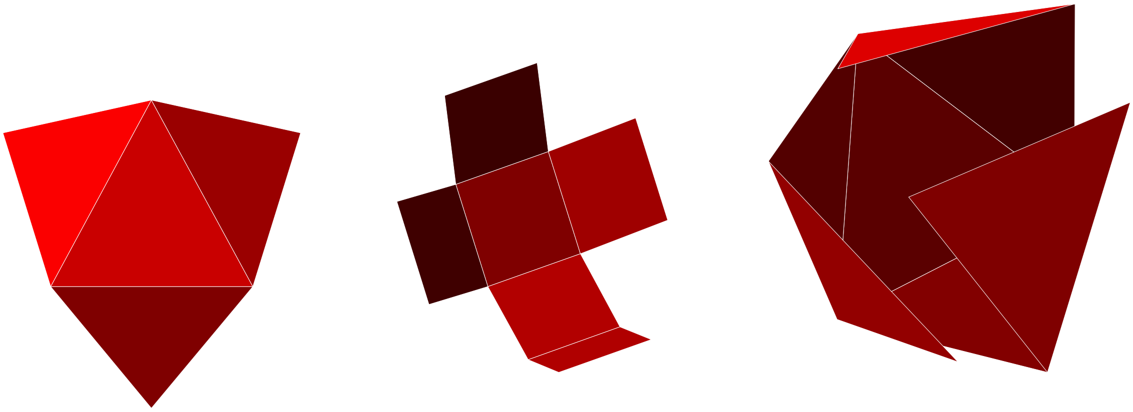


Figure 3 Nets derived from POLY

The posttest was administered at the conclusion of the contact session.

(a) Shape Matching Questions

In this example, you are asked to look at two groups of simple, flat objects and find pairs that are exactly the same size and shape. Each group has about 25 small drawings of these 2-dimensional objects. The objects in the first group are labeled with numbers and are in numerical order. The objects in the second group are labeled with letters and are in random order.

Each drawing in the first group is exactly the same as a drawing in the second group. The objects in the second group have been moved and some have been rotated.

(a) Which shape in Group 2 corresponds to the shape in Group 1?

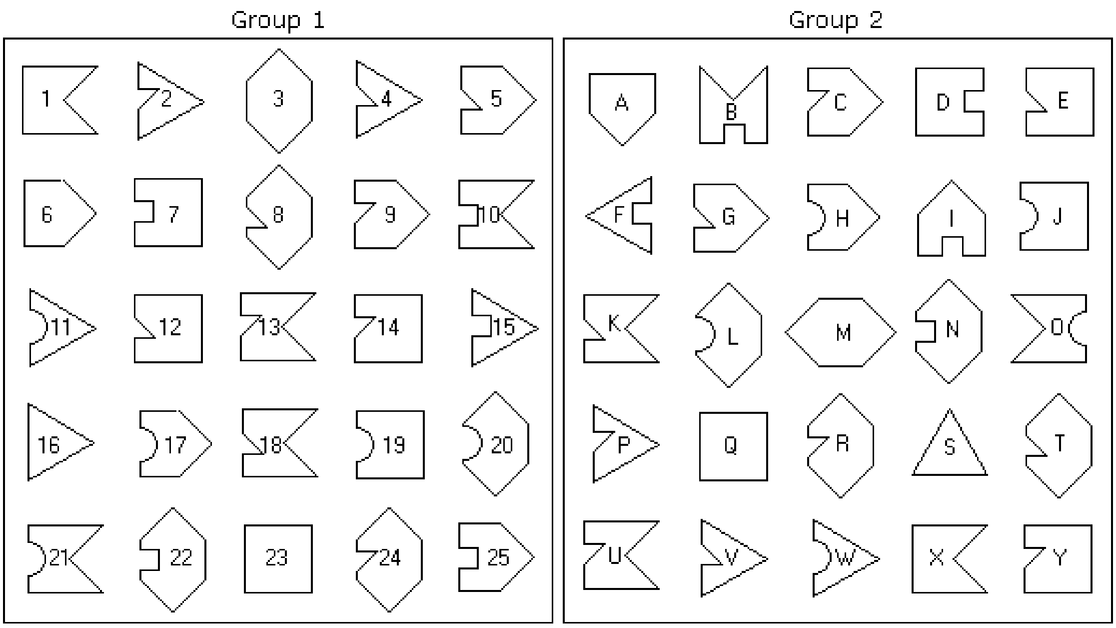


Figure 4 matching 2 d shapes

(b) Nets of a cube

Take a cardboard box like this:

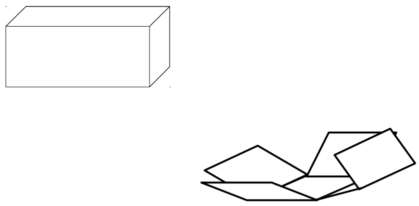
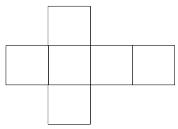


Figure 5 Cutting a box into a net

Cut the edges of the box so that you can open it up and lie it flat The flat box looks like this figure.



(c) Which of the following nets will fold into a cube?

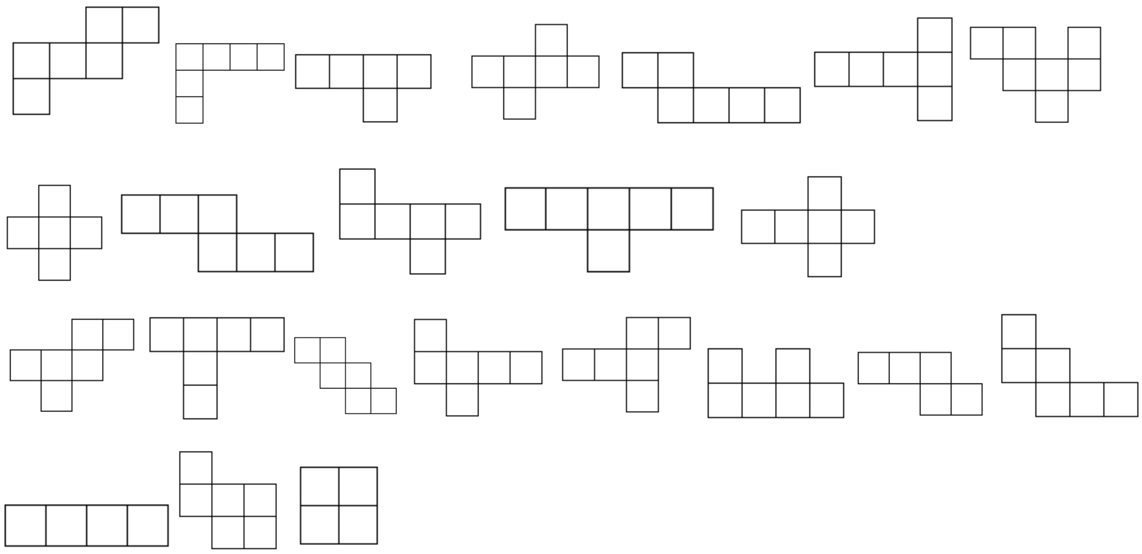


Figure 6 nets of cubes

(d) Nets of tetrahedra

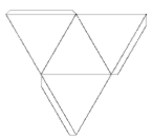
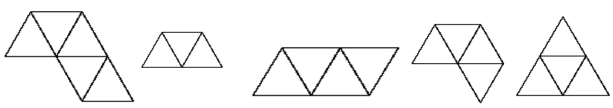
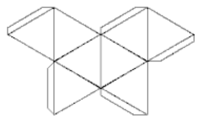


Figure 7 Nets of tetrahedra

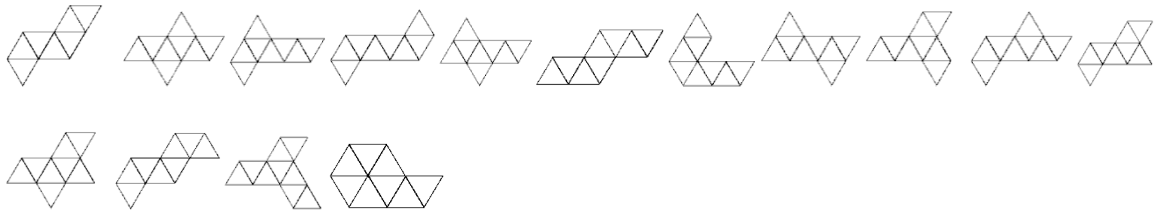
See if you can work out which nets will make a tetrahedron



(e) Nets of octahedron Of course, for an octahedron, you must have eight triangles in the net. There are also four triangles round a point. See if you can find all the nets below.



Nets of octahedra



4 Analysis of the results

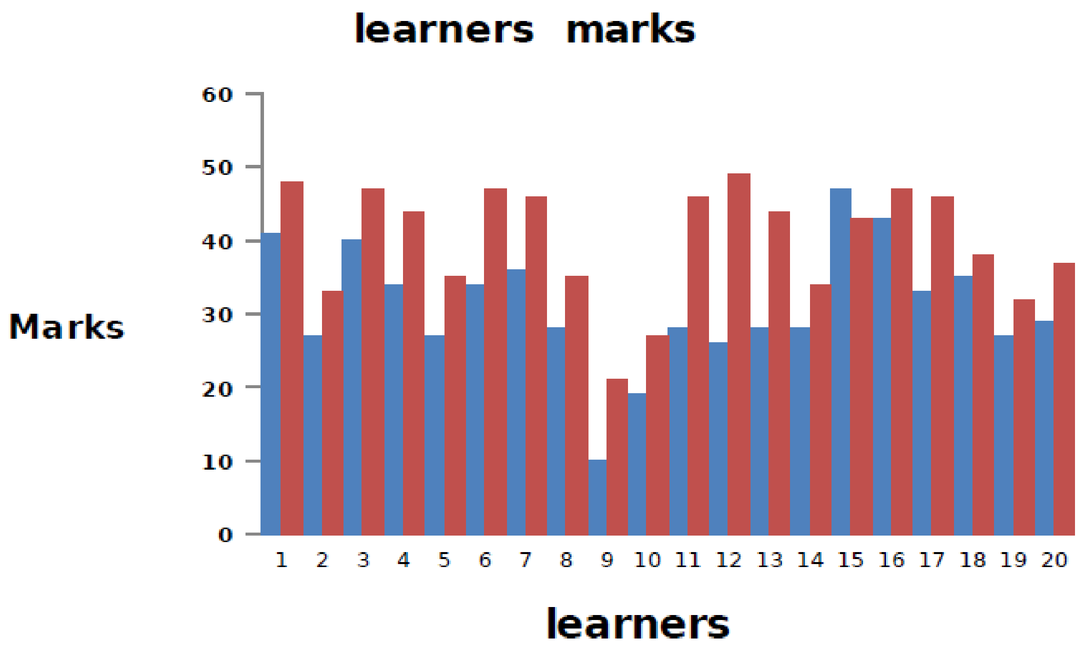


Figure 8 : Learners pre and post-test performance

There are a number of potential benefits of using the computer as a tool for instruction in an educational setting. First, technological tools help to support cognitive processes by reducing the memory load of a learner and by encouraging awareness of the problem solving process. Second, tools can share the cognitive load by reducing the time that learners spend on computation. Third, the tools allow learners to engage in mathematics that would otherwise be out of reach, thereby stretching learners opportunities. Fourth, tools support logical reasoning and hypothesis testing by allowing learners to test conjectures easily. Instructionally, computers allow for a record of problem-solving processes-the fits, starts, and different pathways children followto be recorded and replayed as a window into childrens thinking.

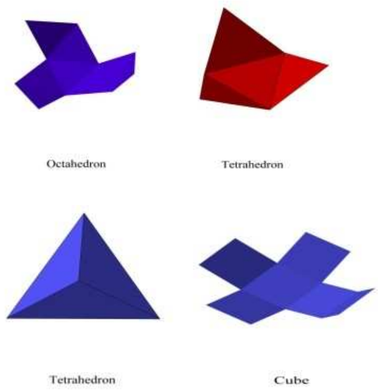


Figure 9 : Interactive nets derived from POLY

This research aims to develop an instructional model to be implemented in an effort to improve the geometric understanding of primary school learners. The model will be work shopped with educators to evaluate its appropriateness and to assess whether it can have an impact on learners performance.

The use of manipulatives has been studied over a number of years. The advent of virtual manipulatives has rekindled interest in the role that manipulatives have to play in the learning of maths. The following is a brief outline key research findings.

Learners who use manipulatives outperform those who dont [(see [3])] . Kennedy noted that: "Although no single study validates the claim that children should use manipulative materials as they learn maths, the collective message garnered from many studies is that materials are worthwhile" [(see [9])] . Hands-on materials may be useful in teaching 3d geometry, at all levels, for the following purposes
To develop visual thinking
To see real applications
To solve realistic problems
Fair Division
The Art Of Packaging
To Model Reality
To make real measures
Using scales to make models
To measure exterior angles
To explore 2 dimensional and 3 dimensional relations

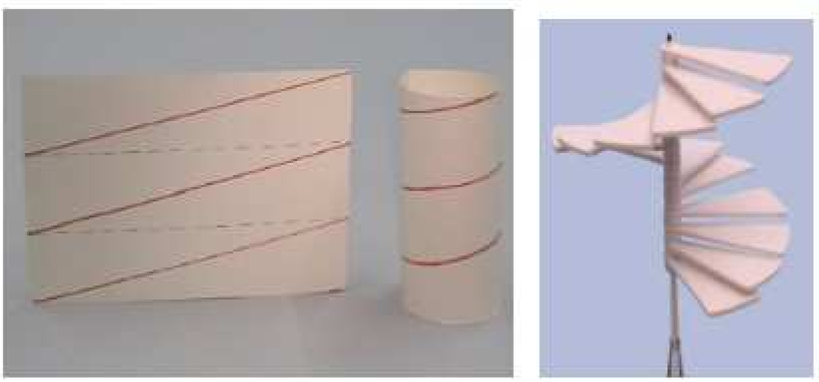


Figure 10 : 2 d and 3 d display

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Cryptology, isoperimetric problems and shadows

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The practical problem is the following. Objects should be labeled with some geometric pictures. To avoid easy falsification, the pictures are chosen randomly. That is, a space S with a distance d and a measure μ is given. The label of one object (picture) is a randomly chosen element of S. More precisely we will mark out some subsets A_i of S and if the random point falls in A_i then it can be used as a label of an object numbered i. The sets A_i must satisfy certain properties as described below.

If $A \subset S, 0 < \varepsilon$ then define $n(A, \varepsilon) = \{x \in S : d(A, x) \leq \varepsilon\}$. The family of subsets $A_1, \dots, A_m$ is called a *geometric code* with parameters ε, ρ and α if (1) $\mu(A_i) \leq \rho$ holds for every i $(1 \leq i \leq m)$, (2) the sets $n(A_i, \varepsilon)(1 \leq i \leq m)$ are pairwise disjoint, (3) $\alpha \leq \mu(\cup A_i)/\mu(S)$.

A geometric code can be applied in the following way. Choose random elements of S according to μ. If $x \in A_i$ then the let its code $c(x)$ be the binary form of i. On the other hand, if $x \in A_i$ holds for no i then x is a waste. (1) ensures that a random choice of $c(x)$ (not knowing x) reproduce it with a small probability. (2) implies that reading x with an error at most ε $c(x)$ still can be recovered. Finally assumption (3) is needed to lowerbound the probability of the waste. The problem is to find the maximum of m, given S, ε, ρ, and α. We give an inequality what has to be satisfied among these parameters, supposing that (1) the measure of a ball is not changed by moving its center, (2) the space satisfies the Brunn-Minkowski inequality.

In our implementation a label is a rectangle containing many small circles (they have a three-dimensional nature, this is why it is hard to copy them), what can be represented by their centers. Because of the computational approximation, it can be supposed that these centers are elements of a grid in the rectangle. Therefore an element of the space S is a subset of the set of the points of the grid where the sizes of the subsets are between a lower and an upper bound. From practical experiences we know that some of the points can be "lost" during the control, therefore the distance d should be defined accordingly. One A_i is therefore a family of subsets of the grid points. Roughly speaking the largest number of such families should be found in such a way that deleting a small number of points from one member of a family is different from a subset obtained by deleting the points from the member of another family. This leads to the usage of the theory of extremal problems of finite sets, especially the "shadow theory".

Let us illustrate the problem in a very-very special case. Choose two families, $\mathcal{A}$ and $\mathcal{B}$ of 3-element subsets of an n-element set in such a way, that deleting one element from an $A \in \mathcal{A}$ and one element from $B \in \mathcal{B}$, the so obtained two-element sets are different. Determine $\max \min\{|\mathcal{A}|, |\mathcal{B}|\}$. The complete asymptotic solution of this "easy-looking" problem will be presented.

ON CERTAIN GENERALIZATIONS OF SKEW NORMAL DISTRIBUTION

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Abstract : The skewness is one of the significant aspects for departures from normality. In order to model departures from normality O'Hagen and Leonard (Biometrika, 1976) modied the density function and proposed a new class of distributions namely skew normal distribution (SND). Recently there is a growing interest in the construction of generalizations of the SND. Here we discuss certain recently developed classes of generalized versions of the SND, which are suitable for handling both asymmetrical and multimodal data sets.

TOMOGRAPHIC IMAGE RECONSTRUCTION: AN INTRODUCTION TO INVERSE PROBLEMS, ANALYSIS AND COMPUTATIONS

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Abstract : Tomogrpahy is a cross sectional imaging of an object or tissue from the mathematical modelling and measured experimental data. The study of tomography involves mathematical modelling, analysis, development of numerical algorithms and its implementation, experimental validation. Thus tomography is an image reconstruction based on mathematical models and experimental data and it has wide applications in many areas of engineering including medicine. In this lecture, we plan to present a general introduction about various tomographic reconstructions. We present X-ray tomography and other tomographies like diuse correlation tomography (DCT), ultra- sound modulated optical tomography (UMOT). Most of the talk will be addressed to cater the general audience both from engineering and science, but if time permits, we also present certain mathematical analysis and convergence together with computational results.

MATHEMATICS IN MEDICINE

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Abstract : Mathematics is regarded as the language of science and Medical Science being an evolved branch of science, is invariably dependent on mathematics though it might be inconspicuous. Form simple arithmetic operations to complex transform series are used in various levels of healthcare, besides the statistical tools used in medical research. In this talk, a brief outline of how mathematics is, or can be utilized in different levels of life science and healthcare research is given. In day to day clinical practice, a clinician is using simple arithmetic operations in dose calculation of drugs according to the body weight, body surface area etc, which is evident, and the application of set theory in deriving at a clinical diagnosis for a given set of symptoms and signs, is ignored or is unknown, even to the clinician itself. Normal functioning of

various systems in the body is governed by the physical and chemical laws which indirectly are dependent on mathematics, like the fluid dynamics, microfluidics of blood flow. Imaging science in medicine is fast growing as the most advanced technology is introduced in market which uses back projection calculations, Fourier transforms, coincidence statistics etc in image reconstruction. Probability and statistics are used in its full potential in epidemiological studies to find patterns, causes, and effects of health and disease conditions in defined populations. Biomathematics is an interdisciplinary scientific research field with a range of applications in biology, biotechnology, and medicine where calculus, probability theory, statistics, linear algebra, abstract algebra, graph theory, algebraic geometry, topology, dynamical systems, differential equations and coding theory are applied in biology. Biological mathematical modeling, relational biology/complex systems biology (CSB), bioinformatics and computational bio-modeling/bio0-computing are the main subdivisions of it.

NON-LINEAR BOUNDARY VALUE PROBLEMS IN THE ANALYSIS OF FLOW AND HEAT TRANSFER OVER A CONTINUOUS MOVING SURFACE: A NUMERICAL APPROACH

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Abstract : Nature is in essence non-linear. Many fundamental laws in science and engineering are modeled by Linear/non-linear differential equations. The origin of non-linear differential equations is very old but they have undergone remarkable new developments in the last few decades. One of the main impulses, among others, for developing non-linear differential equations has been the study of boundary layer phenomenon in fluid motion. At high Reynolds number the effect of viscosity is confined to a layer near the wall where the velocity changes are very large. Prandtl in1904 proposed, what we now call the Prandtl boundary layer equations. The main assumptions Prandtl makes are the following According to Prandtl flow about a solid body can be divided into two regions:

i. A very thin layer in the immediate neighborhood of the body known as the boundary layer in which the viscous effects may be considered to be predominant.
ii. The region outside this layer where the viscous effects may be considered as negligible and the fluid is regarded as inviscid.

With the aid of this hypothesis the Navier-Stokes equations are simplified to a mathematically tractable form, which are then called boundary-layer equations. The theoretical derivations using the mathematical analysis based on these assumptions; agree very well with the experimental observations. in many fluid mechanics problems One such example is the concept of, among the others, the boundary layer behavior over a continuous moving surface. The theory of continuous moving surface was initiated by B.C. Sakiadis in 1961. Using the similarity transformations, the governing equations are converted into nonlinear coupled ordinary/partial differential equations with or without variable coefficients. Their solution structure demands sophisticated analytical/numerical schemes.
The analysis of flow and heat transfer over an infinite range occurs in many branches

of science and engineering. The fluid velocity generally satisfies third order non-linear differential equations depending on the modeling of the stress-strain relation. The corresponding heat transfer satisfies second order, coupled differential equations. In very few instances one is able to obtain exact analytical solutions. In many situations one is compelled to develop a good numerical scheme, fast, as well as accurate, in order to obtain an approximate solution to these coupled equations. Obtaining such numerical schemes to solve these coupled ODEs/PDEs for all governing physical parameters is the key point. For example, all convection and stretching sheet problems under different physical situations arising in flow and heat transfer are normally brought to these boundary value problems. One such topic is the mixed convection flow over a permeable non-isothermal wedge.

EFFECTS OF CHEMICAL REACTION IN A VERTICAL CHANNEL WITH A PERFECTLY CONDUCTIVE THIN BAFFLE

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Abstract : The effect of chemical reaction in a vertical channel with a perfectly conductive thin baffle filled with purely viscous fluid is analyzed. The transport properties are assumed to be constant. The channel is divided into two passages and each stream will have its own pressure gradient and hence the velocity will be individual in each stream. After placing the baffle the fluid in one of the passage is concentrated. The temperature and concentration fields are observed to be governed by complex interactions among dispersion and natural convection mechanisms. The velocity, temperature, concentration profiles for various physical parameters such as ratio of Grashof number to Reynolds number, modified Grashof number to Reynolds number, and Brinkman number, chemical reaction parameter at different position of baffle has been discussed in detail.

EFFECT OF FIRST ORDER CHEMICAL REACTION IN A VERTICAL DOUBLE-PASSAGE CHANNEL

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Abstract : Fully developed laminar mixed convection flow in a vertical channel in the presence of first order chemical reactions has been investigated. The channel is divided into two passages by means of a thin, perfectly plane conducting baffle and hence the velocity, temperature and concentration will be individual in each stream. The coupled, nonlinear ordinary differential equations are solved analytically using regular perturbation method valid for small values of Brinkman number. The effects of thermal Grashoff number, mass Grashoff number, Brinkman number and chemical reaction parameter on the velocity and temperature field at different positions of the baffle are presented and discussed in detail. The increase in thermal and mass Grashoff number and Brinkman number enhances the flow, whereas the chemical reaction parameter suppresses the flow at all baffle positions.

TRIPLED BEST PROXIMITY POINT THEOREMS IN METRIC SPACES

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Abstract : In this paper the concept of a tripled best proximity point is introduced in metric spaces. As an application of these results we prove tripled best proximity point theorems in metric spaces. Presented theorems are generalizations of the recent best proximity point theorems due to Sintunavarat and Kuman [1]. Few examples are given to validate our results.

A RECENT DEVELOPMENT OF NUMERICAL METHODS FOR SOLVING BOUNDARY VALUE PROBLEMS ARISING IN SCIENCE AND ENGINEERING

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Abstract : During the past half-century, the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science and engineering. Computational Analysis naturally finds applications in all fields of engineering and the physical sciences. Numerical analysis is the area of mathematics and computer science that creates, analyzes and implements algorithm for solving numerically the problems of complicated mathematics. Such type of problems originate generally from real-world applications of algebra, geometry and calculus, and they involve variables which vary continuously; these problems occur throughout the natural sciences, social sciences, engineering, medicine, and business. In this talk, numerical methods with computer applications will be discussed.

A PREY-PREDATOR MODEL WITH AN APROACH TO CONSERVATION

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Abstract : In this paper a prey-predator model is considered where there is a single prey and two predators. A suitable taxation policy is introduced for an attempt to save the prey as well as the natural predator from the clutches of the mighty third predator, i.e., human being. The existence as well as stability of the dynamical system is examined. With the help of the numerical analysis the detailed theoretical results was analyzed.

TECHNOLOGICAL APPLICATIONS OF FRACTALS

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Abstract : Benoit B. Mandelbrot is the best appreciated for his first broad attempt to describe irregular shapes in nature. He founded fractal geometry in 1975. Since then the whole fractal theory has been developed using one-step feedback system. In 2002, an attempt was made to study and analyze fractal objects using two-step feedback system. Researchers used superior iteration method to implement two-step feedback system, which was the beginning of a new iterative approach in the study of fractal models and seems promising to solve many problems in fractal theory. The purpose of this paper is to look into technological applications of fractals.

CONTINUITY AT FIXED POINTS

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Abstract : The question of continuity of contractive maps in general and of continuity at fixed points in particular emerged with the publication of two research papers by R. Kannan in 1968 and 1969 respectively in the Bulletin of the Calcutta Mathematical Society Vol. 60 and the Amer. Math. Monthly Vol. 76. These two papers generated an unprecedented interest in the fixed point theory of contractive maps and the question of continuity of contractive maps at their fixed points emerged as an open question. In this survey, we discuss some developments during the last four decades on the question of continuity at fixed points.

EFFECT OF ELECTRIC FIELD ON DISPERSION OF A SOLUTE IN AN MHD FLOW THROUGH A HORIZONTAL CHANNEL WITH AND WITHOUT CHEMICAL REACTION

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Abstract : The longitudinal dispersion of a solute between two parallel plates filled with two immiscible electrically conducting fluid is analyzed using Taylors model. The fluids in both the regions are incompressible and the transport properties are assumed to be constant. The channel walls are assumed to be electrically insulating. Separate solutions are matched at the interface using suitable matching conditions. The flow is accompanied by an irreversible first-order chemical reaction. The effects of viscosity ratio, pressure gradient and Hartman number on the effective Taylor dispersion coefficient and volumetric flow rate for open and short circuit are drawn in the absence and in the presence of chemical reactions. As the Hartman number increases the effective Taylor diffusion coefficient decreases for both open and short circuits. When magnetic field remains constant, the numerical results show that for homogeneous and heterogeneous reactions, the effective Taylor diffusion coefficient decreases with an increase in the reaction rate constant for both open and short circuits. The results of Gupta and Gupta (1972) and Wooding (1960) are obtained in the absence of Hartman number.

SOME APPLIED ASPECTS OF FIXED POINT RESULTS IN GENERALIZED METRIC SPACES

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Abstract : Inspired from the impact and utility of metric space, several generalizations of this notion have been attempted such as pseudo metric space, semi-metric space, fuzzy metric space, probabilistic metric (Menger) space, G-metric space, cone metric space. The fixed point theory as a part of non-linear analysis, is a study of function equation in metric or non-metric setting. It provides necessary tools for the existence of theorems in non-linear problems. The classical Banach contraction principle in metric space is one of the fundamental results in metric space with wide applications. This principle has also a big impact on establishing fixed point results for non-expansive mappings in Banach and Hilbert spaces. The main purpose of this presentation is to discuss some developments of classical metric subspaces in functional analysis with applications to other disciplines.

ON SERIES APPROXIMATIONS AND CORRECTION FUNCTIONS

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Abstract : The presentation deals with derivation and analysis of correction functions and error functions for approximation of some of the infinite series say, for *pi*, *log2* etc. A technique for extraction of some rapidly convergent series from the error functions is also discussed, evaluated and analyzed.

EXPLAINING THE UNREASONABLE EFFECTIVENESS OF MATHEMATICS

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Abstract : Ever since Eugene Wigner highlighted the problem of the unreasonable effectiveness of mathematics in the natural sciences numerous attempts have been made to explain this phenomenon. This paper argues that the resolution to the problem can be achieved by combining the deductive mode of reasoning from first principles pioneered by the ancient Greeks with the classical Indian view of mathematics as a system of conjectures to be validated by appeal to experience. In particular the paper will consider the examples of non-Euclidean geometry and imaginary numbers as vehicles that later turned out apparently fortuitously - to be effective for the articulation of relativity theory. The approach adopted suggests that Greek rationalism can be fruitfully combined with Indian mathematical empiricism as a framework for a new philosophy of mathematics that can show why such anticipations of mathematics are quite reasonable.

WAVE PROPAGATION IN A DOUBLE POROSITY MEDIUM: REVIEW OF CONSTITUTIVE RELATIONS

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Abstract : This study considers the propagation of harmonic plane waves in a double porosity solid saturated by a non-viscous fluid. Existence of three longitudinal waves and a transverse wave is explained through the Christoffel equations, which define the phase velocities and the polarizations of constituent particles. Reflection of plane waves is studied at the stress-free plane surface of the composite medium. A numerical example is solved to calculate the partition of incident energy among the reflected waves. The conservation of incident energy could be made only through the share of interaction energy. The presence of interaction energy is something unexpected, when the medium behaves non-dissipative to the propagation of elastic waves. Reason lies in the constitutive relations being used for double porosity medium, which are not symmetric in elastic coupling, as required by the Bettis reciprocal theorem. Key words: double porosity medium, constitutive relations, reflection, interaction energy.

HEAT AND MASS TRANSFER FOR A CHEMICALLY REACTING PERMEABLE FLUID IN A VERTICAL DOUBLE PASSAGE CHANNEL

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Abstract : The heat and mass transfer characteristics of mixed convection about a vertical double passage channel in a saturated porous medium is presented. The governing equations of continuity, momentum, energy and concentration which are coupled and nonlinear ordinary differential equations are solved analytically using perturbation technique and using differential transform method. Numerical calculations for the analytical expressions are carried out and the results are shown graphically. The effects of the various dimensionless parameters such as porous parameter, thermal Grashoff number, mass Grashoff number, Brinkman number and chemical reaction parameter on the velocity, temperature and concentration fields are discussed in detail at all the baffle positions.

ANALYTICAL STUDY OF VELOCITY DISTRIBUTION BETWEEN NON PARALLEL WALLS IN JEFFERY HAMEL FLOWS

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Abstract : In this study, a steady laminar flow of a viscous incompressible fluid between non parallel plane walls is considered. Unlike plane parallel walls such as channel ows where the flow is governed by parabolic velocity proles, it is not so for the non- parallel plane walls. The flow velocity is governed by strong non linear ordinary deferential equations with the Reynolds number(R) and the convergent and divergent angles as the main parameters. With the availability of excessive computing power and Computer algebraic systems, the problem formulation is revised to give a more accurate analytical solution for the velocity prone using Homotopy Analysis Method. It was found that choosing a suitable initial prole; linear operator and non linear operator corresponding to the problem plays a crucial role in determining the convergence of solution. A recurrence relation is formed in terms of the higher order deformation equation and it is solved for successive values of the velocity function to get the result to a desired accuracy.

SOME GRAPHS WITH n-EDGE MAGIC LABELINGS

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Abstract : In this paper a new labeling known as n-edge magic labeling is introduced. Let G (V, E) be a graph, where V and E represents the set of vertices and edges respectively .We denote the graph G by G (p, q) , having p vertices and q edges. Graph, graph labeling, magic labeling, edge magic labeling, vertex magic labeling, 0-edge magic labeling and 1-edge magic labeling have been discussed. It is proved that some graphs such as Pn, Cn (n being odd +ve integer) and many other graphs have n-edge magic labeling. In this paper the problem of graphs having n- edge magic labeling is studied.

EFFECT OF THERMAL RADIATION ON MHD FLOW WITH VARIABLE VISCOSITY AND THERMAL CONDUCTIVITY OVER A STRETCHING SHEET IN POROUS MEDIA

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Abstract : It is proposed to analyze the steady of two dimensional stagnation point flow of a viscous incompressible electrically conducting fluid over a linearly stretching sheet in porous media with variable viscosity and thermal conductivity. The viscosity and thermal conductivity are taken as inverse linear functions of temperature. The medium is influenced by a transverse magnetic field and volumetric rate of heat generation in the presence of radiation effect. The governing boundary layer equations are transformed into ordinary differential equations by taking suitable similarity variables. The resulting coupled nonlinear differential equations are solved numerically by using fourth order Runge Kutta method along with shooting method. The effect of various parameters such as radiation, porosity, viscosity, thermal conductivity etc have been discussed in detail with computer generated graphs and tables.

ANALYSIS OF A SYSTEM WITH THREE UNITS UNDER PRE-EMPTIVE REPEAT REPAIR POLICY ATTENDED BY TWO REPAIRMEN

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Abstract : In the present paper we have considered a system with three dissimilar units namely A, B and C (with different failure rates). The system is in operational state if either all units are working perfectly or the unit C and at least one of the units A or B are working efficiently. The considered system is repairable and two repairmen having different skills and efficiency are available for repairing. The subunit A has got priority to repair over the subunit B and pre-emptive repeat repair policy is followed. When both the repairmen are involved in repair jointly with different failure rates, copula technique is applied to obtain the reliability measures of the complex system. The model has been solved using supplementary variable technique, Laplace transformation and Gumbel-Hougaard family of copula. Different reliability measures of the system like availabilities, reliability, asymptotic behaviour, M.T.T.F. and cost effectiveness have been obtained. Numerical examples are then studied in detail to demonstrate the theoretical results developed in the paper.

PERFORMANCE EVALUATION ON DIFFERENT STEGNOGRAPHIC ALGORITHMS

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Abstract : This paper describes a universal approach to steganalysis that relies on building a statistical model of rst- and higher-order magnitude and phase statistics extracted from multi-scale, multi-orientation image decompositions. We are able to reliably detect, with a fairly low false-positive rate, the presence of hidden messages embedded at or near the full capacity of the underlying cover image. We expect that as universal steganalysis continues to improve, steganography tools will simply embed their messages into smaller and smaller portions of the cover image. As a result, hidden messages will continue to be able to be transmitted undetected, but high-throughput steganography will become increasingly more difcult to conceal.

ON EXISTENCE OF COMMON FIXED POINT FOR WEAKLY COMPATIBLE MAPS SATISFYING MEIR-KEELER TYPE CONTRACTIVE CONDITION

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Abstract : The aim of this paper is to prove common fixed point theorems for two pairs of weakly compatible maps satisfying Meir-Keeler type contractive condition without continuity and containment requirement of underlying subspaces. Example is also furnished in to demonstrate the validity of results obtained.

CERTAIN FINITE MIXTURES OF INTERVENED STUTTERING POISSON DISTRIBUTION

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Abstract : In this paper, we consider certain finite mixtures of the intervened stuttering Poisson distribution (ISPMD) and derive some of its important properties. The estimation of the parameters of the ISPMD by the method of factorial moments and the method of maximum likelihood are discussed. A simulation study is attempted for comparing the performance of the estimators obtained by both the methods of estimation. The significance of the number of components of the ISPMD is tested. Also, these estimation and testing procedures are illustrated with the help of certain real life data sets.

ON A CLASS OF HYPER-POISSON AND ALTERNATIVE HYPER-POISSON DISTRIBUTIONS

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Abstract : Here we propose a class of hyper-Poisson and alternative hyper- Poisson distributions, and study some of its important aspects by deriving expressions for its probability mass function, mean and variance, and obtain conditions under which the distribution becomes under-dispersed or over-dispersed. Certain recurrence relations for probabilities, raw moments and factorial moments are also developed. Further, the estimation of the parameters of this class of hyper-Poisson distributions is attempted by various methods of estimation and shown that this new class of distribution gives better fit to certain real life data sets compared to the existing models.

AN EXTENDED WEIBULL DISTRIBUTION AND ITS PROPERTIES

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Abstract : A new class of distribution named as extended Weibull distribution is introduced here as a generalization of the well-known extreme value distributions - Weibull, linear failure rate distribution, modified Weibull distribution etc. We study several properties of this new class of distributions by deriving expressions for its characteristic function, recurrence relation for moments, mean residual life function, Shanon's entropy measure etc. The maximum likelihood estimation of the parameters of the distribution is discussed and consider some real life applications of the distribution. The asymptotic behavior of maximum likelihood estimators are also studied by using simulated data sets.

THE DEVELOPMENT OF A SAGE GRAPHER AND ITS APPLICATIONS FOR MATH COURSES

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Abstract : While the usage of powerful mathematics software packages plays a key role in Calculus, Linear Algebra, and higher level mathematics courses, shortcomings in these softwares exist: namely, issues of price and portability. A free online function and concept grapher has been developed using Sage, a mathematics software based primarily in the Python computing language, that students may access and utilize without the need to purchase or install supplementary (and often times expensive) programs. This Sage Grapher, optimized for mobile environments, does not require users to learn an entirely new coding language and is completely open source. Graphs can be manipulated and students can save their work directly on the server, their personal computers, or their mobile device to utilize them at their convenience. A total of eight different versions have currently been developed: graphers for general functions, parametric functions, implicit functions, polar functions, and other essential concepts of Calculus. The abilities of this novel math tool will be demonstrated. As well, these graphers have a .STL file printing capability that students may use for 3-D printing purposes within or outside of the classroom. Application of these graphers can serve as an effective method of concept visualization for all students of mathematics, regardless of age or level. http://matrix.skku.ac.kr/grapher-html/sage-grapher.html

GRAPH THEORY APPLICATIONS

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Abstract : Graph theory continues to evolve rapidly under the growing interest evinced by mathematicians, scientists and engineers in the past few decades in its powerful combinatorial methods, which have been put to use in diverse and often unexpected applications in science, computer science, engineering and technology. In this talk we discuss some applications of graph theory in mathematics and in other sciences

THE OPTIMAL JOINT HARVEST OF A PRAWN FISHERY AND A POULTRY OF BIRDS IN A LINKED BIOECONOMIC SYSTEM

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Abstract : This paper considers the optimal joint harvest of prawns and poultry in a linked bio-economic system. Through the cultivation process, poultry and prawns are reciprocal predators of one another. Prawns of non-marketable quality are fed to the birds, and birds which perish (in greater numbers in the face of increased density) are fed to prawns, along with a lot of other things that one doesnt usually consider prawns to eat (hogs, broken rice, etc.). The paper derives optimality conditions for the joint effort imposed in each of these industries, where effort is somehow analogous to the control variable in classical Gordon-Schaefer fishery problems. Growth of both species is governed by parameters as well as externally applied nutrients and the biomass of the other species available as supplemental nutrition. Analysis of the boundedness of this dynamical system is discussed. The conditions for local and global stability are derived. Finally, an optimal harvesting policy is discussed by applying Pontryagins Maximal Principle. Due to linearity of the objective function with respect to the control variable, the solution is bang-bang in this control and the best policy is to reach the singular equilibrium as quickly as possible by switching to the singular control.

COMPARING MATHEMATICS OF DIFFERENT CULTURES: DIFFERING PREOCCUPATIONS, PROOFS AND PRACTICES

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Abstract : In this seminar, I propose to examine critically the ways in which different mathematical traditions of the past have been characterised by historians of mathematics. A litmus test of a valid mathematical practice today is 'proof' and a number of criticisms have been levied against certain traditions because of the perceived absence or lack of rigour in their proof procedures (seen today as the litmus

test of whether we are "doing" real mathematics or doing it well). By taking specific examples, I propose to examine the validity of these criticisms and indicate the relevance of alternative proof traditions today.

SOME MORE DIVISOR CORDIAL GRAPHS

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Abstract : A divisor cordial labeling of a graph G with vertex set V vertex G is a bijection f from V to $\{1, 2, 3, \ldots, |V|\}$ such that an edge uv is assigned the label 1 if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1differ by atmost 1. If a graph has a divisor cordial labeling, then it is called divisor cordial graph. In this paper we prove that flower graphs and helm graphs are divisor cordial. We also prove some special graphs such as switching of a vertex of cycle, wheel, helm; duplication of arbitrary vertex of cycle, duplication of arbitrary edge of cycle.

CRYPTOLOGY NUMBER THEORY AND DIGITAL SIGNATURE

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Abstract : This is one exciting application of number theory and is one of the emerging areas in mathematics. The paper presents some preliminaries into this branch, digital mechanism its types, and its applications.

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