Pedagogical guidance for mathematics in Wales

Section 1 Principles

There is no research that says one way of teaching mathematics is better than any other in all situations and for all school mathematics. This paper therefore gives a range of pedagogic elements, drawing on international research to describe justifications and limitations, so that in Wales teachers of mathematics can make professional decisions about their teaching, on their own and with others. The educational context is the generic pedagogical principles of Wales as set out in Successful Futures, namely:

Good teaching and learning

1. maintains a consistent focus on the overall purposes of the curriculum
2. challenges all learners by encouraging them to recognise the importance of sustained effort in meeting expectations that are high but achievable for them
3. means employing a blend of approaches including direct teaching
4. means employing a blend of approaches including those that promote problem solving, creative and critical thinking
5. sets tasks and selects resources that build on previous knowledge and experience and engage interest
6. creates authentic contexts for learning
7. means employing assessment for learning principles
8. ranges within and across Areas of Learning and Experience regularly reinforces
9. cross-curriculum responsibilities, including literacy, numeracy and digital competence, and provides opportunities to practise them
10. encourages children and young people to take increasing responsibility for their own learning
11. supports social and emotional development and positive relationships
12. encourages collaboration

These generic principles do not address the detail and special features of teaching and learning mathematics.

For mathematics there are necessary strands of mathematical proficiency[1]

- conceptual understanding
- comprehension of, and capability with, the symbol systems of mathematics
- ability to formulate problems mathematically in order to solve them (problem solving)
- fluency of memory and skill with facts and techniques
- logical reasoning about relations between and within concepts and relationships, justifying and proving them (mathematical thinking)

These aspects of proficiency contribute to the ultimate goal of learners becoming able and willing to use and apply mathematics to solve unfamiliar problems within mathematics, in other subjects, and throughout life and work. The challenge in mathematics is to combine these facets of mathematics within the overall principles for Wales, and also to ensure that all students gain an adequate knowledge of the concepts, facts and processes that are technically necessary for their overall mathematical progression.

This means that the choice and mixture of appropriate pedagogical strategies has to relate to the mathematical idea or behaviour being taught, and overall has to enable development of all the components of mathematical proficiency.

[1] These are adapted for the Welsh context from various sources, mainly *Adding It Up: Helping Children Learn Mathematics* Kilpatrick, J., Swafford, J. * Findell, B. (2001) Mathematics Learning Study Committee, National Research Council, US.

Section 2

Pedagogical issues for mathematics

In the tables that follow, pedagogical issues are presented that relate particularly to **the cognitive work of learning mathematics as a structured, cumulative body of knowledge.** These are ways to enact **SF principles 2, 3, 4 & 5**, and also to ensure intellectual development in the subject matter of mathematics.

Questions are suggested to support PD, development of practice, and individual planning. Detailed discussion of advantages and limitations associated with these issues is given in Section 3.

Coherently structured mathematical content	**No amount of good general pedagogy can make a lasting difference to learning unless the mathematics on offer is conceptually coherent**
Pedagogical issue	**Guiding questions**
Make connections	Why are we learning this new thing now? How does it connect to or extend what went before? Does it build on, or alter, what went before? What is it like that we already know? How is it different?
Mastery	What has to be secure before new ideas can be built using this concept? What is the intended endpoint of this sequence of lessons?
Starting points	How can we start thinking about this idea, given what learners already know?
Consistency	How does the teaching of this idea relate to methods, language, images used in previous and future teaching? If these have to be changed, adapted, extended, how can this be done?
Lesson sequences	What sequence of experiences is necessary for learners to move from what they currently know and can do to understanding, being fluent in, and being able to use and build on a new idea? What language forms, images, materials and recording are necessary for this process to be coherent?
Representations and learning media	What different experiences, materials, representations, language forms, dynamic and digital images, formats and symbols should learners have to fully grasp a new idea, its rules of manipulation, and its potential?
Digital tool use	How can their understanding of this mathematics be influenced, enhanced, altered or reorganised by using digital technology as an exploratory tool, or a tool for doing the procedural work, or a tool for displaying the outcomes of reasoning? This includes a range of mathematical software: computer algebra; dynamic geometry; statistics; spreadsheets; turtle control; and graphplotting.

Problem solving	What is the role of problem solving? Is there a problem that creates a need to meet and use a new idea? Is there a problem through which a new idea can be discovered or developed? Can this new idea be used to solve, and pose, further problems?
Mathematical thinking	What can be explored? What conjectures might arise? What situations can be modelled? What relationships have to be understood? What has to be demonstrated, justified or proved? What new questions can be posed?
Direct instruction or mathematical enquiry	What aspects of this new mathematical idea are best learnt by being told? (e.g. symbolism, conventions, unexpected methods, special cases) What can be discovered, derived, conjectured, explored, modelled, reasoned about?
Example choice	What sequence of examples would make this new idea clear for learners? What examples would expose a need for this new bit of maths? What examples might introduce limitations and misconceptions? Who provides the examples: textbook, teacher, internet resource, or students themselves?
Explanation	What has to be explained and how? What can be deduced or derived or conjectured?
Connections outside mathematics	What authentic connections can be made that would aid understanding of (a) the mathematics and (b) the context?

Modes of working	**The mode of working should fit the intended mathematical learning**
Pedagogic issue	**Guiding questions**
Discussion	What aspects of a particular mathematical idea need to be discussed? Is discussion between teacher and students, or students in pairs of groups, or teachers in front of students? What happens to the outcomes of discussion?
Independent work	What aspects of mathematics need quiet, independent thought?
Exercises	What is being practised? What is becoming fluent? What is being noticed?
Groupwork	What aspects of mathematics need to be done in groups? Is this about task completion or about everyone learning a new idea? How will individual group members contribute to the mathematics?
Interactive animations	What aspects of mathematics are best learnt by trying things out in animations: dynamic apps, digital software, calculators? Should this be done by the teacher, by learners, in whole class, in groupwork, individually?
Sustained tasks	Is this task worth spending extended time on, such as homework, a sequence of lessons, or frequent return to the task? Is there an ‘aha!’ discovery that students can make after some time contemplating and experimenting?
Written work	Who writes what, where and how? Does it need to be permanent or temporary? Who is the audience? Does time have to be spent drawing formats such as grids, graphs, etc. that could be given on paper or screen?

Classroom and mathematical norms	**Mathematics is particularly challenging in terms of classroom atmosphere, expectations about behaviour and a mathematical mindset**
Pedagogic issue	**Guiding questions**
Questioning	What kinds of questions are used frequently in the classroom? What kinds of questions enable learners to develop all aspects of mathematical competence? Who asks those questions: teacher or learners?
Mathematising situations	Are learners regularly encouraged to: identify variables in situations; explain how variables relate to each other; represent these relationships using: algebra, graphs, spreadsheets, programs, statistical arguments?
Logical and mathematical reasoning, justifying and proving	Are learners regularly encouraged to: express their reasoning and describe stages in their thinking in mathematical situations? present arguments and decisions based on mathematical relationships? Are learners regularly encouraged to say why describe how they know give examples, extreme examples, non-examples and counter-examples?
Self-check	Do learners have to depend on the teacher or textbook to know they are right? Do learners have a range of self-checking strategies? Do mathematical tasks have their own inbuilt checking mechanisms?
Shifts in understanding	Are learners given opportunities and support to move from simplistic views of mathematics as 'techniques you do' to 'structured concepts to be understood and used'?
Habits of mind	What habits could be inculcated in learners so that they become better at mathematics?
Curious and enquiring minds	Are learners regularly asked: What puzzles them; What questions they have; What they can find out about a situation?

	And are they given the opportunity and support to follow up their ideas?
Thinking time	How long do learners need to think about something? What classroom norms support extended thinking?
Mathematical talk	Is it the norm to use mathematical language when discussing mathematical ideas?
Positive attitudes towards mathematics	Is this a place in which they are safe and will be listened to? Is everyone welcome? Are they in the presence of someone who enjoys maths? Is mathematics promoted as a human endeavour?
Avoiding maths anxiety	Who is having real emotional difficulty with the subject?
Classroom layout	Does layout support everyone’s learning and a positive attitude? Can everyone see, hear, move appropriately? Does everyone have access to materials, equipment, and digital technology to support learning?

Section 3 Elaboration of pedagogic issues

Coherently structured mathematical content	
Pedagogic issue	**Background**
Make connections	Connecting/relating new ideas to existing ideas is how we learn. Most mathematical ideas are not totally new for learners but are extensions of previous ideas: new notations; new kinds of input; new question-types to answer; new contexts; 'what if…?' movements from familiar situations to less familiar situations. Connections help maths to make sense to people. This kind of connecting provides starting points for new ideas. Using models and diagrams that embody underlying mathematical meanings, and therefore extend beyond the current use, helps meaningful connections to be made, e.g. using numberline instead of hundred square, using balance model for equations instead of 'change the side change the sign' etc. Connecting algebraic expressions to numerical rules aids understanding. Connecting equations to graphs, spreadsheets, dynamic diagrams gives meaning to solving, manipulating, transforming etc .
Mastery	'Mastery learning' originally meant that all, or nearly all, learners could come to understand an idea but some might need more teaching time, and more associated activities, than others. It does not assume that some learners are incapable, but that they may need to be taught more, differently, better. Learners cannot progress safely to dependent, more complex, ideas until the current one is 'mastered', which takes time and multiple experiences. It was not, originally, a recipe for a particular kind of teaching or classroom/group organisation. Sometimes our understanding of an idea develops over a long time, during which we use it in many different contexts and more complex aspects of mathematics. Not all mathematics is hierarchical. Mathematical habits of mind are not hierarchical. Assessment for learning is important as it shows who needs more teaching and of what kind. The nature of teaching for those who need more support has to be carefully developed so that they experience the same forms of reasoning, language, imagery, use, that others do. Usually teachers 'triage' through assessment and provide extensions/challenges as well as further support. It is also important to ensure everyone is prepared for a new idea.
Starting points	Although mathematics is all joined-up, it has to be taught in timed chunks and learners are learning lots of other subjects at the same time, so the starting point of mathematical work is a marker for what is going to be the focus and how the learners can make sense of it. Starting a lesson with a title or learning objective that makes no sense to learners does not support coherence. A lesson or sequence of lessons could start with a question that is left over from a previous lesson, or a puzzle that depends on extension of a familiar idea, or a rehearsal of something that is going to be needed,

	or a ‘what if …?’ from previous work, or a story of some kind, or a real problem that is current: (how would you measure hurricanes?). The starting point has to intrigue, or fulfil a shared need (tomorrow is your exam so …).
Consistency	If learners are given different explanations, different models, different ways to think about a mathematical idea by different teachers they may become confused. Examples: when subtracting by decomposition do they ‘borrow’ or do they ‘exchange’? When solving linear equations do they ‘change the side, change the sign’ or ‘do the same thing to each side to undo the expression’? On the other hand, learners can handle different manifestations if they have an organising context. For example, if they have the word ‘function’ they can compare linear, quadratic, exponential functions as examples of the same mathematical class; if they have the word ‘multiply’ they can compare repeated addition, division by fractions, and scaling as examples of the same idea.
Lesson sequences	Obviously in mathematics teaching things are taught in an order, chunked according to lesson times. Where a concept involves several critical points it makes sense to think about how best to organise those points so that the whole concept can be built. When people say that ‘mastery’ is about ‘step by step’ they need not assume that these are instructions such as ‘step one’ has to be achieved before ‘step two’ and so on; instead they can recognise that mathematical concepts entail contributory concepts (sometimes called critical aspects), and teaching involves choices about how and when these are addressed. After a concept has been worked on for some time learners will have formed strong connections between the idea, its representations and meanings, associated techniques, and how it can be recognised and used. The endpoint can always be a deep familiarity with a mathematical structure. Taking care over the small contributory concepts to build a higher-order concept takes time, but it leads to better learning than, say, diving into a whole new concept and then having to deal with misconceptions and those who have not understood. In the medium term it saves time because the concept does not have to be retaught, and the structure of mathematics is secure enough for later pieces of the jigsaw to be inserted. For example, spend time discerning angles before considering questions of equality of angles; spend time thinking about ratios before diving into trigonometry.
Representations and learning media	To learn new concepts about anything, and about mathematics in particular, people have to be able to manipulate it – either through objects or diagrams or dynamic software or with examples; create some kind of way to record what happens, such as with diagrams, pictures, or tallies, or scribbles of various kinds; and symbolise the idea with formal numbers or algebra or formal forms of language. This is true for all new ideas – not only for young children. Sometimes the enaction can take place with mental images in the imagination. This idea is sometimes called ‘CPA’ (concrete-pictorial-abstract) but this has lost some of the flexibility of meaning and order. Ideas can be presented in an abstract/symbolic form and explored using materials; some diagrams act both as objects and icons; not all mathematics needs pictures but some other kind of marks on paper or arrangements of objects (counters, rods etc.) Some people prefer to use ‘CRA’ where R stands for representations, since this formulation gives flexibility.

	Teachers have to be aware of how to 'detach' the idea from the materials or icons in which it was first met; language helps to create 'inner speech' which works instead of objects. As well as needing to learn mathematical conventions, learners need to be able to translate between representations in order to grasp the background idea that is the basis. So, just as they need to experience examples that represent the background abstract idea they also need to use several representations. This includes manipulable materials, diagrams, software etc. Carefully chosen layouts can expose patterns and relationships between parts of a calculation (e.g. grid multiplication, relating to area), or between different versions of the same idea (e.g. expressing an addition and its associated subtractions). Teachers may need to model the use of different representations and materials, but some (numicon, numberline, rods, geometric shapes and diagrams, spreadsheets…) can be explored by learners before any demonstrations by the teacher because the way they interact exposes mathematical ideas. Learners may need time to make their own connections between representations, and develop 'inner talk' to embed these connections.
Digital tool use	The use of digital tools is totally embedded in all workplace mathematics and all further study in mathematics and in other subjects. For decades, studies have shown that learners who have access to digital tools become more knowledgeable about mathematical ideas, in particular about numbers, quantities and the behaviour of geometrical objects. Dynamic capability enables questions to be posed about covariability that would otherwise be out of reach. Functions and graphs can be studied from the pint of view of classes of objects related by transformation. Real data sets, and therefore big data sets, can be used routinely to pose authentic statistical questions and make predictions. Digital technology should be fully integrated into mathematics teaching and learning.
Problem solving	Posing a problem within mathematics, or from outside mathematics, that needs new techniques or developments of old techniques is a genuine and intriguing way to introduce new concepts, since this is how they were developed historically. This approach requires the use of problem-solving skills and develops them into a hybrid of mathematical thinking and reasoning. A typical difficulty in finding suitable problems is that those that can be solved by trial-and-adjustment, or by ad hoc methods, do not require the adoption of new methods by all learners. Problems that can trigger a variety of directions of work are not helpful in achieving specific mathematical learning goals.
Mathematical thinking	To develop mathematical thinking, a question, task or situation can be offered with a 'finding out' goal, or an initial result to be found before further questions can be posed and generalities sought. Learners can become independent mathematical thinkers, with flexible approaches to unfamiliar situations and applications, with a sense of mathematics as a human activity, and a mature understanding of concepts. Exploratory work can reveal unexpected connections, and provide raw material for conjecturing, generalising, application and adaptation of abstract ideas. Often explorations or open-ended tasks vaporise towards the end as learners complete tasks at different rates or in different ways, or find out different things. The final phase of work is crucial – to ensure everyone has made mathematical

	progress in the direction for which the task was set, which could be a core mathematical idea or developing an important way of working, or both.
Direct instruction or mathematical enquiry	Direct instruction is a very efficient way to ensure that all learners meet necessary information. There are many definitions of 'direct instruction'. The meaning that is most useful in mathematics teaching is that some things are only learnt by being told, and telling everyone at once and checking they have heard and understood is an efficient way to share the information. Research results have often been extended to say that mathematics is best learnt through direct instruction. This may be true of some facts, technical terms, conventions, representations, actual instructions about what to do to get started in a task, and so on. However, teaching has more dimensions than 'instruction' implies, and no high-performing jurisdictions depend only on direct instruction. Arithmetical methods can be taught this way but other work is needed to ensure understanding, recognition, flexibility etc. Learners can become dependent on teachers and textbooks if most teachin is by direct instruction. The SFPP cannot be enacted if all teaching is explicit about target knowledge, and all teaching is led by the teacher. Research into teaching problem-solving using direct instruction is inconclusive and tends to focus on simple applications.
Example choice	A feature of teaching in high performing countries is that the examples a teacher uses are deliberately chosen and sequenced to build conceptual understanding. This applies to any worked examples the teacher presents, and also to examples that learners work on themselves. In a well-designed set of examples there is more to be learnt than 'doing' maths. Examples expose mathematical structures and relationships, from which conjectures and generalisations can be formed and discussed. Use of non-examples is important to show what is, and what isn't, relevant for a concept. This need not be taken to mean that trudging through examples is a good idea. Traditionally, 'examples' have meant 'use the method just taught to work out answers to these questions'. But examples are what we all use to present mathematical ideas, because the ideas are abstract. Examples need to be chosen, or generated, so that learners get an idea of the range of objects, or situations, that relate to the concept. Too often they are a list of work to be done with harder numbers, or with negative signs, rather than a designed pathway into understanding.
Explanation	It is the teacher who knows most maths (usually) so has the responsibility of helping learners come to know it. Explaining can be necessary at several points in a lesson, and in maths we have a wide range of tools for explaining: words, diagrams, pictures, graphs, dynamic representations, demonstrations, and numerical examples. Getting learners to explain their current thinking is valuable for them and informative for the teacher. An explanation must 'make things plain' so has to lay out the meaning of something in ways that learners can interpret and connect with what they already know. Explanations have to be planned. When learners explain to each other they often 'show how to' but if the teacher is a good explainer, learners can pick up more complex ways of explaining. Using examples plus diagrams is often the most easily understood kind of explanation, since generalising from examples and experience is how we learn everything in life.

Connections outside mathematics	Connections outside maths give it purpose; also connections to familiar situations enable learners to imagine relationships. This kind of connecting can appear as applications after a new idea and its associated skills are learnt, or can provide the purpose for a new mathematical idea, e.g. what if the temperature is at zero and falls another two degrees overnight? Connections can be contrived and unrealistic, and hence unconvincing. Not all maths connects 'outside'; some exists because maths itself is interesting. Contexts that turn some learners on can turn other learners off.

Modes of working	
Pedagogic issue	**Background**
Discussion	Discussion can be used to compare methods; practise explaining mathematics; listen to and critique arguments; extend knowledge, strategies and understanding by hearing others. Discussion can be dominated by individuals; can lead to a sharing of ignorance. Learners need to learn how to discuss maths; how to develop precise language; how to use examples; how to evaluate contributions. 'Discussion' led by teachers is often not real discussion but orchestration, evaluation and guided talk.
Independent work	All learners have to be able to function mathematically on their own. They need time to think without being swamped by others' talk; time to internalise methods, facts etc.. Individual needs to read, practise and review will differ. If working on your own is the main mode there is no opportunity to learn from each other; maths can be seen as a solo activity; many learners experience being stuck with nowhere to turn; there is a risk of practising, and hence embedding, errors. They need to develop personal motivation and resilience to sustain independent work.
Exercises	Doing exercises (several unconnected question using versions of the same technique) is a traditional mode of working in mathematics lessons and little research has been done about it, except to observe that the last hundred years do not provide much positive evidence from practice. Recently, there is more understanding about how exercises can be structured to promote fluency, understanding and mathematical thinking together (see example in Appendix 1) and that studying well-chosen worked examples might be more beneficial. Well designed exercises, reflected on as a collection of examples, can provide raw material for mathematising and conceptualising.
Groupwork	Collaboration and group functioning are worthy aims in themselves. Some mathematics depends on generating several sources of data, forms of representation, separate steps of multi-step tasks that can be shared work in a group. All can contribute to completion of something beyond what any individual could have done. The task has to need the group, i.e. needs contributions from several people to be completed. Do all participants move forward mathematically? Can groups help support resilience? How are groups constituted? Do some students always end up supporting others rather than moving forward mathematically? Do some groups and

	individuals always get stuck with lower level approaches (e.g. I'll do the measuring)? Groupwork develops communicating and listening about maths, requires enough understanding to explain, and sometimes developing communication aids understanding.
Interactive animations	Cycles of observation-conjecture-test or action-reflection-generalisation-justification can be used regularly as a mode of learning, i.e. mathematical thinking. When used on the main screen the whole class can participate in these cycles of learning and initiate action. The stages of 'reflection-generalise-justify' need to be included – it is easy to miss these out. The teacher can provide the conventional language and symbols for expressing what is observed. Who initiates changes in an animation, teacher or learners? Can *all* ideas be pursued or only some, and who chooses?
Sustained tasks	Mathematical problem solving and exploration typically takes time, and may involve several attempts before an insight is achieved that enables solution. Learners can develop resilience and persistence if they are given time to return to a suitable problem or exploration over a sequence of lessons and thus experience being stuck, becoming unstuck, gaining insight, developing suitable notation for their findings.
Written work	It is worth questioning whether all mathematics has to be done in books on squared paper. Benefits can be gained if learners use whiteboards for trying out ideas, rough work books, different kinds of paper unless squares are mathematically necessary. Different decisions about written work might be made if the work is for personal revision, for presenting to the teacher, for recording groupwork and so on. Digital printouts can also be valuable. Are there facilities in your classroom for groups to work together on whiteboards, or flip charts, rather than on A4 paper?

Classroom and mathematical norms	**Mathematics is particularly challenging in terms of classroom atmosphere, expectations about behaviour and a mathematical mindset**
Pedagogic issue	**Background**
Questioning	The questions teachers pose, whether verbally or in tasks, whether to individuals or whole class, shape learners' thinking and also their view of mathematics. Teachers can develop and extend a repertoire of questions that influence the mathematical atmosphere in their classrooms. Frequent use of standard questions can form mathematical habits, e.g. what is the same? What is different? How does A change as B changes? What changes? What stays the same? Now I know this, what else can I deduce? To do this, what do I know? what do I need to know? A distinction between open and closed questions is not helpful; more valuable are distinctions between questions that require: recall; conjecture; explanation; comparison; justifications; connections; suggestions about procedures; anticipation; analysis; rapid response; exemplification. Also, the purpose of questions could be for: testing; reflection; driving the lesson on; getting a sense of participation; genuine shared puzzlement; etc. Because questioning plays such a central role in learning, key questions should be planned.
Mathematising situations	Identifying variables in situations, constructing the relations between what is known and what is not known, deciding what operations or techniques to perform, manipulating the relations to find more information, is a main use of mathematics in employment and everyday life, and in other school subjects. The process of analysing a situation in terms of relationships is at the heart of many traditional word problems, and also of understanding algebraic expressions and equations. Sometimes this process is called 'formulating' and its importance is now recognised in PISA international comparison tests. Digital software can be a useful support, since different formulations can be tested with actual data. Mathematising is also a key component in quantitatively informed decision-making.
Logical and mathematical reasoning, justifying and proving	"If, then, because …" reasoning is met through mathematics and is different from everyday reasoning. Where everyday reasoning often depends on generalising from examples, mathematics requires statements to be fully justified from facts, known relationships, or derivation from previous results using agreed reasoning rules. For many learners, mathematics lessons are the only place they will meet this kind of reasoning.

Self-check	In maths there is always another way to work things out, check using digital tech, or reason backwards. Otherwise learners are dependent on teachers and textbooks to find out if they are right or not. If questions are set that have an inbuilt checking device (such as the answers forming a pattern, or all being the same) this helps develop the habit of reflecting on answers.
Shifts in understanding	Most school mathematics depends on moves from doing something to expressing and using an abstract idea. Examples of such shifts: skills to methods; representations to symbolism; tangible objects to imagined objects; familiar to unfamiliar ideas; intuitions to reasoned arguments; immediate reactions to longer considerations.
Habits of mind	Engendering mathematical habits of mind is an important facet of shared classroom culture. A substantial list of these is given in Appendix 2. Most of these relate to issues described in earlier sections of this document.
Curious and enquiring minds	Learners who have an enquiring approach to mathematics, and draw on a repertoire of strategies as well as knowledge, do better in exams (because they approach questions with a problem-solving mindset) as well as enjoying the subject more. Many mathematical concepts can be found by extending earlier concepts, or by enquiring into phenomena. What happens? Does it always happen or only sometimes? Why does it happen? Can I make it happen? ICT enables phenomena to be generated either as multiple cases or as dynamic events, e.g. 'discovering' scientific notation while using a calculator; seeing effects of transformation on geometric shapes; using a spreadsheet to generate covariation; doing and undoing a sequence of operations etc. Most school mathematics can be developed through curiosity. In practice there isn't time to learn everything this way, but in theory the only things that need to be told are conventions, definitions, axioms, names, and the rules of symbol systems.
Thinking time	Giving explicit thinking time, perhaps a 'hands-down think' allows more learners to collect their thoughts and give considered answers (see 'think-pair-share' above). Thinking time can also be used after an answer has been given, without comment from the teacher. The traditional model of: factual or technical question-hands-up- answer- response from teacher is not conducive to participation, nor to develop thinking, nor to give space for learners to make their own judgements about the answer that has been given.
Mathematical talk	Talk about meanings, methods, reasoning, justifications, reasoning, relationships etc. enables all learners to attain a higher level of mathematical thinking and reasoning; mathematical vocabulary should be developed. The final talk in a lesson is often

	a list of answers, but more helpfully would be a synthesis of what has been discussed in a lesson.
Positive attitudes towards mathematics	A positive atmosphere about the subject is important for willing engagement by learners. Feeling safe enables learners to take risks Teachers generate enthusiasm, and so do TAs. It is important that the whole school should also do so – not merely saying 'it is important for your future employment' but more positive messages about doing maths here and now.
Avoiding maths anxiety	In maths, learners can be wrong more times and more publicly than in any other subject if the teaching interactions are based on short, quick questions that have to be correct. Recognise the panic symptoms: raised pulse, sweat – these can be real for a few people. Environments that are more about thinking, discussion, conjecture, reasoning and suggestion – and questions with alternative answers – are more conducive to positive engagement and also lead to stronger development of mathematical thinking, and are more authentic in terms of future uses of maths in study and employment. Although it is not yet known for sure if maths anxiety is a particular condition with maths, it IS known that some people find maths lessons very stressful and may physically go into panic mode, needing deep breathing etc. to calm down.
Classroom layout	Everyone needs to see and hear clearly, and be able to work on their own, or in a pair, or with others in a group as appropriate. A mathematics environment is helpful to generate positive attitudes. Are the wall displays incorporated into teaching? Can learners' own work, or work in progress, be displayed so that the nature of 'mathematical work' is understood? Can learners move around to get equipment? Is there easy access to computer(s), tablets, calculators when a need might crop up during a lesson for individuals or for a whole class?

Section 4

Theories of mathematics pedagogy

Some theories of mathematics pedagogy and their implications for teaching mathematics. All of these theories can be found on the internet.

Readiness to learn	Being 'ready' for new mathematical learning is a combination of knowledge, experience, disposition and a repertoire of strategies and methods. Think about what is entailed in a new idea. The idea that you have to wait for children to become 'ready to learn' through natural development is no longer thought to be an appropriate model of learning – you need to prepare them. Identify what prerequisite knowledge or habits of mind are necessary as you would prepare a toolkit. Sometimes an idea, method, bit of knowledge is not necessary beforehand, but becomes necessary and purposeful during a task, and learners get a better sense of it as a result.
Conceptual development	Rather than 'becoming ready' it is now understood that the use of language, models, diagrams, software can create situations in which learners get the curiosity, language, methods of manipulation and toolkit to build new concepts for themselves and cement these through talk with each other and with teachers. There needs to be discussion, reflection, in order to embed the connections and relationships that are made during a task that presents new ideas. Just 'doing the task' is seldom enough.
Cognitive conflict	We form ideas based in our experiences and the examples we have seen. This can lead to limited or faulty understanding, such as 'multiplication makes things bigger' or 'Squares are not rectangles' or 'continuity means you can differentiate'. To extend understanding and avoid misconceptions, situations can be offered that challenge existing assumptions and habits. This creates conflicts that have to be resolved to get a stronger sense of mathematical meaning.
Growth mindset	Our capacity to learn can change, particularly if learning is understood to be complex and needing effort – not a straight line of getting right answers and doing things correctly and quickly. Our brains are constantly reforming internal links and connections; our ways of thinking about a mathematical idea should change and grow. All our experiences contribute to development, including those that have to be rethought such as misconceptions or dead-end reasoning. Research about explicit application of Growth Mindset strategies in schools shows no special effects, except the common effect that if teachers believe something will help more children learn maths – then it usually works.
Flipped classroom	Flipped classroom is a strategy in which homework provides some kind of direct instruction and practice or problems, and lessons involve discussing the problems and concepts. 'Flipped classroom' assumes that homework is normally practice of what has been taught directly in lessons. It is based on the belief that time spent working on practice examples in lessons could better be spent discussing concepts and reflecting on answers with the teacher. Many teachers set homework that is preparation for the next

	lesson rather than leftovers from the previous one. Use of online videos demonstrating mathematical techniques is associated with 'flipped classroom' and, while some of these might have use, some teachers prefer to create their own as they would rather be 'in control' of how an idea is demonstrated.
Blended learning	Concepts are learnt in a variety of ways: visual, verbal, diagrammatic, practical, dynamic, static, alone and with others. The word 'blended' reminds teachers that there needs to be some gathering of threads from time to time to make connections and understand equivalent ideas – blending the experiences towards coherent understanding.
Realistic mathematics	Realistic mathematics education (RME) is a way of teaching in which a sequence of tasks is given based in situations that can be imagined in a realistic way. The situations have a similar underlying mathematical structure, so that learning maths takes place through adopting the ideas and structures that have been manifested in the tasks.
Variation theory	Variation theory says that we notice what varies against a background of invariance, or what is invariant against a background of change. It leads to careful choice of examples (for example, using systematic variation of important aspects), representations and tasks, and deliberate sequencing so that relationships unfold through seeing what is the same and what is different in what is on offer. For example, a new idea needs to be compared to old, but similar, ideas so that definitions are purposeful. This avoids learners getting limited conceptions, e.g. all rectangles have to have sides parallel to edge of a page or screen; multiplication always makes things bigger, etc. Relevant questions are: What's the same, what's different? What changes and what stays the same? When one thing changes, what else changes and how? What is, and what is not? How can important patterns and connections be made obvious? What different representations are there of the same idea?
Learning styles	Mathematics can be accessed through physical objects and actions observations, words, diagrams, numbers, abstract statements, rhythmic recitations etc. Learning is best when many of these are combined. The idea that individuals have 'preferred' styles may or may not be true, but to learn mathematics people need to learn to use all these ways in, whether they come naturally or not. The founder of this theory (Howard Gardner) rejects absolutely that it is an instruction for teaching individuals differently.
Cognitive load theory	The mind can only attend to a few things at a time, so giving too much variety at once, or too much information, can cause overload. Teaching needs to take into account what is necessary for the core idea, what is germane to the core idea, and what is extra load. Heavy load can be managed by 'chunking' ideas into larger ideas – for example, the idea of proportion 'chunks' the multiplicative relationships between variables; the idea of a total 'chunks' counting an accumulation of separate sets. In early adolescence we generally become capable of handling more bits of information, about 4, where younger children handle fewer bits. Some

	theorists claim that load should be minimised to make room for working memory to make sense of things. However, one of the desired outcomes of mathematics teaching is that learners can take a complex situation and sort out what is necessary within it – breaking the task down into steps. Many of the concepts met in secondary school: algebraic equations; functions; trigonometric ratios; probability – 'chunk' simpler ideas so that they can be handled. If time has not been given to conceptualisation then learners will be overloaded later on.
Intelligent practice	This phrase has been introduced by NCETM to describe careful and purposeful collections of examples that reveal patterns, relationships and concepts. A balance has to be found between becoming fluent (i.e. able to do things quickly and accurately without much thought) and reflecting intelligently on what has been done, to learn from it.
Teaching for mastery	NCETM suggests five principles for teaching to achieve mastery: coherence, variation, fluency, representation & structure, and mathematical thinking. All these have been addressed in earlier parts of this document.

Appendix 1

Example of exercise

Example of a well-structured exercise that is not only about doing techniques, but also gives raw material for discussion, reflection on patterns and behaviour, extension to other numbers and powers, and deepening understanding of indices. This is adapted from *Extension Mathematics* by Tony Gardiner.

If $C = -3$, find

i. C^2
ii. $-C^2$
iii. $(-C)^2$
iv. C^3
v. $-C^3$
vi. $(-C)^3$
vii. C^4
viii. $-C^4$
ix. $(-C)^4$

Appendix 2

Mathematical habits of mind

From: Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, *15*(4), 375-402.

Students should be pattern sniffers
Students should be experimenters
Students should be describers
Students should be tinkerers
Students should be inventors
Students should be visualisers
Students should be conjecturers
Students should be guessers

Mathematicians talk big and think small
Mathematicians talk small and think big
Mathematicians use functions
Mathematicians use multiple points of view
Mathematicians mix deduction and experiment
Mathematicians push the language
Mathematicians use intellectual chants

Geometers use proportional reasoning
Geometers use several languages at once
Geometers use one language for everything
Geometers love systems
Geometers worry about things that change
Geometers worry about things that do not change
Geometers love shapes

Algebraists like a good calculation
Algebraists use abstraction
Algebraists like algorithms
Algebraists break things into parts
Algebraists extend things
Algebraists represent things

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