Pergamon

Journal of Mathematical Behavior
20 (2002) 461–475

Mathematical Behavior

Instances of mathematical thinking among low attaining students in an ordinary secondary classroom

Anne Watson*

Department of Educational Studies, University of Oxford, 15 Norham Gardens, Oxford OX2 6PY, UK

Abstract

This paper is a report of a classroom research project whose aim was to find out whether low attaining 14-year-old students of mathematics would be able to think mathematically at a level higher than recall and reproduction during their ordinary classroom mathematics activities. Analysis of classroom interactive episodes revealed many instances of mathematical thinking of a kind which was not normally exploited, required or expected in their classes. Five episodes are described, comparing the students' thinking to that usually described as "advanced." In particular, some episodes suggest the power of a type of prompt which can be generalized as "going across the grain." © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Mathematical thinking; Low attaining students; Underachievement in mathematics; Mathematical generalization; Mathematical exemplification

1. Introduction: deficiency approaches to low attainment

"Low attaining students" are generally classified on the basis of competence on routine tests. Perhaps it would be more accurate to say they are classified on the basis of accumulated incompetence in tests and other written work. Rarely do teachers have time to search actively for their mathematical strengths which may not be manifested in the usual ways. Teachers and authors tend to talk of low attaining students in terms of what they lack or what they cannot do. The effect is to negatively label the learner rather than the learner's behavior, and to dwell

* Tel.: +44-1865-274024; fax: +44-1865-274027.
E-mail address: anne.watson@edstud.ox.ac.uk (A. Watson).

PII: S0732-3123(02)00088-3

on negative aspects of academic performance. Typically most comments, whether positive or negative, are about general behavior rather than specifics concerning learning. Haylock (1991) is in most respects a very positive writer about low attaining students, but in case studies his positive comments are usually about behavior and low level skills, and only very rarely about features of mathematical thinking which might accompany such skills. Denvir, Stolz, and Brown (1982) list statements made about low attaining students by teachers: there are a vast number, all of them negative. Perhaps this is inevitable since it is low attainment according to some expected norm, communicated in an expected way, which they have in common with each other, yet dwelling on general lack can draw attention away from individual capabilities.

In the UK, teaching mathematics to low attaining students in secondary school often involves simplification of the mathematics until it becomes a sequence of small smooth steps which can be easily traversed. Even in exploratory situations, little is expected beyond generating data. Wiliam (1998) points to an assumption that low attainers *should* not, rather than *could* not, progress to high levels of mathematical thought. Frequently the teacher will "take the pupil through the chain of reasoning" and the learner merely fills in the gaps with arithmetical answers, or low-level recall of facts and so on (Pimm, 1987, p. 53; Watson, 2000a). Achievement in such situations is identified with getting to the end of the work, completing something successfully, filling in the required partial answers, and maintaining some concentration throughout. But none of these definitions of success necessarily indicate that anything new has been learnt. All they signal is completion. Learning, at best, might be characterized by the reproduction of algorithms with the associated problems of that approach (Duffin & Simpson, 1999). Path-smoothing is unlikely to lead to learning on its own, since the strategy deliberately reduces a problem to what the learner can do already in terms the learner already recognizes. No cognitive processing is required. Learning, presumably, is about deciding what to do and stringing the steps together — but the helper has already done this for the learner! Furthermore, the learner is reinforced in the view that if she sits there doing nothing for long enough, the helper or the teacher will provide the appropriate task-transformations (Bauersfeld, 1988).

For secondary students, accumulated past failure may lead to expectation of future failure, both by the student and by the teacher. There may also be utter confusion about some or all arithmetical procedures. The learner may have experienced a melee of concrete images, money calculations, helpful hints, special cases, partially-remembered rules and so on, but have no access to any secure conceptual framework on which future number work can be developed, or which can be called into play when necessary in other mathematical contexts. It is tempting for secondary school teachers to try a piecemeal approach, concentrating on remediating what cannot be done rather than building on what can be done.[1] This brings students into renewed contact with the site of their previous failures while offering them very difficult learning tasks — to learn, recall and use facts and procedures which are

[1] Upgrading programs, such as Springboard 7, take this approach of remediation and repetition.

unconnected to anything about which the learner feels secure. Often this takes place in the context of test preparation in which short-term performance-boosting may take priority over long-term effective learning. Low attaining students may consequently conceive of mathematics as fragmentary, illogical, difficult and alienating. To identify and remediate the difficulties of individuals in large classes is impractical, unless the difficulties turn out to be wide-spread, so the individual attention required for a fully successful remedial approach is very hard to organize.

2. Proficiency approaches to low attainment

The Cognitively Guided Instruction Project, reported in Fennema, Franke, Carpenter, and Carey (1993), encourages teachers to build on learners' existing understandings. That is, it develops from existing proficiencies and concept images rather than focusing on filling gaps in knowledge or remediating weaknesses. The project depends on teachers' identification of learners' cognitive states and support for the use of research in devising teaching strategies. In this paper I am taking an alternative approach to proficiency which focuses on mathematical thinking rather than on cognitive growth. I am looking for evidence of thinking more commonly associated with advanced mathematical thinking, rather than with performance on standard test items. Strategies which emerge as useful in this study can be incorporated both into macro and into micro teaching and do not require close attention to individuals.

Freudenthal (1971) saw mathematics as an activity involving organizing and mathematizing, not as a body of knowledge to be 'learned.' Learning objectives derived from this view release low attaining students from the frustrations of needing to KNOW more and more things, which keep slipping away from the memory, and replace these with opportunities to DO mathematics in the kinds of ways described by Boaler (1997) and Ahmed (1987).

Low achieving students in a school studied by Boaler (1997) learnt mathematics by becoming involved in explorations of mathematical situations. Alongside higher achieving peers, they found their own pathways through discovery to solution. They made their own sense of tasks, and hence of mathematics and of mathematical thinking. They did significantly better in national examinations than comparable students at a comparable school using more traditional methods and groupings, but they still achieved in only a small proportion of the mathematics on offer in the curriculum, and their results were low compared to the national cohort. A problem with the approach described in Boaler's book is that ad hoc mathematical methods do not provide a foundation for mathematics at higher levels of abstraction. *Everyday* thinking skills were well-exploited in the mathematics lessons (which is more useful than a pretence that mathematics lessons support everyday thinking), but *mathematical* thinking was not necessarily developed systematically in all students.

Ahmed's (1987, p. 72) approach emphasizes proficiency rather than deficiency. He concentrated on general learning skills such as curiosity, problem-solving and willingness to work and found that *all* students display these in *some* contexts. The results of his project showed how teaching which incorporates open-ended problems, extended exploratory tasks, groupwork and discussion could fundamentally influence the attainment of students, simultaneously raising expectations and making achievement in mathematics seem worthwhile. He also found that what was effective for those classified as low attaining students was also effective for higher attainers.

In their concentration on investigative tasks, social interaction and classroom organization, the studies of Ahmed and Boaler yield little information about ways of thinking which can be applied to learning curriculum mathematics, with its techniques and concepts. Research studies by Tanner and Jones (2000), Adhami, Johnson, and Shayer (1998) and Adhami (2001) show that students who are explicitly encouraged to take on mathematical challenges and follow them through in special tasks with explicit attention to thinking skills may do better across the spectrum of school mathematics, but neither of these studies attend explicitly to the question of using and developing mathematical thinking during normal curriculum tasks. It is as if standard topics are seen to have a special status which is independent of what an expert would identify as mathematical behavior, and in which success is not dependent on expertise in mathematical modes of thought.[2]

3. What is special about mathematical thinking?

Some of the extant descriptions of mathematical thinking concentrate on problem-solving heuristics (Feuerstein, 1980; Mason, Burton, & Stacey, 1982; Polya, 1962; Romberg, 1993; Schoenfeld, 1985) while others relate more directly to the development of conceptual understanding in mathematics (Krutetskii, 1976; Tall, 1991).

Tall (1991) says that those who succeed are those who, without being taught, can reflect on processes, abstract entities from them (Dubinsky, 1991), manipulate these, and hence gain an image of concept, while overcoming obstacles, such as inherent difficulties in the subject, and countering unhelpful intuitive tendencies to gain a conventional understanding and acceptance. Students for whom the usual logical sequence of presentation of formal mathematical products matches their own cognitive development are in a highly advantaged position because the matching of conceptual development and formal, generally-accepted mathematics is more likely for them than for others who have taken a different route. Dreyfus (1991) adds the ability to move fluidly between representations as an additional feature of mathematical success.

Krutetskii (1976) provides a description of common features of the ways of thinking and working manifested by gifted mathematicians. These include a grasp of structures, mental flexibility, an inclination to generalize, facility with manipulation of mathematical entities and a memory for the characteristics of members of classes. Michener (1978) comments that mathematicians see "understanding" as including "knowing to fool around with examples"

[2] I am not suggesting that these authors believe in this separation themselves. They have taken a pragmatic approach in order to work with teachers on developing one area of practice.

(p. 382). Thus, doing mathematics successfully involves certain identifiable "habits of mind" (Cuoco, Goldenburg, & Mark, 1996).

One might think it irrelevant to refer to these skills in an article about low attainment, since they occur in books about advanced mathematicians, but Haylock (1991, p. 24) reports how one low attaining student (Ben) voluntarily produces and uses a pictorial representation of division of whole numbers which he then manipulates in order to get a correct answer. This illustrates more than merely an ad hoc approach; it shows the ability to create and use an appropriate visual image, that is, a kind of concept image which is manipulated until a conventional result is obtained. Since symbolization and transformation are two mathematical actions, and since this student performed them voluntarily, I would say that he was doing some mathematical thinking.

It is noticeable that some aspects of mathematical thinking developed above are also ordinary ways of thinking and problem-solving which play a specially important part in mathematics. For instance, it is common to exemplify in order to illustrate complex descriptions; it is common to switch from one representation to another, such as when one draws a map to accompany verbal directions. Thus, it is plausible that we could expect to find the roots of these explicitly mathematical behaviors in the behavior of most human beings, even those labeled and classified as low attaining students. Consequently, we could look for evidence of these powers, and try to employ and exploit these in standard mathematical topics and concepts.

Vergnaud (1997), Tall (1991) and Freudenthal (1971), from different perspectives, all support the notion that students need to experience personal conceptual progress towards understanding, and that this journey is describable in terms which are mathematical. Vergnaud (1997) says that what is needed to describe progress towards conceptual understanding in mathematics is within the mathematics itself, rather than as a separate psychological or socio-cultural commentary. An example of this might be to regard negative numbers as an extension of the number line, and the associated arithmetic as an exploration of what it might be possible, or sensible, to do with these new objects. Thus, one might approach a calculation such as $2(1 + (-1)) =$ as an extension, backwards, of the pattern: $2(1 + n) =$ and the pattern in the answers for positive and zero values of n leads you to suppose that the answer for $n = -1$ might be zero. This in contrast to multiple attempts to make negative numbers concrete by contextualizing them within questions about temperatures and debts. Progression towards knowledge of negative numbers can be seen as mathematical, a desirable extension of positive numbers, rather than contextual (as in "this is how temperatures go up and down") or social (as in "this is an accepted convention").

To suggest that this approach, rather than an artificially concrete one, is appropriate for students is to suggest that they can see and extend patterns within whole-number mathematics, and can use whole numbers and associated operations,[3] and that they might be interested in doing this.

[3] I am not suggesting that this is *all* that is required, or that this is sufficient; there may be motivational, emotional and contextual factors which also affect ability to make the extension.

4. Mathematical thinking and low attainment

For this paper mathematics is seen as sequences of: objects and their properties; classes of objects with their associated properties; generalizations about classes; abstractions and relations which become objects for more complex levels of activity. These can be represented by a variety of symbols which can, like abstractions and relations, be further manipulated in their own right. Drawing on this view, I set out to look for evidence that low attaining students could:

1. identify and use patterns;
2. abstract through reflecting on processes;
3. exemplify and counter-exemplify in ways which do more than imitate what a teacher has offered, since this implies some level of generalization;
4. develop and use images of concepts;
5. change and manipulate representations;
6. perhaps work with abstractions and relations, which is generally a feature of advanced study.

My aim was to show also that these attainments can be manifested during work on standard topics, and hence might be systematically exploited by teachers, leading to further development of their use and power.

Harries (2001) exhibited some proficiencies of mathematical thinking we might find among low attaining students. He placed students in a situation in which they could leave a trace of their thinking, showing clearly how they completed a mathematical drawing task. Using Logo, Harries was able to keep records of the various ways in which they had tried to instruct the computer to draw and move simple objects, and then to use those objects as new elements in the creation of further shapes and patterns. He found evidence that most of them were able to transcend a purely manipulative approach through seeing objects they had made at one stage of their work as tools/objects to use in the next stage (foci 5 and 6 above). That he found such a shift among some low attaining students suggests that there might be other forms of mathematical thinking which have been characterized as features of the work of high achievers, but which some low attaining students *can* do.

5. The school where the research took place

The school in which I undertook my studies had an intake skewed towards below average attainment levels. They had recently been concerned about their underachievement compared to national standards. Some factors in the low performance in national tests had been identified through scrutiny of test papers and clinical interviews during the previous year (Watson, 1999). Many of these were examples of the "misconceptions" recorded in the CSMS study (Hart, 1981). Others were related to how the students constructed their sense of, and response to, test questions, such as: difficulties in "reading" a complex question layout; discrimination between the use of diagrams and pictures; deciding if, and when, it was appropriate to bring everyday knowledge to bear on the test questions. Success in these aspects of test-taking requires flexible use of knowledge and some metacognitive awareness. Nevertheless, the school had restructured its curriculum to raise standards through providing more advanced content to all students, and had paid no special attention to teaching style or metacognitive factors.[4]

Most of the teaching at the project school was of a traditional, text-based type. Since Phoenix Park, the more successful school in Boaler's study (1997), was situated in a similar geographical location to my project school, drawing its students from a similar population, I believed that different teaching approaches, as well as the different curriculum, might also be relevant. However, schools in the UK generally feel vulnerable, due to accountability structures and a rigorous and punitive inspection regime, and so are generally unwilling to engage in innovation. I was only allowed to work in a non-interventionist way with a year 9 low attaining group, because I "wouldn't do any harm."

Their usual teacher was a special learning needs specialist who was assigned to the class because she knew the students well through her general support and pastoral work. As well as having low achievement in mathematics, the students had a mixture of patchy school histories, behavior problems, personal difficulties outside school, low self esteem and disaffection. They had recently entered the school from a variety of feeder schools. They were normally taught, and had been taught in the past, through a mixture of practical activities, repetitive exercises and practice of number-skills. Oral tests were regularly used to monitor the accumulation of skills, facts and their application in simple contexts. The usual teacher talked of 'settling' students at the beginnings of lessons, offering step-by-step approaches to answer routine questions, giving them 'something to do,' and 'keeping it simple.' Even with the school's shift to more demanding curriculum content, these pupils were being given very elementary mathematics to do. During the lessons there were no class discussions; students sat at some distance from each other; there were two support assistants in the room; a calm and quiet tone was the norm; students were usually compliant. Lessons and teaching style seemed to have been selected in the belief that such students could not handle anything challenging or complex, cognitively or socially.

6. The study

I intended to observe lessons, record pupils' responses and comments in field notes, discuss mathematics with the pupils in the naturalistic setting of their classroom and normal tasks and write notes, verbatim where possible, of what they said. The plan was that the teacher and I would eventually team-teach, monitoring responses to particular kinds of task. During the research period, due to unrelated stressful incidents in the school, I was coerced into teaching

[4] This did, indeed, raise achievement in the school overall, but the rise was insignificant for the kind of teaching group studied here and the focus of this study was to identify further potential for achievement.

earlier than I had planned, and the usual teacher and I could not find time to co-plan for team-teaching. Consequently, I often found myself leading the teaching, using tasks selected by the usual teacher, but focusing on using approaches which gave responsibility to the students and offered something for them to discuss as a whole group. The prompts I used, both during one-to-one interaction and whole class teaching, were about seeing and generalizing from patterns, using and generating examples, communicating a sense of mathematical concepts and describing underlying structures. These prompts had been developed previously from observations and experiences in a wide variety of teaching contexts (Watson & Mason, 1998). Some examples, selected to relate to the aspects of mathematical thinking on which I wanted to focus, are:

- *What is the same or different about . . . ?* (encouraging the learner to give attention to pattern);
- *Describe what happens in general* (nudging learners towards abstraction);
- *Can you give me an example from your own experience? Give me an example which fulfils certain conditions* (prompting exemplification);
- *Show me . . . tell me . . .* (eliciting information about images, and other aspects of their understanding);
- *Can you show me this using a diagram/letters/numbers/graphs, etc.?* (prompting flexible use of representations);
- *If this is an answer, what might the question be?* (shifting the focus onto structures rather than answers).

Rather than needing special tasks, special knowledge of how to plan, and prior metacognitive training as are commonly used in intervention projects in mathematical thinking, my approach was to focus on interactions in the teaching context which already existed in the research classroom, giving extra prompts where possible and relevant. I wanted to see if these students, who were believed to have low mathematical ability (and believed that of themselves), could act in a cognitively sophisticated way with mathematics. If they could, then it would make sense to question the general use of teaching styles which avoid the use of such thinking skills.

7. Methods

Field notes were analyzed to identify students' remarks which appeared to show the features of mathematical thinking in which I was interested. Of course there is a problem with interpretation in this approach to research which is hard to resolve satisfactorily. I cannot assume that my interpretation of utterances matches what was intended or experienced by the student. However, in order to operate at all, teachers have to interpret what their students say. Some assumptions need to be made, so I saw it as legitimate as researcher to make similar interpretations. As far as possible I tried to focus on episodes in which it was impossible to find alternative interpretations. For instance, if a student gave me an example of something it is tautological to say that the student was able to exemplify in that situation. If a student made a general statement, then I assumed that some generalizing *had* taken place, even if I did not know *how* it took place or what prompted it. In fact, the only focus of observation which was not directly connected to something one could see or hear incontrovertibly is the idea of developing and using images. It is possible to develop and use images without expressing these, but explicating one's images through diagrams or speech, or even arm-waving, cannot be done unless there is an image to be so expressed. In other words, there were episodes which revealed images to me, but as with any observational research there may have been other images about which I never saw evidence.

As a result of analysis of episodes after each lesson, I planned prompts to use in the next lesson, and decided on whom to focus certain kinds of prompts. For example, after one lesson I found that every student except one had given an example, either prompted or unprompted, during the lesson. Exemplification being a characteristic of mathematical thinking, I decided to ask the exception directly to give an example, when appropriate, in the next lesson. In this way I systematically explored, where possible in the context of their normal tasks, the ability of all students to display the types of response identified above.

Here is a selection of episodes, with comments about the mathematical thinking I took them to illustrate. Numbers in brackets refer to the foci of the study as listed above:

1. Students had been given a sheet of questions about the seven times table, having been given similar ones for other times tables before. There were four columns of calculations to do. Here is a part of it:

$5 \times 7 =$	$7 \times 5 =$	$35/5 =$	$35/7 =$
$6 \times 7 =$	$7 \times 6 =$	$42/6 =$	$42/7 =$
$7 \times 7 =$	–	$49/7 =$	–
$8 \times 7 =$	$7 \times 8 =$	$56/8 =$	$56/7 =$

Students soon realized that the answers to the first and second columns were obtained by adding 7. Answers to the third were all 7, and answers to the last were the natural numbers in order. That is, students had "spotted" and used the patterns which enabled them to complete the worksheet vertically. They reported that this was how they had filled in similar sheets before. I worked with pairs and individuals while they were doing this. To draw their attention to the structure which I saw in the horizontal patterns, I gave them "$23 \times 7 = 161$" as the start of a row and asked them to finish the rest. All could do this after some thought, although their previous patterns working down the page did not help them in this case. Two students then made up their own examples of other rows to illustrate the structure further, after I had suggested this would be a useful thing to do.

This episode illustrates not only the identification and use of empirical patterns (1), but also their ability to shift to observing less obvious, structural patterns, and using them (6). Additionally, some students exemplified (3).

2. June was drawing a square in the normal orientation on a coordinate grid and noting the coordinates. I do not know if she would have reflected on the coordinates if she had not been prompted to do so, but once prompted to say what was the same and what

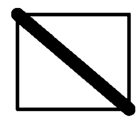

Fig. 1. The first number went up and the other down.

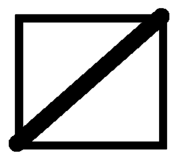

Fig. 2. Both numbers went up.

different about the coordinates she was able immediately to say which ordinates were equal. She was asked if this pattern of matching coordinates would always be true for a square and she replied "yes." She then explained why it worked, so I asked if she could say it using algebra. Although the expressions she developed depended on specified coordinates at one vertex they were in other respects generalizations of such squares, i.e., families of squares sharing a vertex. She had been able to make a transition, on the basis of one example and application of knowledge of squares, from a specific case to a more general symbolic representation, and to justify this (2, 4). In this situation, for this student, expressing a generalization in algebra seemed easy and obvious.[5]

3. Students had been given a printed blank coordinate grid and a long sequence of coordinate pairs which they had to plot. They would then have to join the points they had plotted to make a shape. All students except Almira were plotting points in the order given. She had restructured the task by picking out all adjacent coordinate pairs which would give the same vectors when joined.[6] When questioned, she pointed out that if the first number went up and the other down by 1, she would draw Fig. 1. If both numbers went up by 1, she would draw Fig. 2.

   She could predict other vectors similarly.

   The student had created a more complex task by using pattern, similarity, and classification (1, 2, 4, 6). This led me to suppose that others in the room would also be able to work in this way. I did not work with all the students on this, but managed to spend time with about half the class, working with individuals and pairs. By asking students to look at similarities in the lines joining adjacent points, they came to look at the task as a different sequence of actions: classifying patterns in the coordinate pairs, selecting and drawing the similar vectors corresponding to the classifications (5, 6). This done, they

[5] This was also true for many others in the group with whom I used similar prompts.
[6] She did not yet know about vectors.

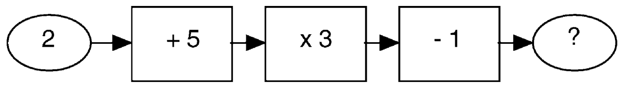

Fig. 3. A flow diagram.

were able to reverse the process and predict generalities about coordinate pairs which would arise from similar lines on the grid (1, 2). For example, to get a line which slanted downwards with a particular slope (these students did not know about gradients) they could say that the x-ordinate of the second pair had to be one more than that of the first pair, and the y-ordinate had to be two less, or three less, etc. Some could write this as (a, b); $(a + 1, b - 2)$.

4. Students had been using flow diagrams to calculate outputs of compound functions involving the four basic operations (see, for example, Fig. 3).

   They were asked to make up some hard examples of their own for the whole class to do. Most students' idea of complexity was to use more operations and bigger numbers, which is a common response (see, for example, Ellerton, 1986). Then Boris suggested constructing one in which the operations and output are known and the input has to be found. Andrew then gave one in which input and output were given but the last operation was missing. These two students were working with the relations rather than the numbers and operations. They saw the structure of the problem as something they could vary, rather than following the template of teacher-given questions (3, 6).
5. Elvira had been asked to round 83 to the nearest 10. She replied "80, but if you had asked me about 87 I would have said 90." In this case, the student seemed to have had an image of what it means to round numbers and had used the image to generate a counter-example in order to indicate to the teacher that she knew more than had been asked (3, 4).

   These examples collectively illustrate all the features of mathematical thinking for which I was searching. The least represented is flexibility with representations, but this is illustrated in the story of Ben and the work of Harries (above).

8. Going across the grain, exemplification and representation

In Watson (2000b) I focused on a feature of the prompts and behaviors which enabled students, as in episodes 1, 2, 3 and 4 above, to shift from superficial to structural thinking (Bills & Rowland, 1999; Goldin, 1998; Reimann & Schult, 1996).

Awareness of structure appears to require reorganizing one's initial approach to a concept by reflecting from another point of view. Good mathematicians do this for themselves; they see generalities and grasp them in several examples; they see global similarities in locally-produced examples; they see possible generalities even in single examples (Krutetskii, 1976). The common experience in episodes 1 and 3 above is that students were comfortably generating answers through repetitive activity which took no account of the whole task. In episode 1, the method of generating answers had little relationship to multiplication, which was the intended mathematical context! I refer to this kind of low-level generative activity as "going with the grain." Once learners' attention is directed *across* the grain, so that they observe a cross-section of their work, using the generating strands of activity as raw material for contemplation, they have the opportunity to see structure and hence transform or reorganize their view of the concept.

Another feature which arose frequently was the use of examples to explain and explore the mathematical topic. In episodes 1, 3 and 4, students demonstrated their willingness to generate and use special examples. Since exemplification requires a sense of structure, domain, or generality the teacher can help learners shift to more complex cognitive levels by asking for examples (Mason & Watson, 2001; Sadovsky, 1999).

In episodes 2 and 3, the students who articulated the relationships found it easy to write them algebraically. This is not a new claim but is mentioned here for completeness (see, for example, Chazan, 1996).

These observations therefore suggest three principles for working with low attaining students:

1. Their attention can be drawn to structure through observing patterns which go across the grain of the work.
2. They can be asked to exemplify, and hence get a sense of structure, generality and extent of possibilities.
3. They can be prompted to articulate similarities in their work, and hence be prompted to represent similarities in symbols.

Each of these principles offers possibilities for task-construction and interactive strategies.

9. Summative observations

In all the reported episodes students were pursuing tasks set by the normal teacher, all of which had the potential to be treated on a mundane level. The particular episodes are not generalizable across the class in the sense that other students always displayed the same thinking in the same lesson. The episodes are not being offered as conclusive proof that all low attaining students can engage in mathematical thinking in all mathematical contexts, nor am I underestimating the difficulties of working with low attaining groups (Chazan, 1996). Instead, I am claiming that some potentially powerful mathematical talents of these students were unrecognized and unused in the teaching of mathematics. This classroom was not unusual in the way mathematics was normally taught, so it is likely that similar findings would arise elsewhere if observers and teachers are attuned to noticing and prompting features of mathematical thinking during ordinary tasks.

In the third illustration, the activity offered the opportunity to approach the task in a variety of ways, from mundane to abstract. Once an approach had been seen and recognized by a teacher who may not already have seen the possibilities, prompts could be devised for other learners to help them make the same shift. In this case, I had proceeded to ask other students "What is the same about these two lines? Can you see patterns in the numbers for these two lines?" and so on. By doing this, I found that nearly all the class could make a similar shift, given the prompting and the chance to do so. Indeed, all the students in this study showed that they were able to make mathematical sense from mathematical experience. All were able to participate in learning interactions which were not based on simplification, step-by-step approaches, learnt procedures, but which expected conjecture, exemplification, generalization, reflection on pattern and other aspects of advanced mathematical activity.

But the simplicity of the mathematical contexts also needs to be considered. Can it really be said that these students were doing mathematical thinking, when the contexts were so simple? Can it be deduced that they could be doing harder mathematics just because they can work in sophisticated ways on simple mathematics?

10. Development of a proficiency-based agenda

It is unlikely that a *deficiency* agenda which focuses on remediation and repetition, which are common practices with low achieving students everywhere, will provide insight into the questions just posed. The provision of low-level activities accompanied by low-level expectations can limit students. Almira, in episode 3 above, may have been exceptional in that she sought out ways to make the work harder for herself, but hence more interesting. However, her behavior puts into question the assumption that low achieving students cannot work at a structural or relational level, even if they do not reconstruct tasks at that level for themselves as she did.

Proficiency approaches such as those reported by Boaler (1997) and advocated by Ahmed (1987) do not attend to how low attaining students can grapple with core curriculum mathematics. They do, however, set out guidelines for helpful social features of the working environment, the importance of choice, flexibility, discussion, freedom to explore and recognition of idiosyncratic approaches.

A *proficiency* agenda with advanced learning as its aim would not dwell simply on the positive aspects of behavior, motivation or attitude, although those would play a part, it would also recognize and emphasize the thinking skills which students exhibit and offer opportunity for these to be used to learn mainstream curriculum mathematical concepts. The low attaining students in this study were already able to think in ways normally attributed to successful mathematicians; they did not need context-free cognitive training; they responded to opportunities and encouragement to use these abilities in mathematical ways. These ways included the ability to exemplify, to generalize, to develop and use images, to abstract from experience through reflection on processes, to work with structure, either voluntarily or when helped to do so, to participate in mathematical discussion, to work in complex situations. This study, which was as systematic as possible in an ordinary classroom in which preserving normal practice was a priority, is therefore a contribution to the development of interactive strategies which enable *all* students to use their ability to think mathematically.

Acknowledgments

Grateful thanks go to the anonymous teacher and school who allowed me to work in their classroom, to a number of teacher educators who have been enthusiastic when offered my data, and to Caroline Roaf and Iben Maj Christensen whose comments on early versions of this paper were very helpful.

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