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The Effects of Variation on Children's Learning

Professional Development Materials for use in Key Stages 1 & 2

by the GLOW Variation Work Group

Contents

Introduction

This booklet has been produced by the GLOWhub Variation Work Group, by teachers for teachers. We started from pupils' work and then looked for ways to describe the variation involved and its effects. This is what variation does for us – it is a tool to understand learning, or learning problems, and to help design more helpful approaches.

The tasks in this booklet are not to be exemplars of best possible teaching. Read our comments. Some of our comments about the variation and learning in these tasks are positive; some are negative. They are chosen to prompt discussion among teachers.

Where we have been negative we have used pupils' work diagnostically to indicate how to develop alternatives.

Our aim is for you to be able to use variation as a lens to plan, analyse, choose, design and use tasks. We have avoided technical terms and background theories, but instead offer our observations, pupils' work and suggestions about how things work and how they might work differently, and examples of questioning. We present these in a structure that has helped us when we discussed the meanings and progressions in the work that our pupils had done.

We focus on some key ideas in number and multiplicative reasoning from KS1 up to the beginnings of KS3, but our uses and strategies for variation are more wide-reaching than that and can be used at all levels and all topics.

We start by providing some meaningful terms for what to look for and what to think about when teaching concepts, and by describing the 'lilypad' structure that became helpful to us. We see the teaching trajectory as a journey, possibly cyclic, across overlapping green lilypads. The work done on each 'pad' contributes to progression from what is known to what is not yet known. The orange lilypads give the reasons for each stage of the journey; the blue lilypads give questions and prompts that enrich teaching and learning at each stage.

The most important lilypad question is number 0: identifying the **key idea** that you would like your pupils to learn.

The design of this booklet is in spreads which show the task and pupils' work, and then commentaries that arise from the task, but are also more general.

The tasks are organised roughly according to progression in mathematical concepts. It is worth looking at all of them, even if you do not teach that level, to get a full sense of how variation can be used.

Do not assume that these tasks necessarily lead to expected learning, some did not; you need to read the commentaries as well.

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Meaning of terms about mathematical learning used in this booklet

Concept: a concept is a network of language, representations, transformations, examples, non-examples, structure, relationships, behaviour, methods and definitions. We illustrate this below using multiplication as an example.

Language: What forms of language go with multiplication? e.g. 3 lots of 4; 4 things, 3 times; 3 groups of 4; 4 groups of 3; enlarge 3 by a scale factor of 4 …

Representations: symbolic, e.g. 3 x 4 = 12; diagrammatic, e.g. numberline, arrays, area model, enlargements, equal-sized groups; pictures; physical objects

Transformations: different expressions (e.g. number sentences) for the same relation, e.g.

12 = 3 x 4	12 = 4 x 3	3 x 4 = 12	4 x 3 = 12
12 ÷ 3 = 4	12 ÷ 4 = 3	4 = 12 ÷ 3	3 = 12 ÷ 4
$\frac{12}{3} = 4$	$\frac{12}{4} = 3$	$4 = \frac{12}{3}$	$3 = \frac{12}{4}$

Examples and non-examples:

3 x 4 = 12 0.3 x 4 = 1.2 0.3 x 4 = 0.12 $\frac{2}{3} \times \frac{12}{16} = \frac{1}{2}$

Structure: recognising anything that looks like two numbers multiplied, or two numbers divided, as a version of: ab = c so that all the above can be used.

Relationships: in the structure pq = r how does a change in one thing relate to changes in another? If p doubles, what happens to r? If p doubles, what has to happen to q to keep r the same?

Behaviour: e.g. when does multiplication make things bigger and when does it make things smaller?

Methods of solution: (depend on the problem): e.g. grouping/counting things in different ways; repeated addition; using multiplication facts; using partitioning; using area calculations; reorganising a situation so that a different number sentence can be used; enlarging the dimensions.

Definitions: [elementary ideas are often hard to define because what makes sense at an elementary level does not easily extend to later mathematics. Here are some attempts to define multiplication in ways that do not limit elementary learning, but do not extend to more than two numbers, nor to higher maths.] Multiplication is an operation between two numbers that produces a product which is a scaling or multiple of each of them by the other; an operation involving two numbers that treats one number as a unit and the other as a number of repetitions, or a scaling of the other

The overlapping lilypad model (see next page for an A4 version; A3 version is available on Glow Hub webste)

The model has several stages:

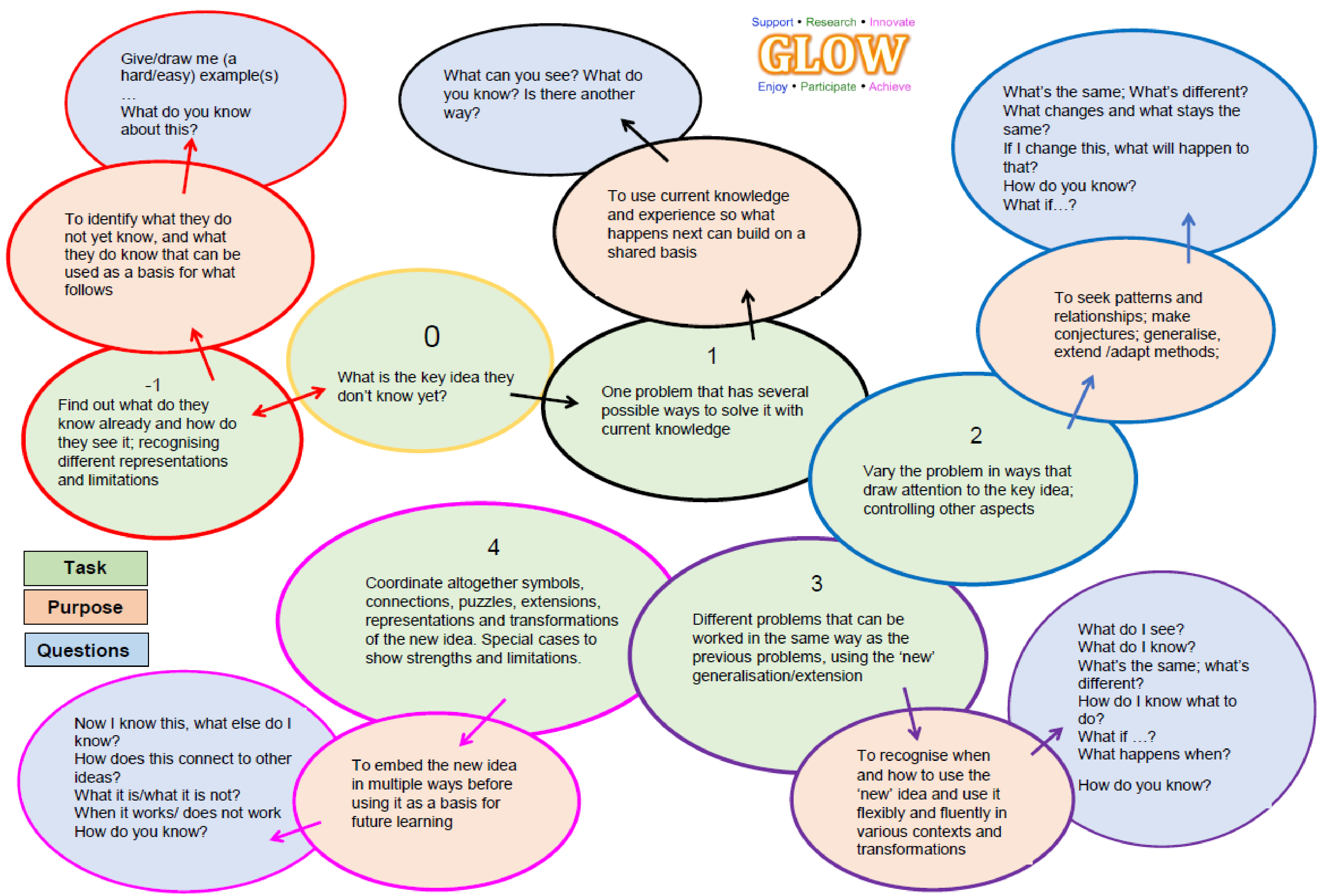
Pad -1, before you plan your teaching, find out what your pupils already know. This helps you to identify the key idea for your next teaching, at **pad 0**. There is no teaching task here as this is teacher-work.

It is generally thought that **pad 0** is the most difficult, and one way to approach it is to anticipate what are called 'common misconceptions'; another way is to look at what goes wrong and work out why it goes wrong; another way is to collaborate with colleagues to identify what pupils typically find difficult; or to analyse the concept to work out what the critical components are; or to see what key features of an idea are highlighted in your most trusted textbook.

You might recognise the remaining stages: **pads 1,2 and 3** correspond to what some people call 'procedural variation', that is the use of variation to scaffold *progression* from what is know to what is not yet known. **Pad 4** corresponds to what some people call 'conceptual variation', that is the use of variation to expose the range of meanings, representations, occurrences, connections and uses of a concept. Contrary to step-by-step, or tick box, approaches we prefer a model of **overlapping** lilypads to support the trajectory of a class through these stages. This model allows for flexibility, individual insights and problems, explorations, and so on.

Don't forget the importance of responsive teaching. Everything depends on what your pupils show that they can do. A task might be planned to be for pad 2/3, but in practice it might turn out to fit pad -1, or vice versa.

For each green pad task-type we give indications of purpose in the pink blobs and typical questions and prompts in the blue blobs.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Give/draw me (a hard/easy) example(s) … What do you know about this?
To identify what they do not yet know, and what they do know that can be used as a basis for what follows
-1 Find out what do they know already and how do they see it; recognising different representations and limitations
0 What is the key idea they don't know yet?
What can you see? What do you know? Is there another way?
To use current knowledge and experience so what happens next can build on a shared basis
1 One problem that has several possible ways to solve it with current knowledge
What's the same; What's different? What changes and what stays the same? If I change this, what will happen to that? How do you know? What if…?
To seek patterns and relationships; make conjectures; generalise, extend /adapt methods;
2 Vary the problem in ways that draw attention to the key idea; controlling other aspects
Task
Purpose
Questions
4 Coordinate altogether symbols, connections, puzzles, extensions, representations and transformations of the new idea. Special cases to show strengths and limitations.
3 Different problems that can be worked in the same way as the previous problems, using the 'new' generalisation/extension
What do I see? What do I know? What's the same; what's different? How do I know what to do? What if …? What happens when? How do you know?
Now I know this, what else do I know? How does this connect to other ideas? What it is/what it is not? When it works/ does not work How do you know?
To embed the new idea in multiple ways before using it as a basis for future learning
To recognise when and how to use the 'new' idea and use it flexibly and fluently in various contexts and transformations

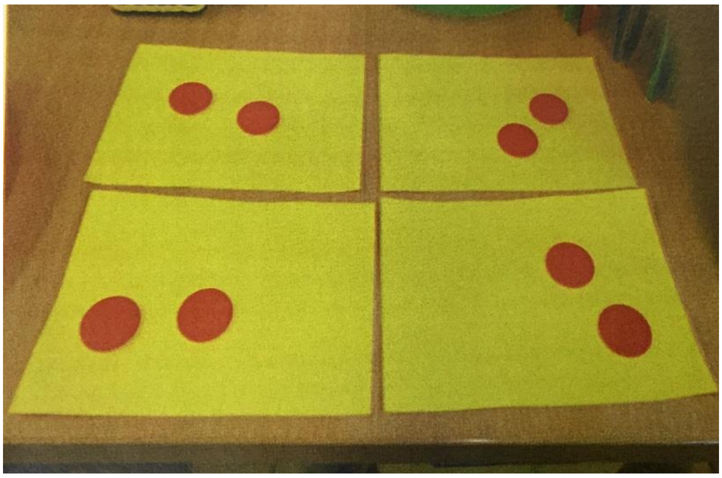
Questions and prompts used with the tasks in this booklet

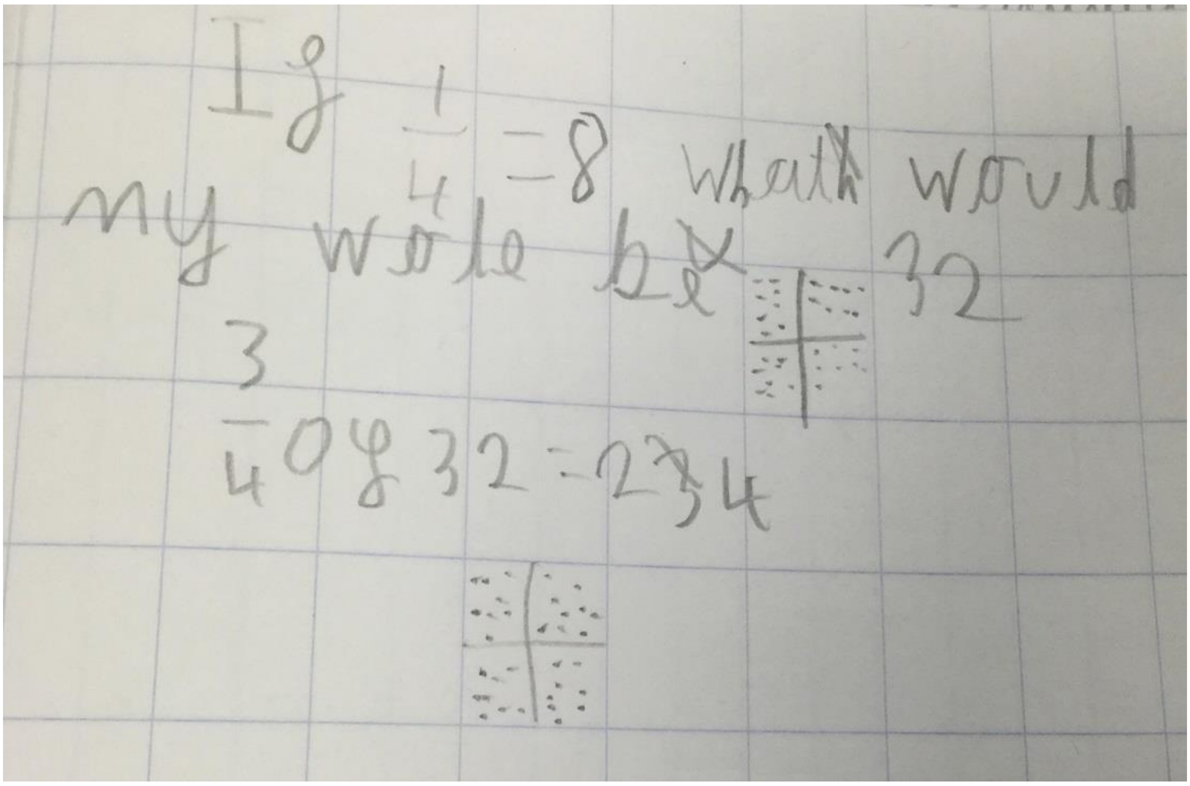
What do you expect?	Describe what you have found out
What do you see?	Is this generally true?
What do you notice?	Can you explain what you find?
What is the same ? What is different ?	Is this sometimes, always or never true?
What changes and what stays the same?	How do you know?
If I change this, what else has to change?	Convince me
Complete or continue a pattern	Prove it
How do you know you have found them all?	Because …?
When you saw this, what did it make you think about?	Make up some of your own like this
What helped you make a connection?	Can you show me one, another one and one your friend hasn't done?
Now I know this what else do I know?	Show me a particular one, a different one and another one?
Any patterns?	Explain the mistake
Do you have a conjecture?	How can you check?
So-and-so thinks this, so-and-so thinks that, what do you say?	Can we work backwards?

Example list

1.

Use of Variation	Mathematical Focus
Varying representations to give different access to same idea. Can we use different representations to check work?	Dividing by 4





Use of Variation	Mathematical Focus
Varying representations to give different access to same idea. Can we use different representations to check work?	Dividing by 4

Pupils have been asked to use a 2by2 grid to divide a number into four equal parts, but one pupil is unable to write so has been given physical materials. His way, with red counters, becomes the place others can go to for checking their work. This is an example where variation of representation can ensure inclusion, without diluting cognitive challenge. Note that, unlike our example 3 (dividing 21 and 20 by factors), these representations do not involve pictures of icecream! The variation here is between diagrammatic and symbolic representations. Dividing by four is invariant.

With one input number this is a pad 1 task; with different numbers divided by 4 but the same representations it could develop into a task for pads 2 or 3

2.

Use of Variation	Mathematical Focus
Same/different so what's the relationship?	Subtracting hundreds, and the meaning of the equals sign

Do these;

756 - 100 = $\boxed{856}$ - 200

587 - 100 = $\boxed{687}$ - 200

923 - 100 = $\boxed{1023}$ - 200

Now do this;

$\boxed{372}$ - 100 = 472 - 200

Arla says ' The number in the red box must be 372.'

Is she correct? Convince me.

She is correct because 472 - 200 = 272 so if 272 is on one side so the ~~other~~ other side should = that so 100 - 372 = 272 wich is = to ~~another~~ 272.

Any patterns? on one side there is bigger numbers and on the other there smaller numbers and all of them are ~~si~~ - 100 or 200 and the first ~~per~~ numbers in the problem the ~~on~~ answer in the box is 100 more ~~then~~ from the first one and the - 100 has the same ~~too~~ like 100 and the rest on is 100 more.

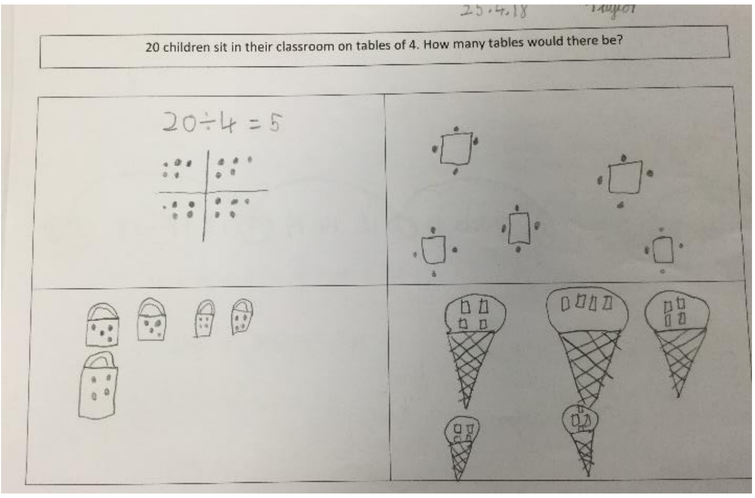
The questions have been designed so that learners can observe what happens to the hundreds digit. It is not clear from this work what materials were available to support the calculation, but it IS clear that this pupil noticed the relationship and probably used it for some of the time. She could maybe check using materials or calculator. The 'Arla says …' prompt points towards reasoning rather than calculation. The teacher prompt 'any patterns?' is interesting, because the pupil has first of all tried to give a justification based on calculation, showing understanding of the equality relationship.

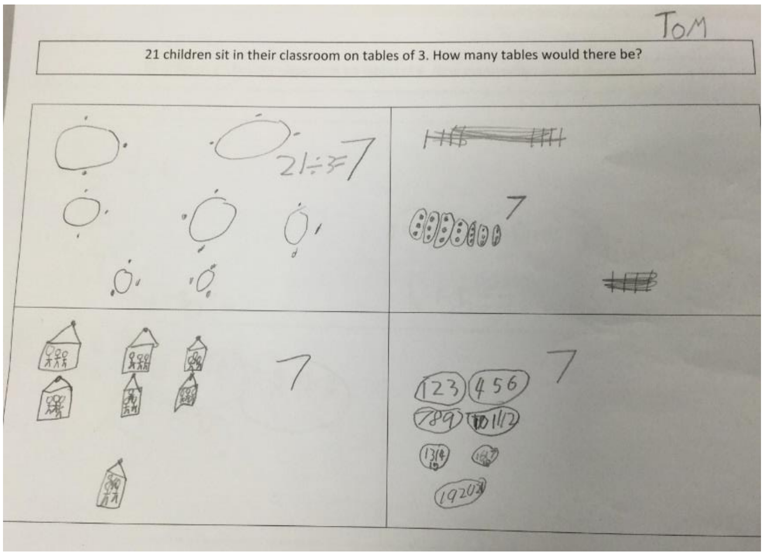
The whole task depends on the role of the layout to show the relationship, and choice of numbers to shape what the pupil notices as same/different; the variation uses the same structure with different digits. If some worked out examples had been given it would be possible to complete the task by noticing how the hundreds digit varies without having to think about why this happens, but this pupil explains in terms of the relationship rather than the 'it goes up' pattern.

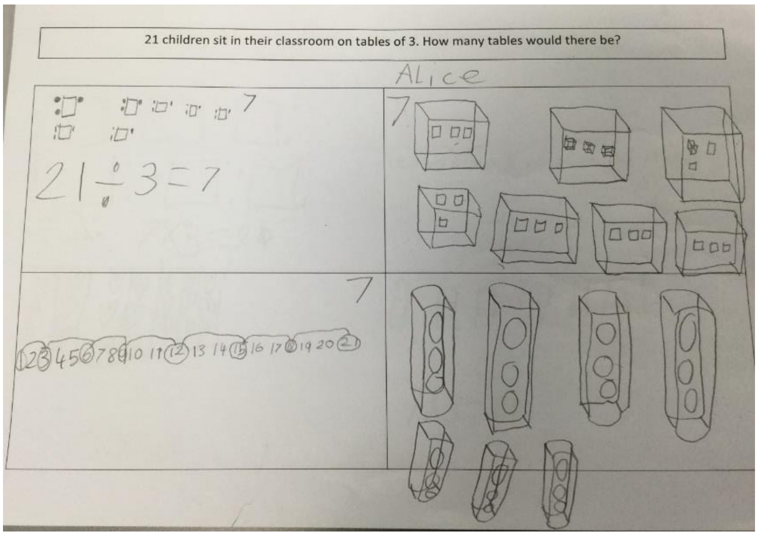
This is a pad 3 task, where a new idea is applied to different problems; the pad 4 aspects would be explanations of the patterns

3.

Use of Variation	Mathematical Focus
Equivalent representations generated by learners. What do they tell the teacher?	Representing a division situation







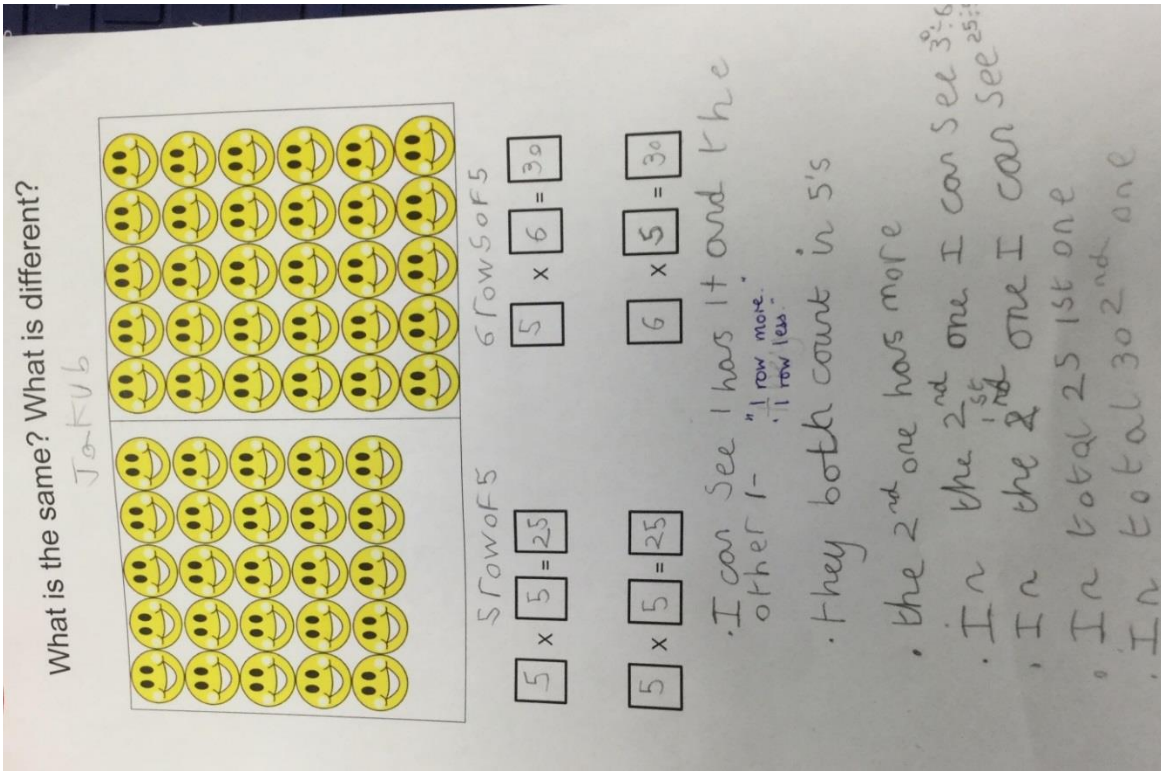
Use of Variation	Mathematical Focus
Equivalent representations generated by learners. What do they tell the teacher?	Representing a division situation

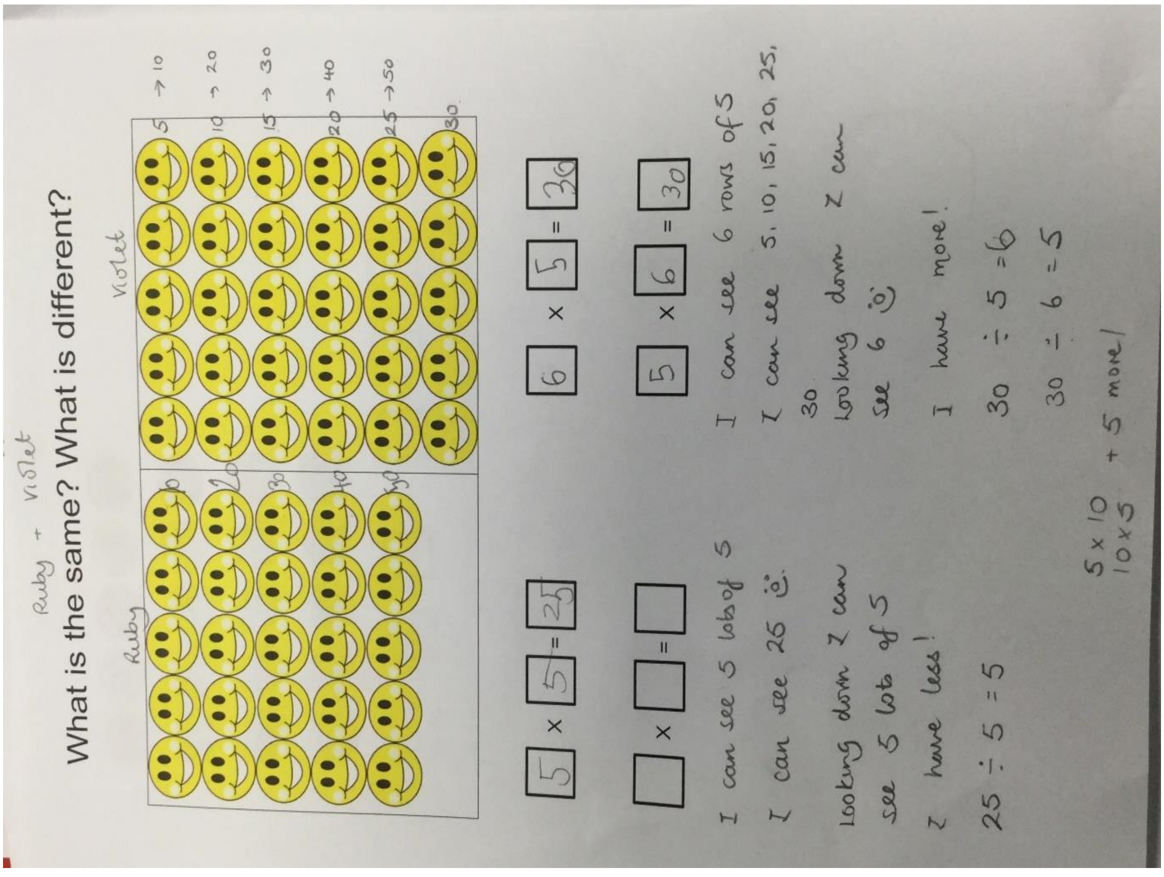
Learners provide the variation. This is VERY valuable for teachers to see how they 'see' a mathematical concept, and what other ways to 'see' it they have. The images offered by the whole class could be cut up and arranged to demonstrate moves from pictures to diagrams, some deriving from materials and some from abstract representations: pictures, sets, arrays, numberline, numerals. All organise the structures of 20 or 21 in useful ways. The variation in representation can be used to develop language about division and multiplication, seeing the representations as transformations of the same situations. (20 or 21). The variation is provided by the pupils, but is also inherent in the ways we represent number relations.

This was used as a pad -1 'finding out' task. It could be a pad 1 task to start moving pupils towards writing exact division as number sentences.

4.

Use of Variation	Mathematical Focus
Same/different so what's the relationship?	Connecting multiples using array model





Use of Variation	Mathematical Focus
Same/different so what's the relationship?	Connecting multiples using array model

Learners show variation of equivalent ways in which this picture can be understood. But the invariant images, and invariant number of columns, draw attention to the varying number of rows. The first one is 5 x 5, so it doesn't matter what meaning you give to which '5'. In the second one, the 5 and 6 denoted different dimensions, so writing 5 x 6 or 6 x 5 would relate differently to the diagram, even though mathematically they are interchangeable. These specific examples can be used to discuss this. The choice of 5 is a small number whose multiples are known; when 6 is introduced the important thing is its relationship with 5.

Think about progression from using a range of representations, as in example 3, to interpreting a given representation that organises the number structure in this way. Learners might draw anything that seems sensible to them; teachers draw diagrams that are structurally informative.

But this is only part of the journey. Children were offered a problem that they could already do and asked to discuss- 'What can you see?', 'What is the same?', 'What is different?' (pad 1)

This task was first to find out what the representation could help the children to understand. This opened up discussions with this group of children about the language linked to multiplication and division, repeated addition, multiplication in any order, related division facts. The children were asked to justify each of their statements using the phrase 'Prove it!' (progression from pad 1 to 2 and 3 through teacher prompts).

The children's responses steered the next steps. There were many different pathways that could have been followed from this activity, such as variation in representation. However as a follow up to this activity the children were set a challenge to use the ideas they had generated (pad 4)

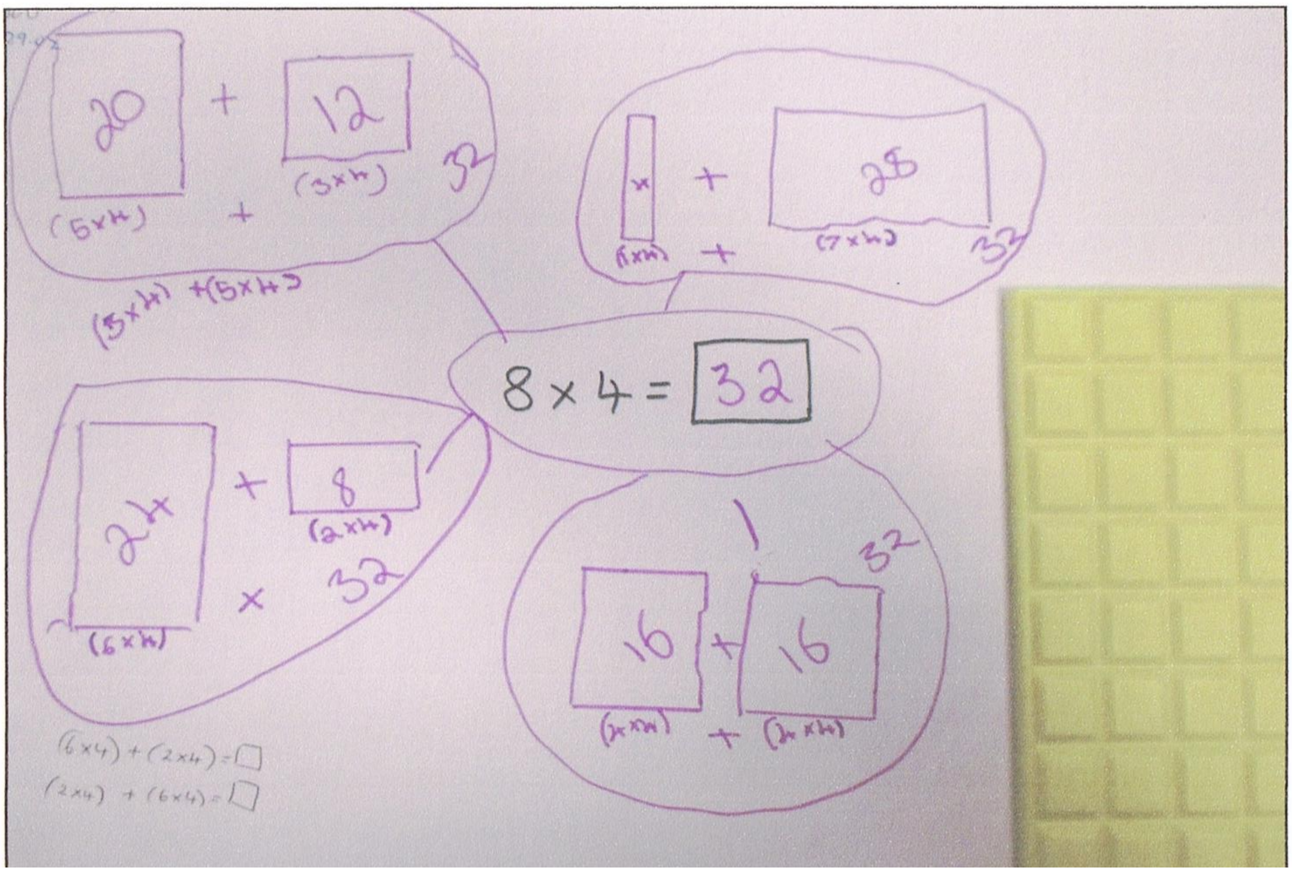
Here are 20 beads. Can you make an array? Can you make an array in a different way?

Again children shared their strategies, and new generalisations were encouraged: 'Can you suggest any other multiples of 5?', 'Prove it!' and 'Convince me!' Maybe invent a sentence stem such as "I know that __ is a multiple of five if it has a zero or 5 in the ones place". By this stage the lesson is settled on pad 4: what can be done with these connections of ideas?

This task was used to move children through a whole lilypad cycle, using children's knowledge and ideas elicited by teacher prompts and questions.

5.

Use of Variation	Mathematical Focus
Equivalent solutions to same situation. How do we know these mean the same thing?	Distributive law/partitioning area



Source: Every Child Counts, Edge Hill University

Use of Variation	Mathematical Focus
Equivalent solutions to same situation. How do we know these mean the same thing?	Distributive law/partitioning area

This task was an episode within a 1-to-1 session. The teacher's aim was to help a (Y4) child to use multiplication facts she knew to derive other multiplication facts, in a way that would help her make connections and see relationships, and so strengthen her understanding of multiplication. In our lilypad model this is pad 2, where connections and relationships can be learnt.

The teacher's starting point was discussing a bar of white chocolate and the child noticing a link with the 4x table (4 pieces in a row). The next step was to use Cuisenaire rods to represent the 8 rows of 4. The child was unable to recall the multiplication fact and the teacher asked what facts she did know. The child offered 3 x 4 = 12 and the teacher modelled partitioning 8 x 4 into 3 x 4 and the rest (5 x 4) with the rods. The child also knew 5 x 4 = 20. The child recorded this pictorially by drawing round the rods and recorded the multiples (photo). The teacher drew the child's attention back to the original 8 x 4 and the equivalence between this and combining the 5 x 4 and 3 x 4 the child had represented and demonstrated recording the two multiplications.

The next step was to consider other ways of splitting the rods, using a known fact as a starting point, then representing and recording, using an 'make another, and now make another…' approach.

Variation here can be considered in 2 ways. Firstly, although the partitioning approach is based on the distributive rule for multiplication, from the child's perspective (which is what matters) she was seeing this as using varied methods to solve the same problem. Secondly, we have multiple representations of the same problem – concrete (chocolate bar and Cuisenaire rods), pictorial, symbolic – so meaning the same but looking different.

In thinking about using this task yourself, you have to decide whether to present this as a 'do anything you like' task or to constrain it to row/column partitions so that there is a strong focus on enumerating arrays and the distributive law (i.e. 4 x 8 = (4 x 5) + (4 x 3) etc.). The invariant is the slab and the 32, but once the idea is understood a different multiple could be presented (maybe 60) with the expectation that learners will be even more creative with their suggestions.

Invariant shape ensures the focus is on the equivalence of different partition methods, equivalence can be the purpose of a pad 2 task

6.

Use of Variation	Mathematical Focus
What's the same? what's different?	Seeing numbers as multiplicative structures

47 48 49

In which ways are they similar/different?

They all go up in 1 so therefore it's a sequence. They also have the numbe 4 at the start. If you add the numbers together you get the sequence 11, 12, 13. So if you times each number eg 4×7=28 it goes up in 4 smallest to largst. The can all be round to the nearest 10.

Similar

all in the forties

all add up to 11, 12, 13 which adds on one each time

47 & 49 are odd

different

ends in 7, 8 & 9

all in a times table except for 47

47 is the only prime number in this group of numbers

Only 48 can be halved, quartered, and split into thirds

~~47 & 49 o~~

48 is even

Use of Variation	Mathematical Focus
What's the same? what's different?	Seeing numbers as multiplicative structures

Using pupils' perceptions is a powerful method for getting insight into what is known and how it is known, and engages pupils far more than answering closed given questions. Of course, they may not show you all they know, but instead they show you some variation in their knowledge. If you want to see if they can do a specific thing, you may need to use a different prompt – one that puts constraints on what is produced.

e.g. which of these numbers has the least/most factors?

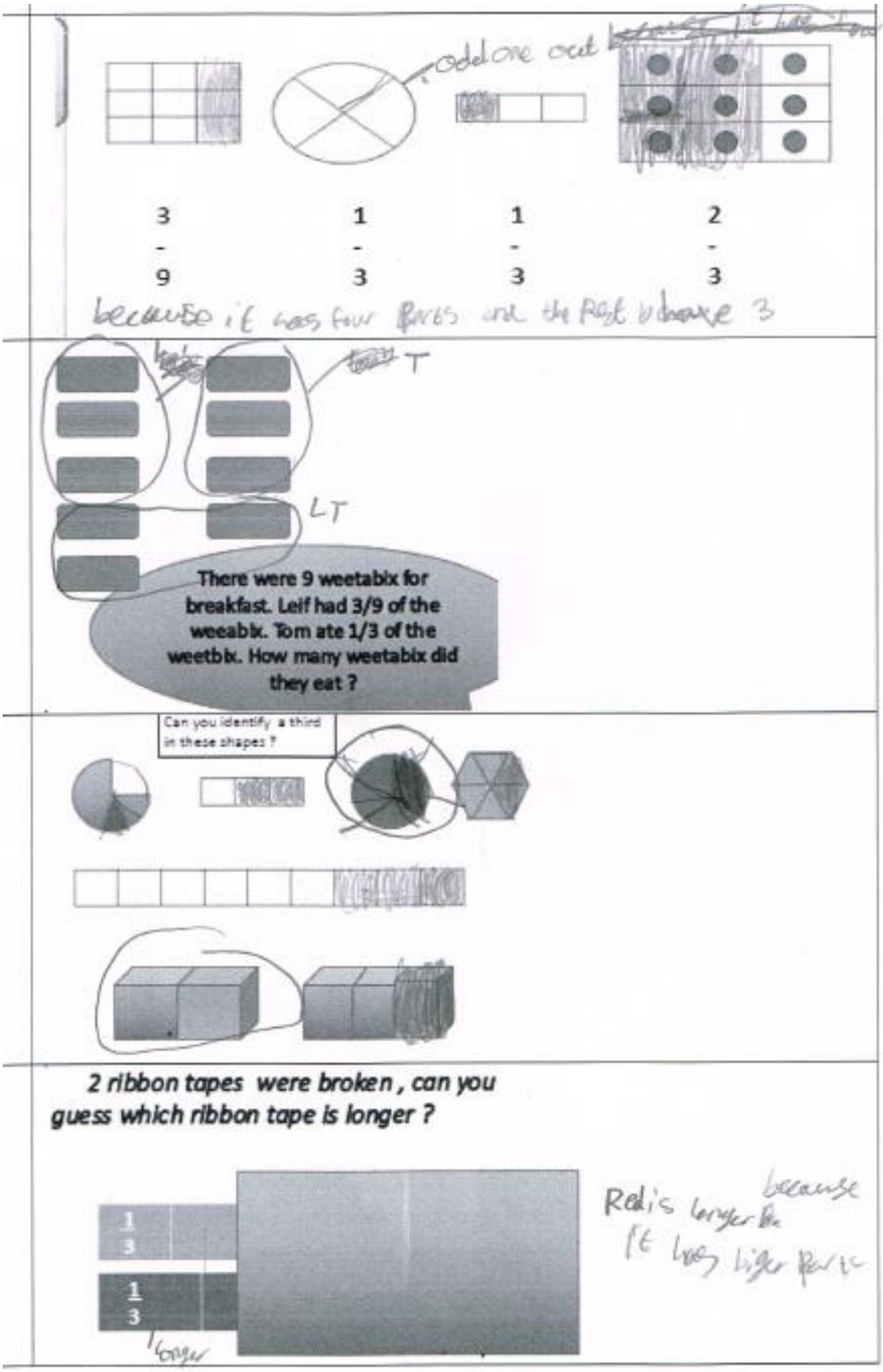
One teacher in our group used the 'same' task with numbers 23, 24, 25. This triple has a similar structure to the original task, i.e. it includes a square number, a prime number, and a number with several factors. The numbers are smaller but the variation within in the triple is similar.

In this task, the teacher wanted to see comments about factors and multiples, i.e. do learners see numbers as multiplicative structures? However, the observation about digit sums being 11, 12 and 13 is interesting and probably unexpected – it looks as if this was an idea that was "going round the class". Is this always true? 49, 50 51 is a triple where this pattern does not apply. This learner's comment provides a starting point for a homework or an exploration in class. It could be displayed on a 'What if …?' board for future consideration. The teacher who used this task followed up the 'digit sum' idea in a later lesson.

This task-type could be used as a pad -1 task to find out, or a pad 4 task to check whether they think about factors, or could also be used as a pad 1 task if the purpose is to start thinking about factors

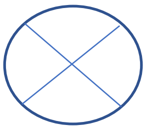
7.

Use of Variation	Mathematical Focus
Balancing variation and invariance. If this is the answer, what was the question?	Fractions: 1/3, different wholes, different representations.



Use of Variation	Mathematical Focus
Balancing variation and invariance. If this is the answer, what was the question?	Fractions: 1/3, different wholes, different representations.

These examples are taken from a powerpoint resource about the idea that 1/3 is the unit you get when you divide the whole of something into three equal parts (adapted from: www.ncetm.org.uk/public/files/23305581/Mastery_Assessment_Y3_Low_Res.pdf). Fractions and division by 3 have varied meanings and manifestations but variation of representation keeps the key idea invariant throughout, i.e. choosing one part when the whole of something is divided into three equal parts. So the 'thing' changes and the 'whole' changes but the 1/3 remains the same. Note that there are some misleading representations included. How can those that do not show thirds be divided into three equal parts? You CAN divide this:



into three equal parts: either ignore the given lines or divide each 'quarter' into three equal parts and 'add them'.

This worksheet is not designed to be done all at once, but interspersed with whole class examples and discussion.

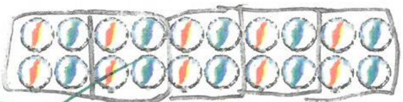
The final task encourages some inverse thinking: if this is a third, what does the whole look like?

These questions were designed to be posed in conjunction with a particular sequence of discussion points. They are pad 2 kinds of question in that they vary ways to look at one task and could encourage understanding structure. Other fractions might follow the same variation sequence.

8.

Use of Variation	Mathematical Focus
Varying representations and other features can hinder. What was the obstacle?	Non-unit fractions using representations

Find $\frac{2}{5}$ of Joe's marbles.

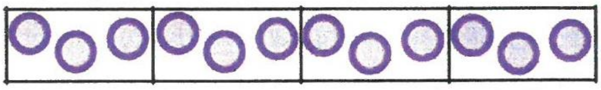


I have divided the marbles into $\boxed{5}$ equal groups.

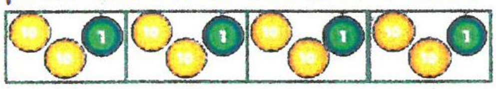
There are $\boxed{4}$ marbles in each group.

$\frac{2}{5}$ of Joe's marbles is $\boxed{8}$ marbles.

Sam has used a bar model and counters to find $\frac{3}{4}$ of 12



Faye uses a bar model and place value counters to find three quarters of 84



Use Faye's method to find:

$\frac{2}{3}$ of 36 $\frac{2}{3}$ of 45 $\frac{3}{5}$ of 65

Use of Variation	Mathematical Focus
Varying representations and other features can hinder. What was the obstacle?	Non-unit fractions using representations

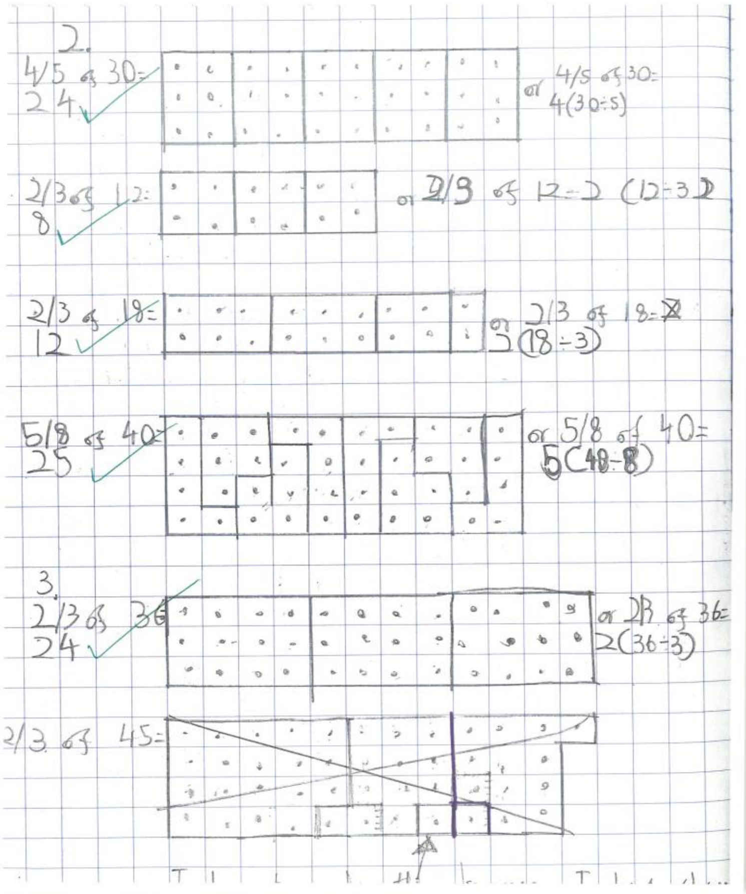
This is an example of the problems that can arise when you use a resource without first analysing what varies and what stays the same. The resource supports a shift from marbles (counters) to a 'bar model' made up of groups of counters, and then to a 'bar model' made up of groups of place value counters.

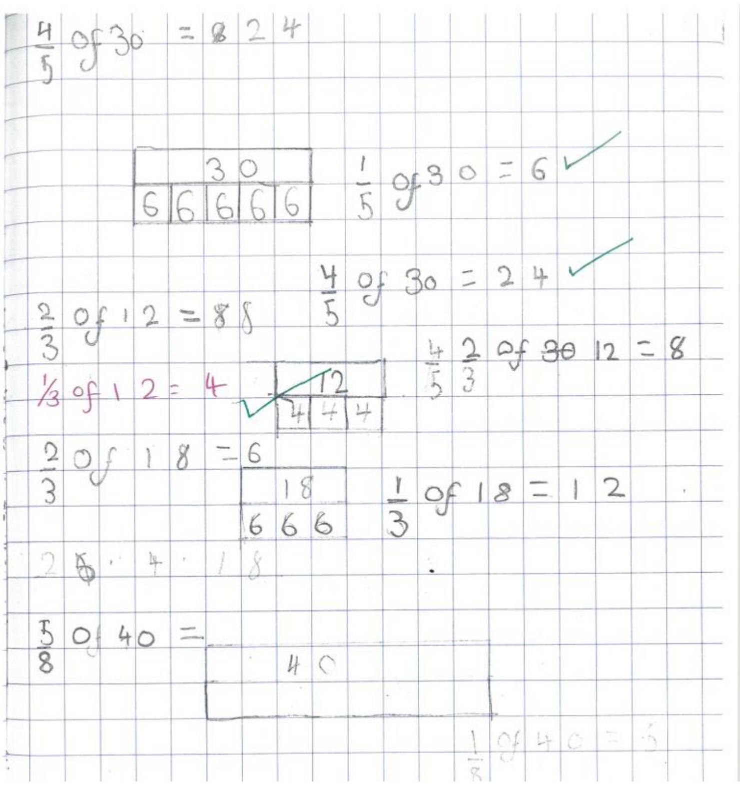
The pupils found it difficult to shift from counters as units, starting with a non-bar-model arrangement, to a bar arrangement and then to 'counters' that represented 5, so the whole resource was abandoned by the teacher. Although the stated focus is non-unit fractions it is not clear whether the key idea on this sheet would be obvious for learners because fractions and representations are both varied. Is it about finding non-unit fractions of numbers or about navigating various representations? Two representations relate counters to the bar model which is sufficient focus in itself, with the units being 3 in the second example and 21 in the third. The teacher felt that it was inappropriate to have all three representations on the same sheet, and varied fractions, without inserting space to discuss relationships and progressions between them. The difficulties pupils had remind us to be very clear about the key idea for a task sequence, and to organise variation around that key idea.

Because the key idea being focused on is unclear, it is also unclear where this might be useful on the cyclic pathway. It could be a -1 task to find out what pupils can do.

9.

Use of Variation	Mathematical Focus
Varying representations and other features can hinder. What was the obstacle?	Finding non-unit fractions of numbers





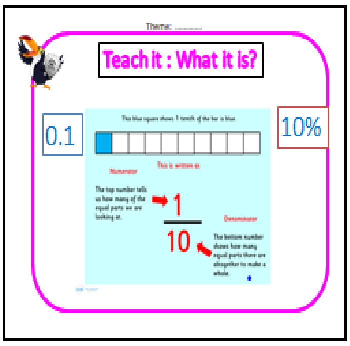
Use of Variation	Mathematical Focus
Varying representations and other features.	Finding non-unit fractions of shapes

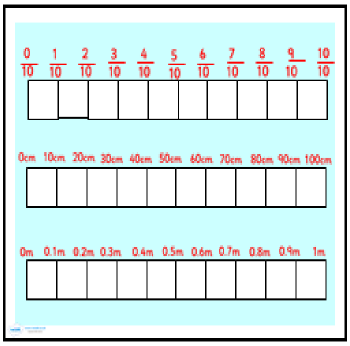
Although the non-unit fractions in the first diagram are mainly correct, it looks as if some pupils used a square-counting strategy and therefore may not have a stronger understanding of fractions. Also, the given tasks varied the total number, the shape and the fraction. It is not clear whether the task was about fractions of the same whole, different fractions of the same whole, varying the whole, fractions of number, fractions of shapes or fractions of an area of a shape. The second set of fraction examples offer a coherent representation and also, therefore, a method to find fractions, but this pupil does not show a clear meaning of 'fraction', and we wonder whether they have an associated language use that helps them focus on whether the fraction is unit or non-unit.

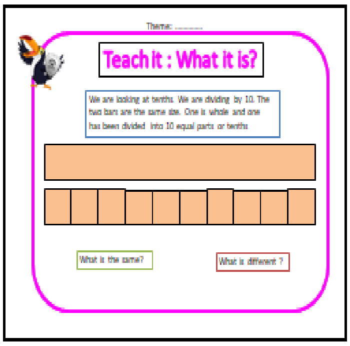
These tasks were offered as pad 3, but have elicited variation in understandings of 'fractions' due to their design.

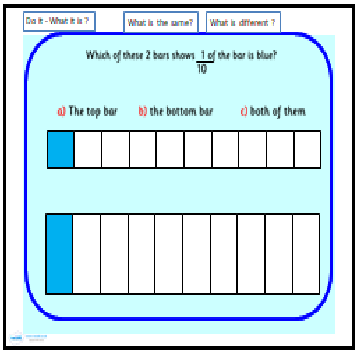
10

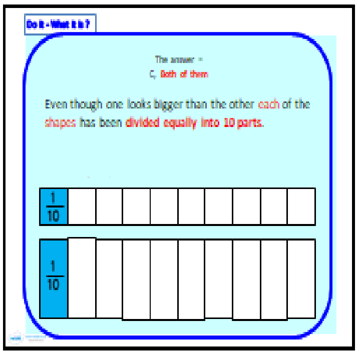
Use of Variation	Mathematical Focus
Balancing variation and invariance. What is the connecting idea between these images?	Tenths

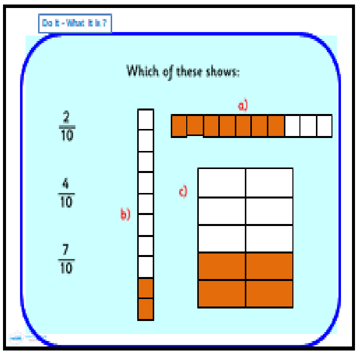


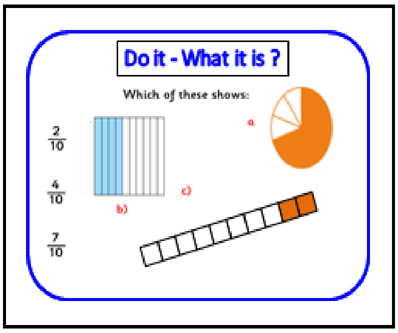


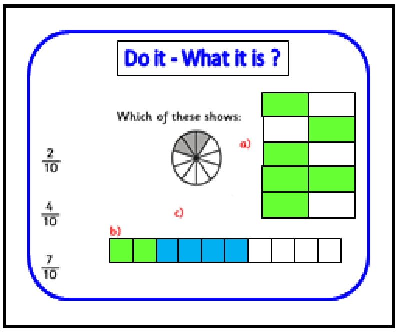


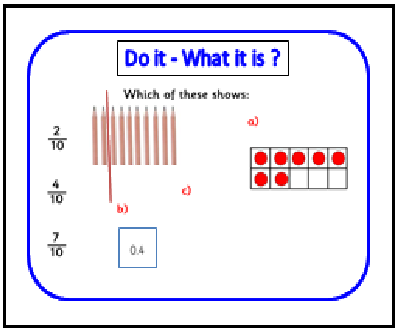


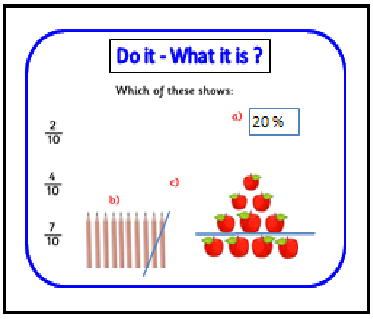


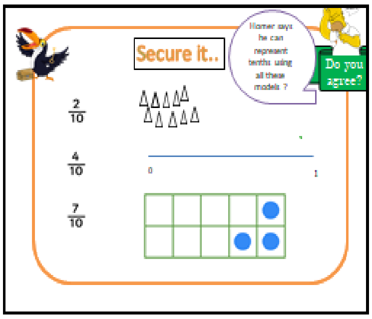


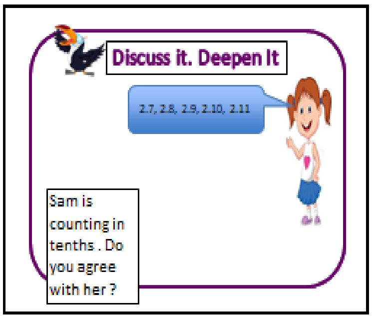












Use of Variation	Mathematical Focus
Balancing variation and invariance. What is the connecting idea between these images?	Tenths

In this slide sequence, (edited from www.twinkl.co.uk/resource/t2-m-405-year-3-fractions-tenths-task-setter-powerpoint-activity-pack and www.kangaroomaths.com/kenny2.php?page=Kschemeks2) tenths are presented in a variety of ways, using careful variation as each new aspect is introduced. At each stage comparisons can be discussed because only one aspect varies. The first slide presents a definition. We think this slide is more use for the teacher than the pupils. An alternative start would be to ask what they think 'tenths' might be. After that bar images are introduced to show, by asking about sameness and difference, one whole without marks, and one whole marked into ten equal lengths. After that, there is a comparison between tenths where the whole is different. The next set of variations involve representations of three multiples of one tenth using different spatial images but keeping the same three multiples. The next sequence shifts from spatial to include other representations, such as discrete numbers of objects, equivalent decimals and percentages, and, eventually, an unmarked numberline segment, thus connecting 'tenths' to measurement. Finally, a classic misunderstanding in counting in tenths and using notation is exposed, but this could be avoided by using a marked numberline model or measuring tape.

Obviously, progression through these slides is dependent on the dialogue with the teacher, who might want to introduce more related tasks, but the overall design controls the development of the idea of tenths to include a range of language, representations, models, and examples. The language of 'one of ten equal parts' works with some models and 'one tenth of the whole distance from 0 to 1' works with numberline models. Including decimals and percentages could depend on what pupils already know, or this could be way of introducing them informally, to be returned to later for place value and proportion purposes.

These slides span ideas that are related to pads 1, 2 and 3. The same approach could be used for other fractions, possibly without the decimal and percentage notations. It would be possible to ask pupils to provide the representation variations once they are used to them.

11

Use of Variation	Mathematical Focus
Balancing variation and invariance. Now that I know this, what else do I know?	Connecting place value to multiplying and dividing by powers of ten

What do you see?

What do you notice?

What is the same/different?

* to the left

1.46 x 10 = 14.6
14.6 x 10 = 146
146 x 10 = 1460
1460 x 10 = 14600

These are all moving 1 dp.* every time its multiplyed by ten.

460 ÷ 10 = 46
46 ÷ 10 = 4.6
4.6 ÷ 10 = 0.46
0.46 ÷ 10 = 0.046
0.046 ÷ 10 = 0.0046

These move 1 dp to the right every time.

Same

Both are moving 1 dp every time

I find that when they are multiplying they're moving to the left but when dividing it moves to the right.

Different

382 x 10 = 3820

What else can you derive from this calculation?

382 x10 = 3820
3820 x10 = 38200
38200 x10 = 382000
382000 x10 = 3820000 ✓

382 ÷ 10 = 38.2
38.2 ÷ 10 = 3.82
3.82 ÷ 10 = 0.382
0.382 ÷ 10 = 0.0382 ✓

382 ÷ 0.1 = 3820
382 ÷ 0.01 = 38200
382 ÷ 0.001 = 382000
382 ÷ 0.0001 = 3820000 ✓

382 x 0.1 = 38.2
382 x 001 = 3.82
382 x 0.001 = 0.382
382 x 0.0001 = 0.0382

382 ÷ 10 = 3820
382 ÷ 1 = 382
382 ÷ 0.1 = 3820
38.2 ÷ 0.1 = 38.20
382 ÷ 100 = 38200
382 ÷ 1 = 382
382 ÷ 0.1 = 3820
38.2 ÷ 1 = 38.2
38.2 ÷ 0.1 = 38.20
38.2 ÷ 0.01 = 38.200

3820 ÷ 10 = 382
3820 ÷ 100 = 38.2
3820 ÷ 1000 = 3.82 ✓
3.82 x 10 = 38.2
38.2 x 100 = 3820
3820 x 1000 = 3820000
3820000 x 10000 = 38200000000
38200000 x 100000 = 3820000000000 ✓

Same

3820 ÷ 10 = 382 x 10 = 3820
3820 ÷ 100 = 38.2 x 100 = 3820
3820 ÷ 1000 = 3.82 x 1000 = 3820

3820 x 10 = 38200 ÷ 10 = 3820
3820 x 100 = 382000 ÷ 100 = 3820

38.2 x 10 = 382 ✓
382 ÷ 10 = 38.2 ✓
3.82 x 10 = 38.2
38.2 ÷ 10 = 3.82

382 ÷ 0.1 = 3820
382 ÷ 10 = 38.2 ✓
382 ÷ 1 = 382
382 ÷ 0.1 = 38.2
38.2 ÷ 0.01 = 3820
38.2 ÷ 0.001 = 38200 ✓

Use of Variation	Mathematical Focus
Balancing variation and invariance. Now that I know this, what else do I know?	Connecting place value to multiplying and dividing by powers of ten

A teacher had noticed that her pupils did not derive a new fact from a known fact in their SATs, despite knowing (she thought) a lot about place value. She therefore modelled generating new facts from old at the start of one lesson, showing how multiplying and dividing by powers of ten could generate 'new' facts, using the digits 1, 4 and 6. Using what she called a 'drip, drip, drip' method (see below) she returned to this idea in a later lesson, with this task:

382 x 10 = 3820

What else .....?

The variation here is about transforming a 'given' into new knowledge using known concepts. You can see that this was not fully guided and pupils were free to follow their own threads of reasoning, showing what they did and did not do about the relationship between powers of ten and place value.

Her 'drip, drip, drip' method is that key ideas in mathematics are inserted throughout a period of time, whether they are the main focus of a lesson or not, so that they are not forgotten but become useful habits.

In terms of variation the key factor in these examples is that the digits remain the same so that the focus is on differences in place value and powers of ten, and the connections between these.

After feedback and more discussion she revisited this idea a week later:

4386 ÷ 3 = 1462

Use this calculation to solve the following:

43.86 ÷ 3 = 14.62 ✓

43.86
14.62

$4386 \div 0.03 =$ 146200 ✓

0.03 goes into 4386 x100 more.
The answer gets bigger than
the question.

$438.6 \div 0.3 =$ 14620

0.3 goes into 438.6 x10 more.

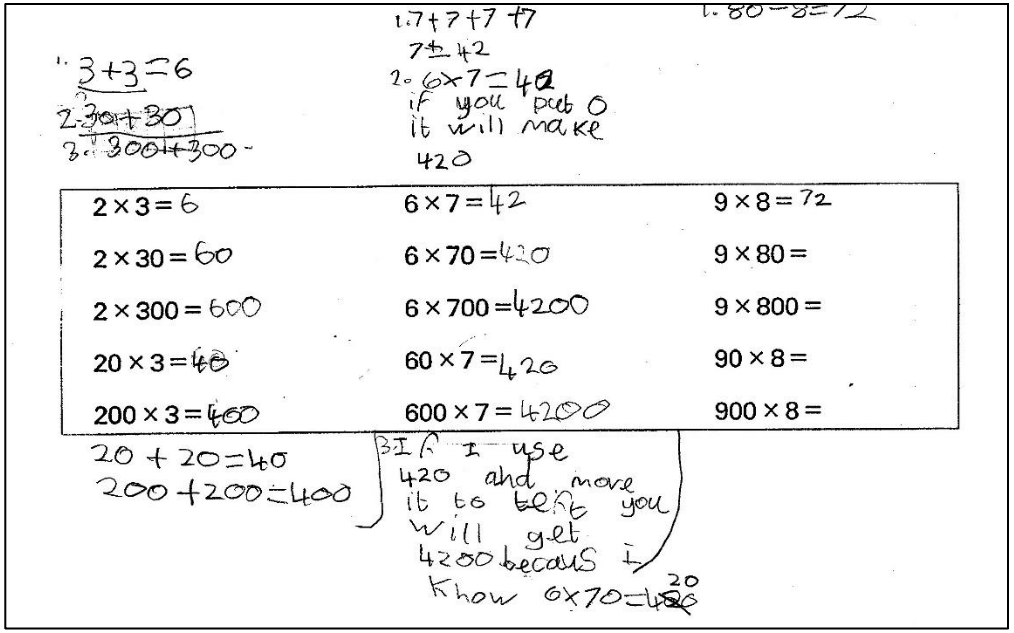
4386 ÷ 3 = 1462
438.6 ÷ 3 = 146.2
438.6 ÷ 0.3 = 1462

I'm finding it easier now.

Later pupils were given some new digits and a sequence of tasks that were carefully varied. You can see that it became very clear what this pupil understood and where their understanding was inadequate. There was some interaction with the teacher to put this right..

These three lesson segments were: teacher demonstration of using variation to focus on a key idea; pupils generating examples; pupils recognising relationships among given examples. In each case the digits were kept the same so that the focus is on the place value and the powers of ten, i.e. the structures, rather than the answers.

These tasks could span all the lilypads. Pad -1 for finding out what is known already. Digits are kept the same and place value is varied (pad 1); then digits are changed and the relationships are kept the same (pad 2)



1. 3+3=6
2. ~~30+30~~
3. 300+300

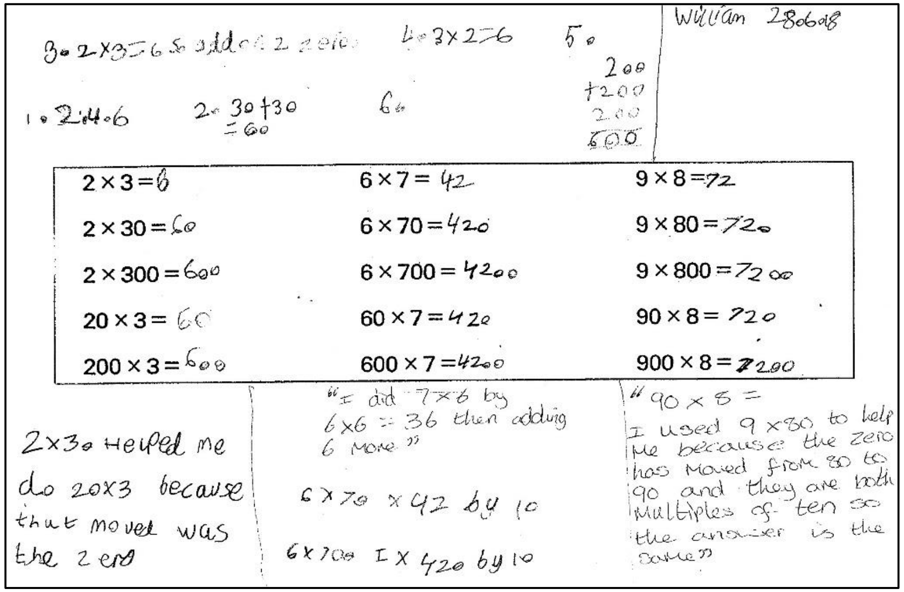
1. 7+7+7+7
7+42
2. 6×7=42
if you put 0
it will make
420

1. 80−8=72

2×3=6	6×7=42	9×8=72
2×30=60	6×70=420	9×80=
2×300=600	6×700=4200	9×800=
20×3=~~40~~	60×7=420	90×8=
200×3=~~400~~	600×7=4200	900×8=

20+20=40
200+200=400

3. If I use
420 and move
it to left you
will get
4200 becaus I
Know 6×70=~~400~~ 20



William 28.06.08

3. 2×3=6 so add a 2 zeros
4. 3×2=6
5. 200
+200
200
600

1. 2+4=6
2. 30+30
=60
6.

2×3=6	6×7=42	9×8=72
2×30=60	6×70=420	9×80=720
2×300=600	6×700=4200	9×800=7200
20×3=60	60×7=420	90×8=720
200×3=600	600×7=4200	900×8=~~7~~200

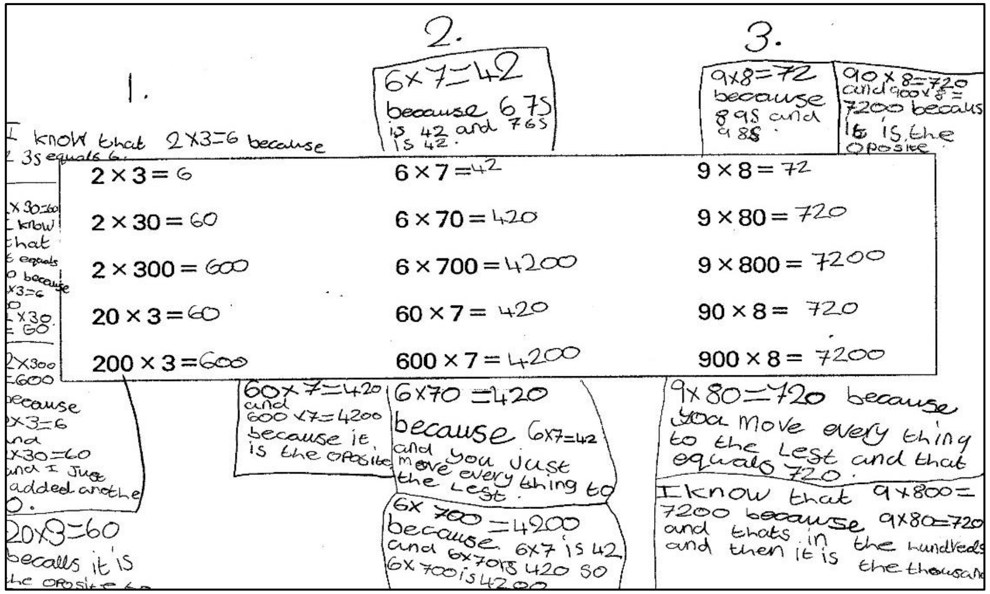
2×30 Helped me
do 20×3 because
that moved was
the zero

"I did 7×6 by
6×6=36 then adding
6 more"

6×70 × 42 by 10

6×700 I X 420 by 10

"90×8=
I used 9×80 to help
me because the zero
has moved from 80 to
90 and they are both
multiples of ten so
the answer is the
same"



1.
I know that 2X3=6 because
2 3s equals 6.

2.
6×7=42
because 6 7s
is 42 and 7 6s
is 42.

3.
9×8=72
because
8 9s and
9 8s.

90×8=720
and 900×8=
7200 becaus
it is the
oposite.

2×3=6	6×7=42	9×8=72
2×30=60	6×70=420	9×80=720
2×300=600	6×700=4200	9×800=7200
20×3=60	60×7=420	90×8=720
200×3=600	600×7=4200	900×8=7200

2X30=60
I know
that
3 equals
60 because
2X3=6
2X30
=60

2X300
=600
because
2X3=6
and
2X30=60
and I just
added another
0.

20X3=60
becaus it is
the oposite

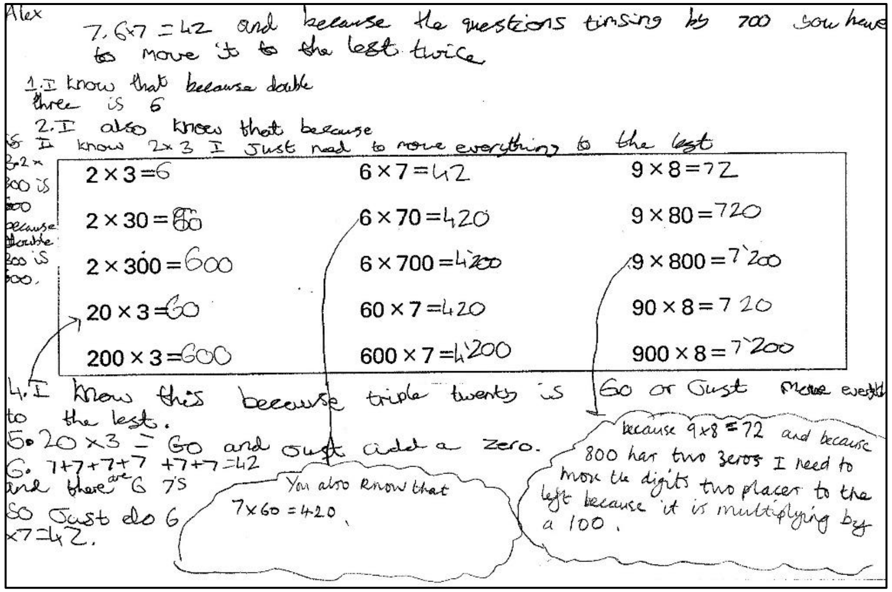
60×7=420
and
600×7=4200
because it
is the oposite

6X70=420
because 6x7=42
and you just
move every thing to
the lest.

6X700=4200
because 6X7 is 42
and 6X70 is 420 so
6X700 is 4200

9×80=720 because
you move every thing
to the lest and that
equals 720.

I know that 9×800=
7200 because 9X80=720
and thats in the hundreds
and then it is the thousands



Alex

7. 6×7=42 and because the questions timsing by 700 you have to move it to the left twice

1. I know that because double three is 6
2. I also know that because I know 2×3 I just need to move everything to the left

3. 2× 300 is 600 because double 300 is 600.

2×3=6	6×7=42	9×8=72
2×30=60	6×70=420	9×80=720
2×300=600	6×700=4200	9×800=7200
20×3=60	60×7=420	90×8=720
200×3=600	600×7=4200	900×8=7200

4. I know this because triple twenty is 60 or just move everything to the left.
5. 20×3 = 60 and just add a zero.
6. 7+7+7+7 +7+7=42
and there are 6 7s
so just do 6
×7=42.

You also know that
7×60=420.

because 9×8=72 and because
800 has two zeros I need to
move the digits two places to the
left because it is multiplying by
a 100.

A similar task was set for younger children, inspired by: www.mathshubs.org.uk/bespoke/april-2015/intelligent-practice/.

These pupils were also asked to make up their own sequences and mainly followed the pattern appropriately.

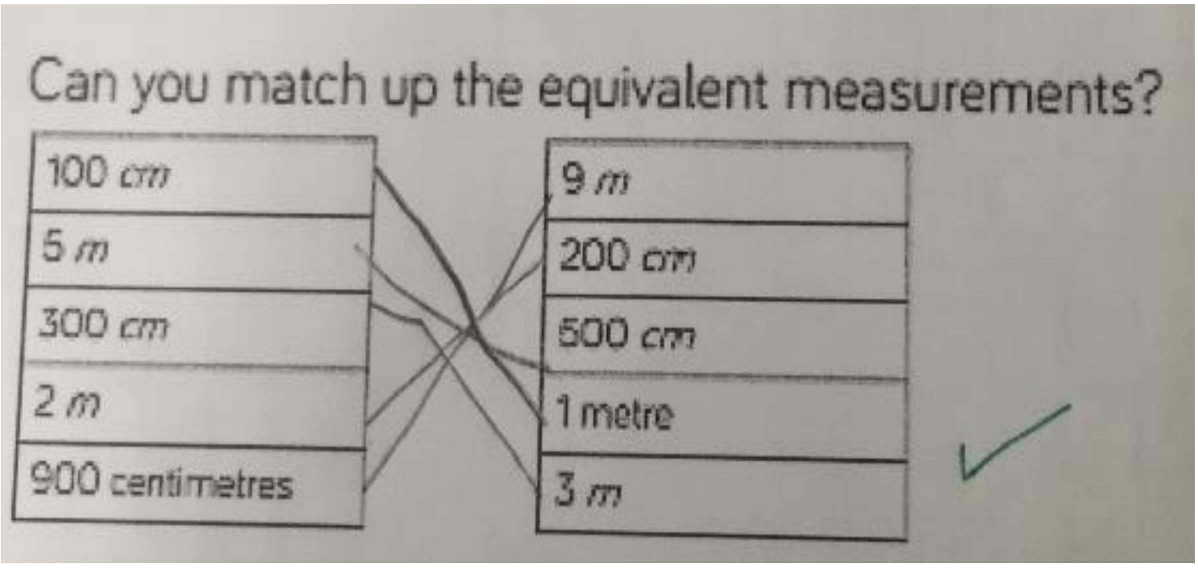
Note that presenting patterned, carefully varied, examples does not ensure learning the key fact you had in mind! It is difficult for children to express the relationship between the zeroes and multiplication by tens and hundreds and words might not capture whether they know WHY this works, or they know THAT it works, and whether the zeroes have a mathematical meaning or are symbols to be moved round.

Any of these ideas can be taken up as key ideas for future teaching.

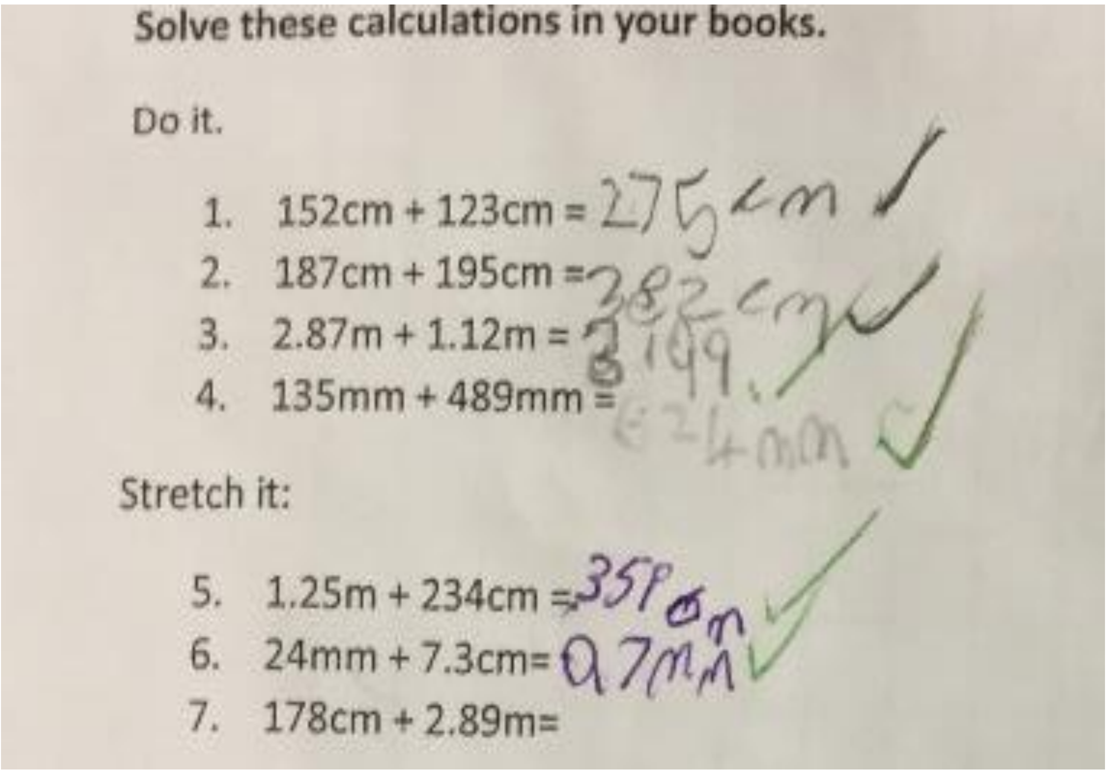
Seeing and using the relationships is a pad 2 task; varying digits move into pad 3; explaining in words and making your own examples is pad 4

12.

Use of Variation	Mathematical Focus
Using variation to critique given tasks and develop new ones. When one thing changes, what else has to change?	Converting and adding different metric units


Can you match up the equivalent measurements?

100 cm	9 m
5 m	200 cm
300 cm	500 cm
2 m	1 metre
900 centimetres	3 m


Solve these calculations in your books.

Do it.

1. 152cm + 123cm = 275cm
2. 187cm + 195cm = 382cm
3. 2.87m + 1.12m = 3.99.
4. 135mm + 489mm = 624mm

Stretch it:

5. 1.25m + 234cm = 358cm
6. 24mm + 7.3cm = 9.7mm
7. 178cm + 2.89m=

Use of Variation	Mathematical Focus
Using variation to critique given tasks. When one thing changes, what else has to change?	Converting and adding different metric units

Teachers were critical of these two resources. In the first case a correct matching could be obtained without any understanding of metric equivalents. However, this could be changed to a 'justify your matching' task with metre sticks or tape measures to back up their arguments, followed by a 'make your own matching question'.

In the second case, it is also possible to get correct answers without thinking about units at all. Some learners might work slowly through the adding and not get as far as the 'stretch it' questions here units really matter.

So we designed our own materials to introduce equivalent metric measures using controlled variation, first starting with some place value tasks relating centimetres to metres with a limited range of values, then adding some of these same values to each other with mixed units, gradually increasing the number of unknowns to be filled in, and therefore increasing the number of decompositions. Throughout, the range of values is limited, thus making it easy to see these on a tape measure as a parallel representation.

Because the key idea being focused on is unclear, it is also unclear where these tasks might be useful on the cyclic pathway

13

Use of Variation	Mathematical Focus
Using variation to design new tasks. When one thing changes, what else has to change?	Converting and adding different metric units

(prior knowledge: tenths and hundredths; Key idea: adding lengths with different units)

100 cm = $\boxed{1}$ m

so 200 cm = $\boxed{}$ m

and 50 cm = $\boxed{}$ m

100 cm + 50 cm = $\boxed{}$ cm = $\boxed{}$ m

200 cm + 50 cm = $\boxed{}$ cm = $\boxed{}$ m

100 cm + 50 cm + 1 cm = $\boxed{}$ cm = $\boxed{}$ m

Explain how these two columns link

125 cm + 50 cm =

125 cm + 0.5 m =

1.25 m + 50 cm =

1.25 m + 0.5 cm =

$\boxed{}$ + 0.5 m = 1.75 m

$\boxed{}$ + 50 cm = 1.75 m

$\boxed{}$ + 1 m = 175 cm

$\boxed{}$ + 100 cm = 175 cm

This is a pad 2/3 task, keeping numbers invariant long enough to see generalisations, and then changing the problem to test and apply them.

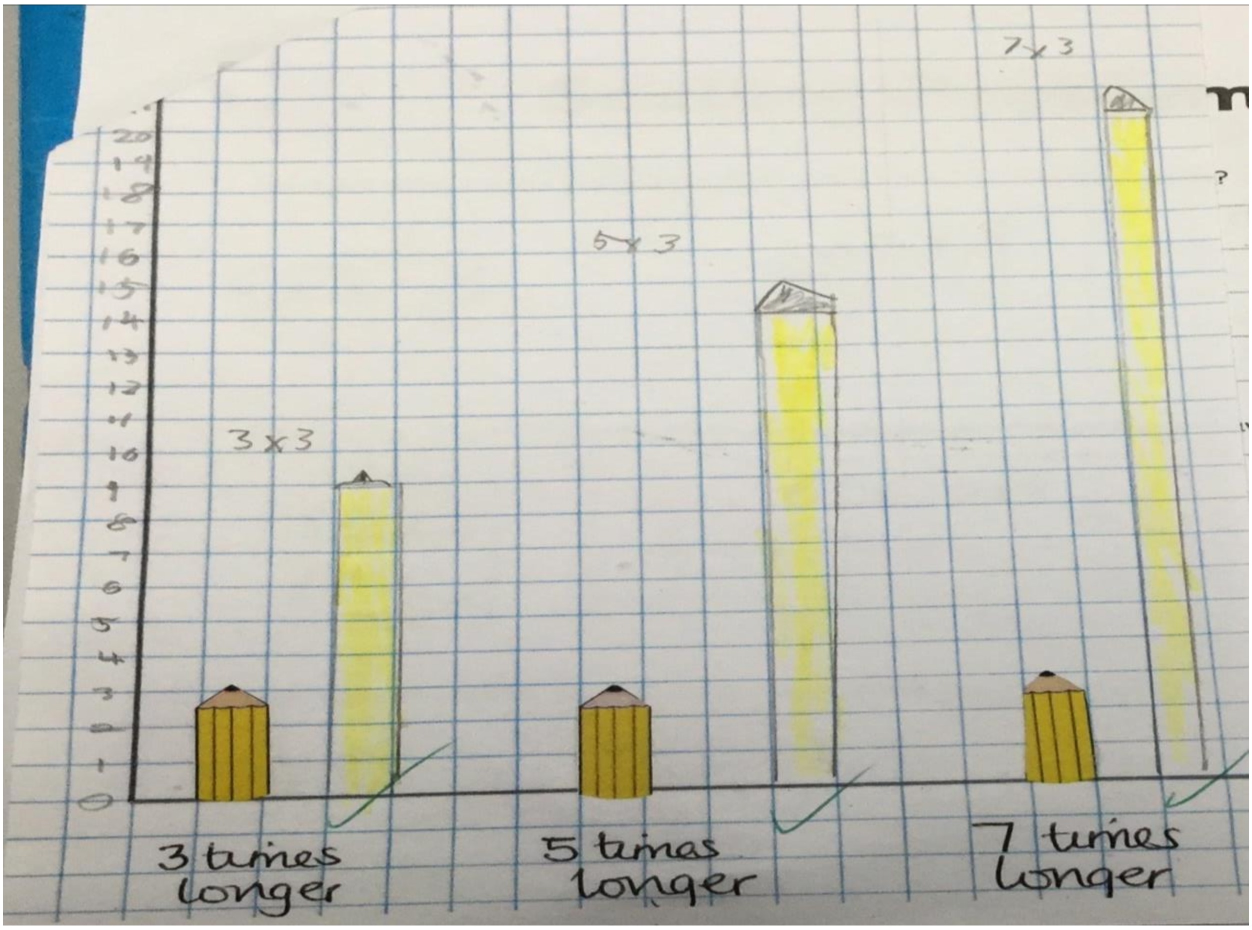
253 cm = 2 m + ___ cm = 2. ___ m

348 cm = ___ m + ___ cm = _ . ___ m

____ cm = 3 m + ___ cm = 3.21 m

14.

Use of Variation	Mathematical Focus
Varying representations to change perception. What will it look like?	Scaling is not repeated addition



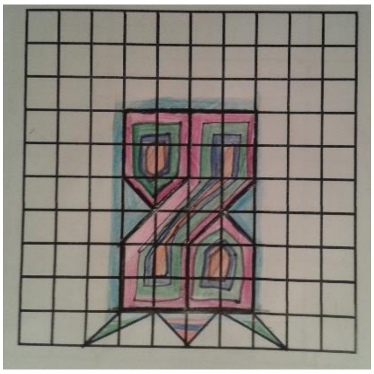
Use of Variation	Mathematical Focus
Varying representations to change perception. What will it look like?	Scaling is not repeated addition

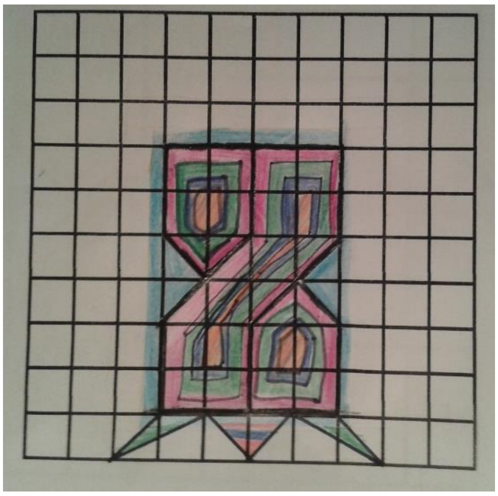
The teacher wanted pupils to shift away from thinking of multiplication as repeated addition or counting in groups, so she provided a change of representation to encourage them to see multiplication as stretching, and hence as a foundation for understanding ratio. Varied representations can present varied, or expanded, meanings. Describing the variation in length between the original pencil and the stretched pencil connects ratio to multiplication. The scaling was only in one direction, and the teacher realised that a fuller understanding of scaling that also retains shape needed some comparison tasks – see the next example.

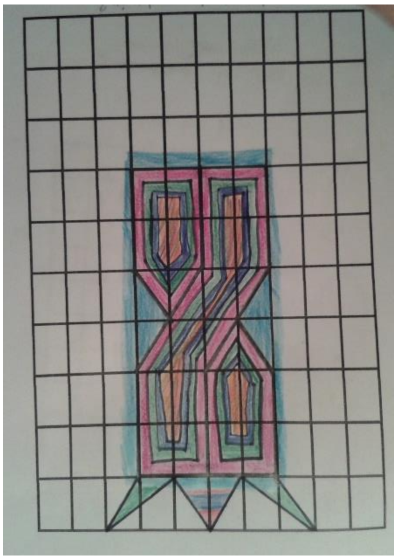
Seeing something a different way could involve pads 1, 2, 3 or 4 depending on the pupils and their existing knowledge.

15

Use of Variation	Mathematical Focus
Comparing different interpretations of similar ideas	Multiplication: scaling and stretching

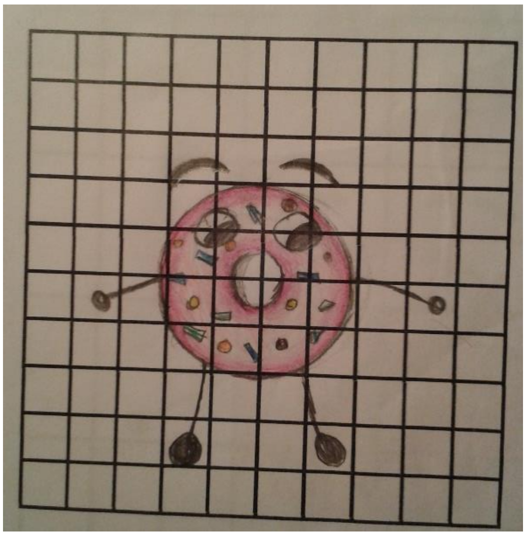


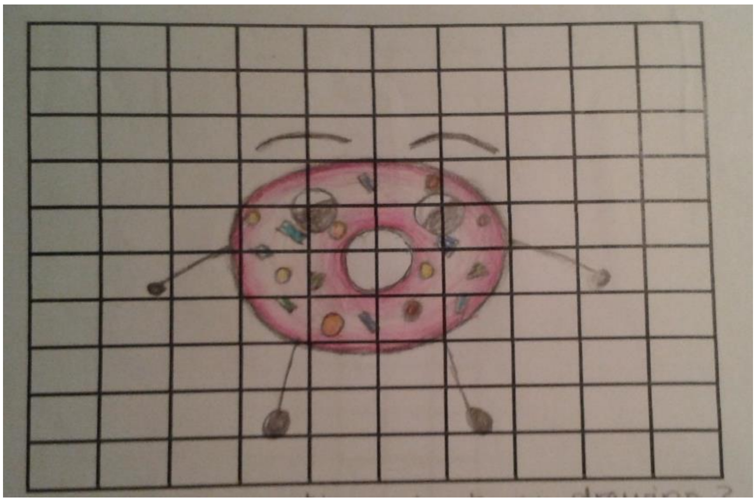




1. What do I notice about my drawing?
I notice that 68% of my drawing is empty and 32% has got something in it.

2. What is the same and different between the first and third drawing?
The third drawing is bigger and flatter than the first one. They are both the same drawings.



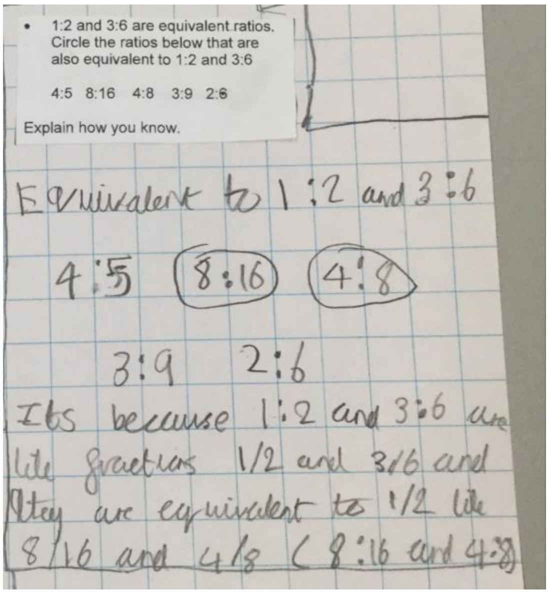


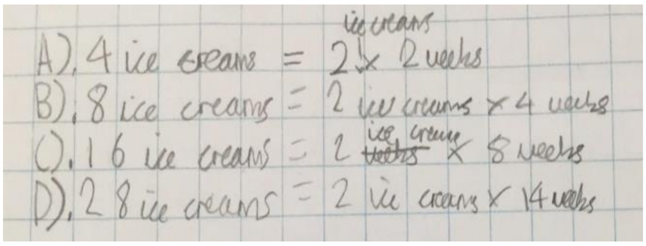
The idea of multiplication as stretching, which started with the pencils in the previous example, was extended with a drawing task. Pupils drew something of their choice on a 10 by 10 grid of 1 centimetre suares. They then transferred their drawing to a 10 by 10 grid of 1.5 centimetre squares and wrote that their drawings were the same but bigger. They also recorded other observations, such as that the percentage of empty squares remained the same. Some were then given grids in which only one dimension had been enlarged, so they were transferring their drawing from squares to rectangles. The writing we show is a typical example of what was written about this. This task addresses a critical feature of scaling and enlargement, that the underlying units change and that may affect the shape. Variation is used to draw attention to this by contrast.

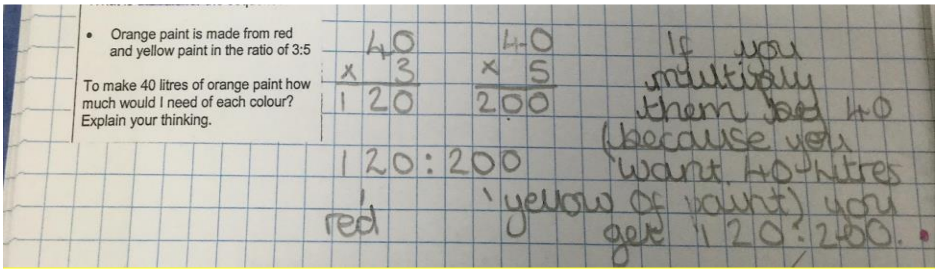
This is a pad 4 task. Once the initial idea has been developed, what it is needs to be contrasted with what it is not.

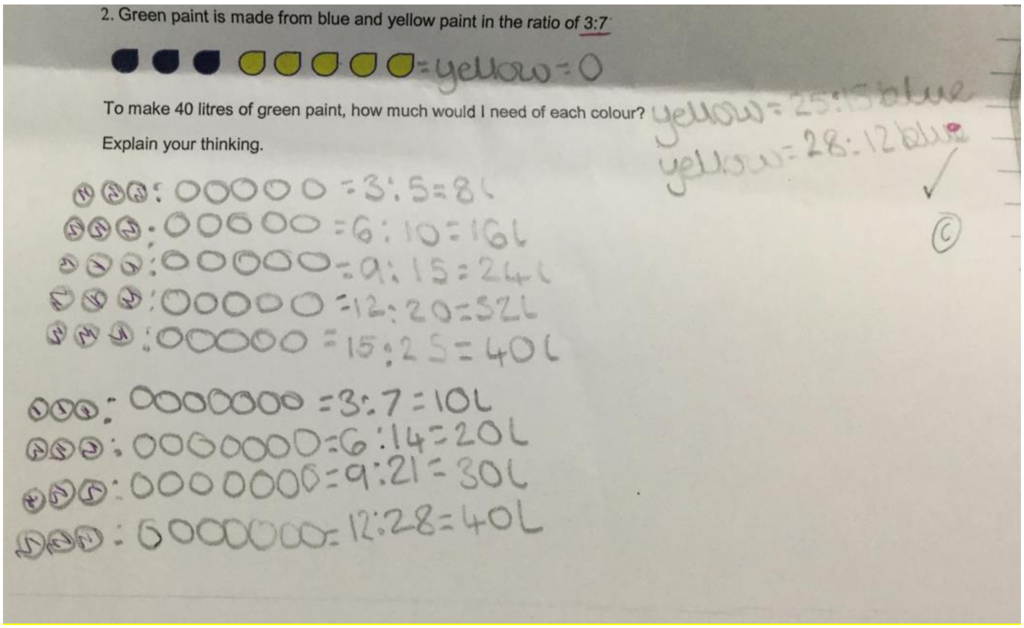
16

Use of Variation	Mathematical Focus
Identifying the important key idea. What is invariant, what is varying, therefore what are learners noticing?	Ratio









Use of Variation	Mathematical Focus
Identifying the important key idea. What is invariant, what is varying, therefore what are learners noticing?	Ratio

A teacher noticed that the worksheet she had given learners on ratio could be completed using counting only, e.g. two for you, three for me, etc. rather than using ratio as a multiplicative concept. This method fell apart during the final questions shown, where multiplying was often applied in a meaningless fashion, and the connection with fractions was superficial.

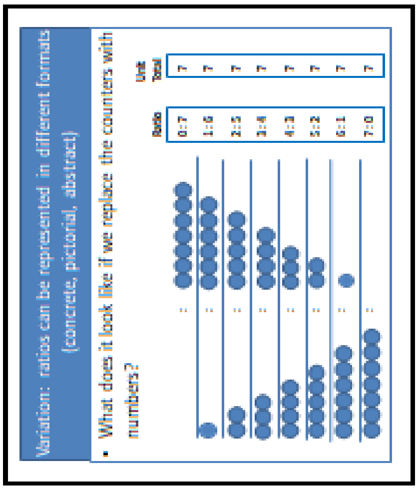
The work that was successful suggests the use of repeated addition, or continuing a pattern down the page, rather than multiplication.

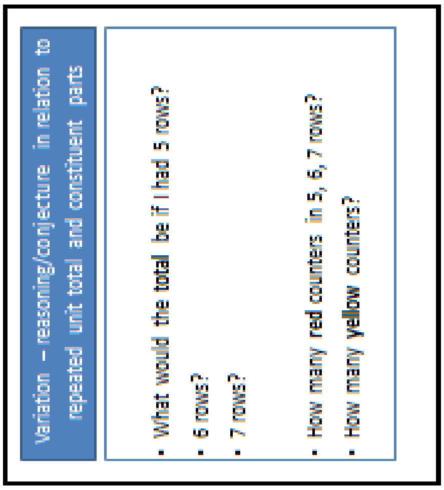
She identified finding the unit from the total number of parts as a key idea to be learnt about, because the questions they had seen before only required multiplying the 'sides' separately to find 'correct' answers.

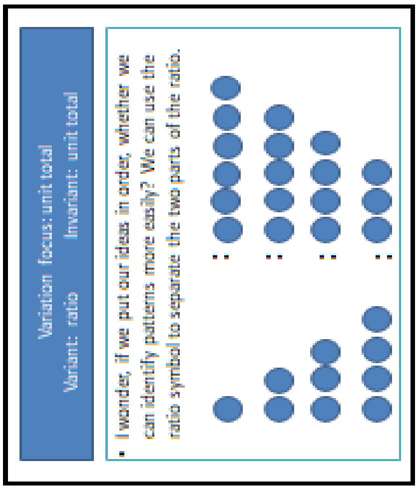
These questions had been intended as a pad 3 task, but turned out to be a pad -1 task from which the teacher moved into stage 0 – identify that the key idea to be learnt was the 'whole' and the unit parts

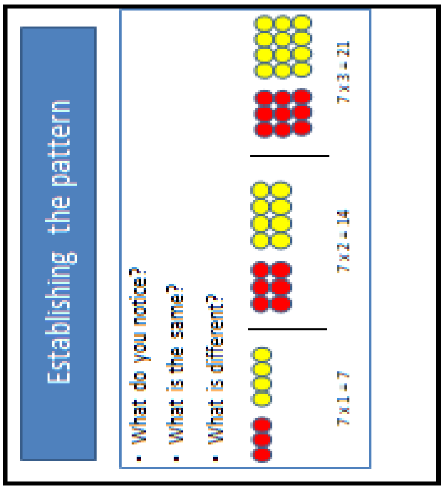
17.

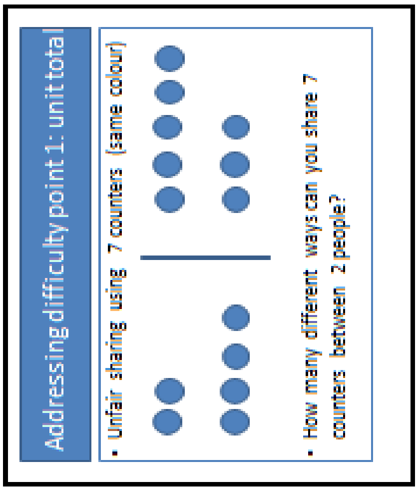
Use of Variation	Mathematical Focus
Balancing variation and invariance What changes and what stays the same?	Identifying the unit from the total number of parts

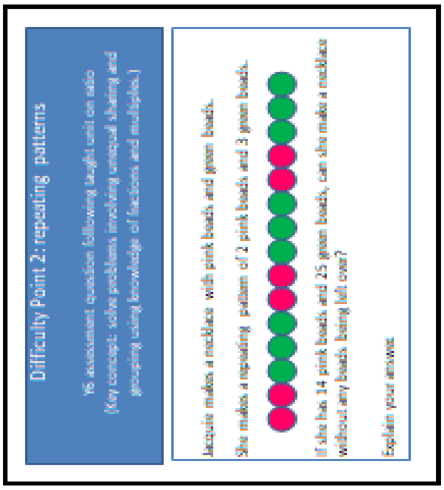




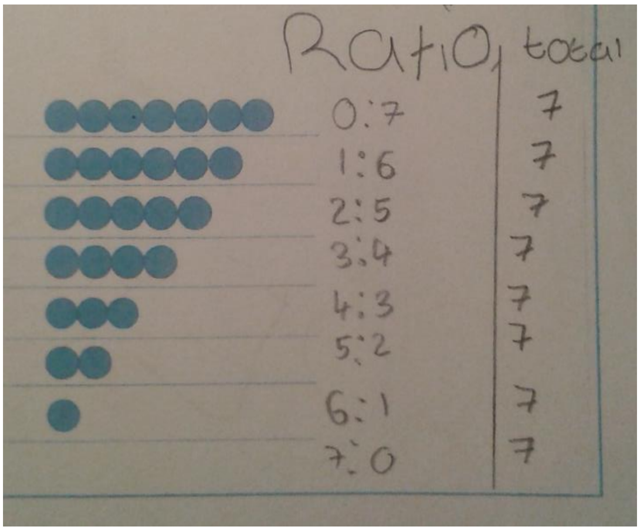


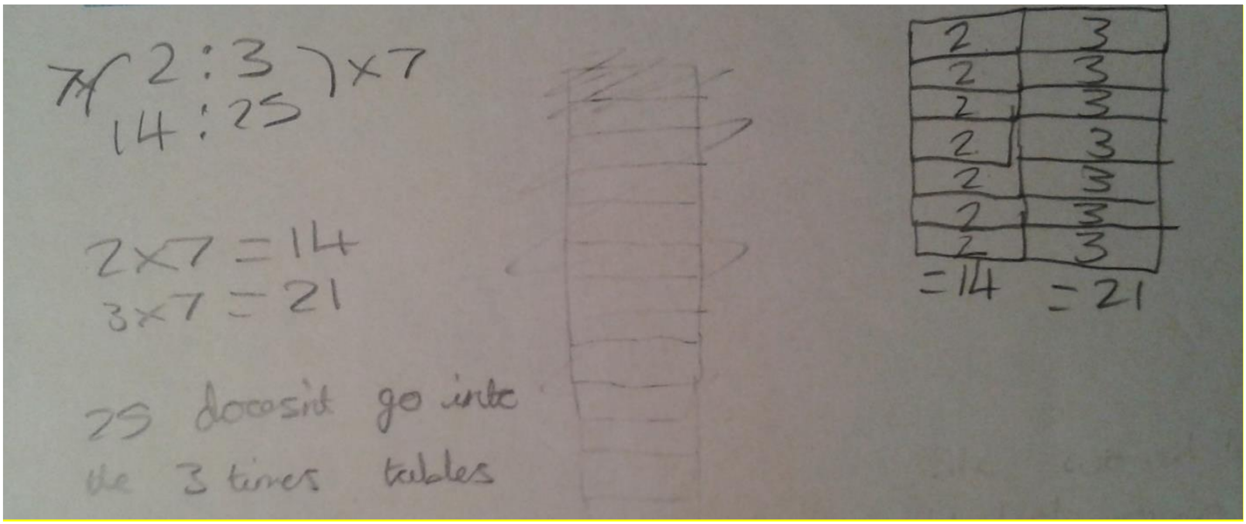


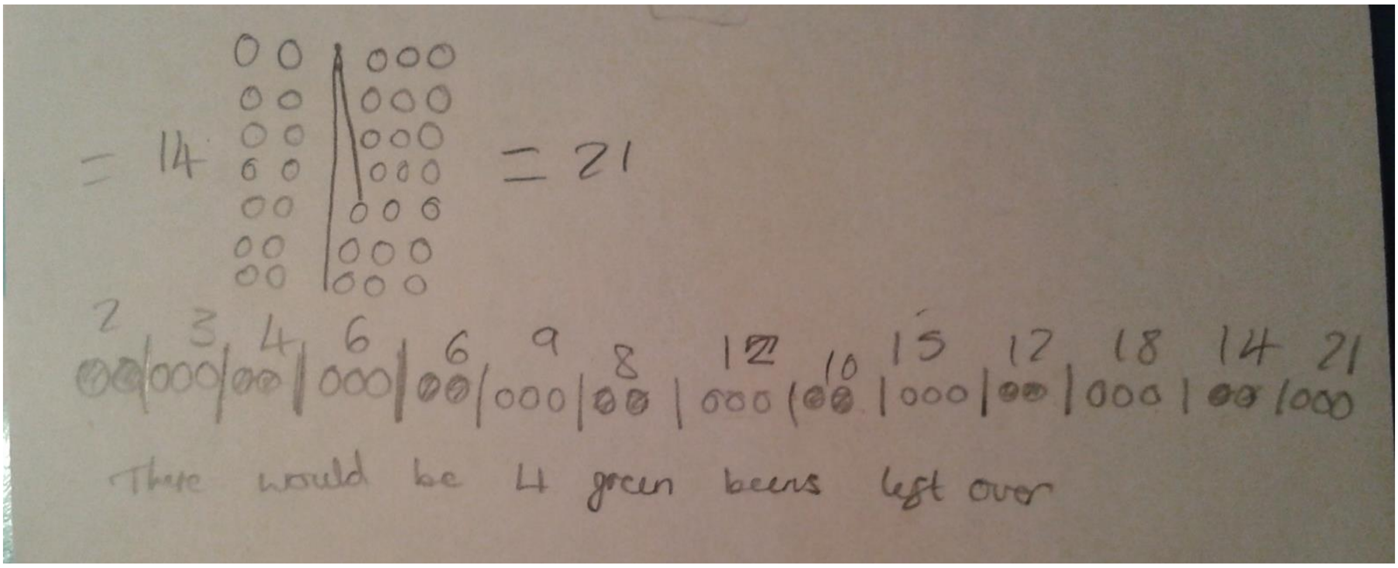




Use of Variation	Mathematical Focus
Balancing variation and invariance What changes and what stays the same?	Identifying the unit from the total number of parts

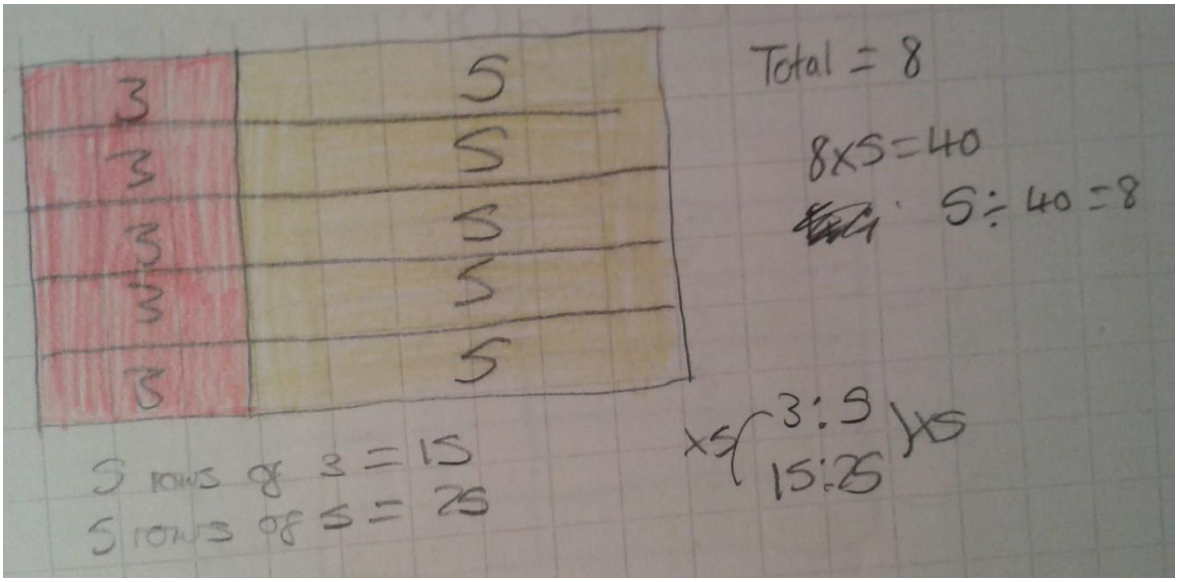






The slides show the use of variation to draw attention to the ways a 'whole' can be split into different ratios. This is done by keeping the whole invariant and collecting different ways of splitting it from the pupils, then organising these ways systematically and comparing them to the conventional notations, with 7 as the constant total.

Repetition patterns are then tackled, and at first it is hard to see how these fit with the progression about totals, especially as the case has changed to 2:3 with 5 as the 'whole'. But this becomes clear. The point is to recognise that each repetition of a pattern holds the same ratios, so changing the numbers ensures that this is thought about and not merely repeated from before. See the second and third examples of pupils' work on the bead question. Then this new reasoning is applied to 3:4 and 7. Note that the layout draws attention to what varies and what stays the same. The repetitions are written as multiples.



The question about mixing paint was given and the pupils were much more successful and most expressed multiplication and division in their written work.

This sequence shows progression from pad -1 and pad 0. Pad 1 gets pupils started, and then pad 2 crossed by providing comparisons, language, formats and symbols to focus on 'whole' and 'unit part'.

18. Wet mixtures

Use of Variation	Mathematical Focus
Invariant proportion: different representations, same context, varying numbers	Scaling, ratio and proportion

Making salad dressing: in the context of a Year 2 science project incorporating growing salad leaves and designing salad dressings. The children were confident with repeated addition in arrays.

Session 1

The first of five sessions involved children engaging in a practical 'real life' problem-solving task, providing an opportunity to apply their recent maths learning over a sequence of lessons on multiplication within a context related to their class project on Growing Things. They were to make small bottles of salad dressing to be given to parents as part of an end of topic celebration.

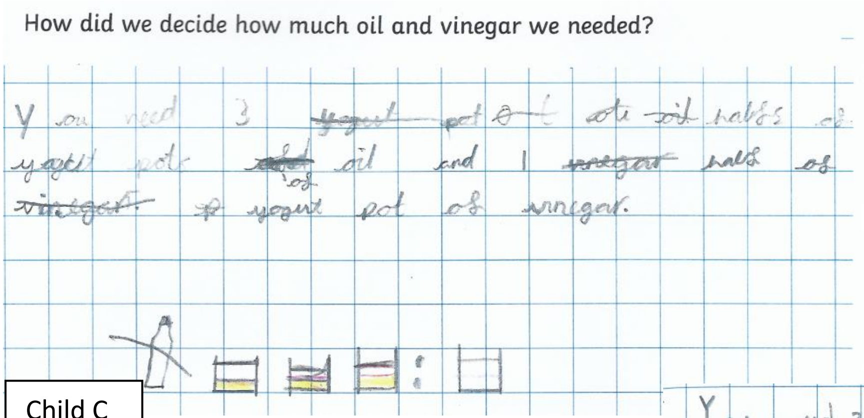
Key mathematical aspects were the proportional relationship between the oil and vinegar in the recipe and expressing this as a ratio, and between the quantity of dressing made in the jar and the number of bottles required.

The lesson started with the teacher sharing the task, ingredients and equipment with the children. She discussed the recipe with them and represented the 3:1 relationship between the oil and the vinegar pictorially on the whiteboard. The children worked practically in small groups to make the dressing, with the adults supporting asking questions to prompt children's thinking… Will it fit in the jar? How can we find out? What did you see before we shook the bottle (separate layer oil and vinegar in the correct proportion?)? What has stayed the same, what is different? Do we have enough for everyone? How many little bottles will 1 jar make? What do we need to do so everyone has a bottle?

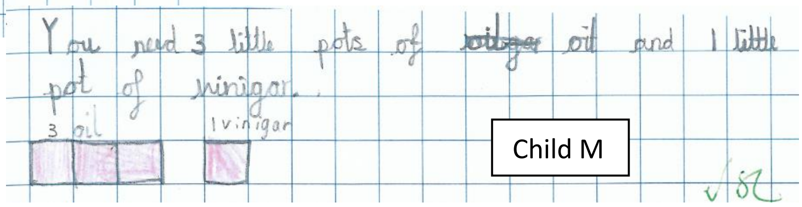
The following sessions focused on drawing out and developing the mathematics learning from their practical experience.

Session 2

Writing up the practical in their own words. This was more of a narrative supported by pictorial representation.



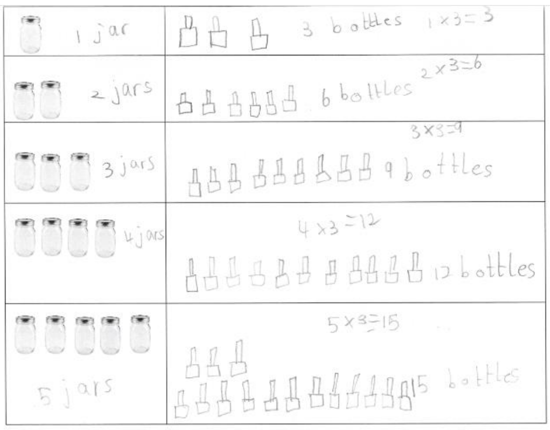
Child C

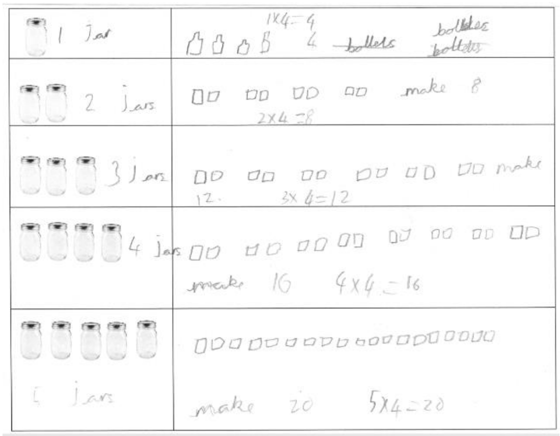


Child M

Session 3

Exploring the relationship between jars and bottles of dressing using a structured table format.
Children used their own recording. Child C moved swiftly from a picture of a bottle to a symbol. Initially he talked about going from 1 jar to 2 jars as doubling, and then had to revise his explanation for 3 jars.





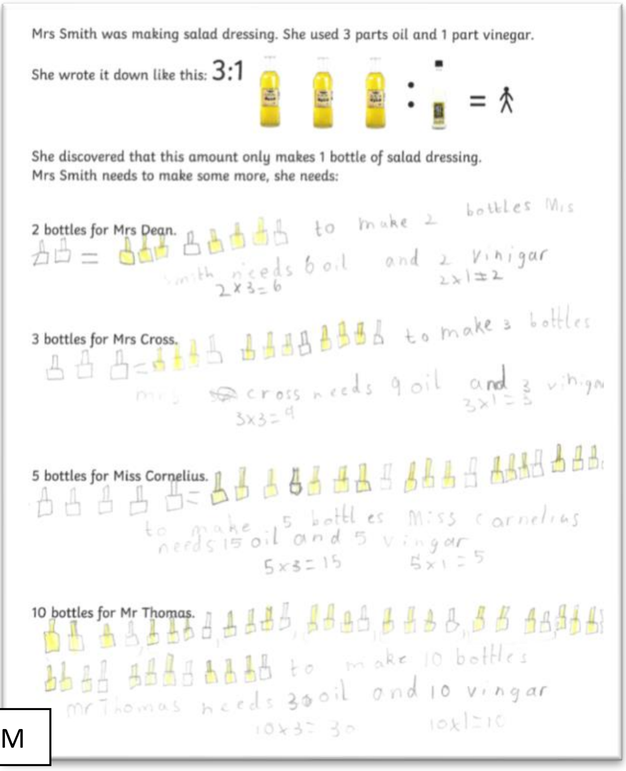
Child M

Child C

Session 4

Variation in the problem, returning to the oil to vinegar ratio in the recipe. Same context, different problem.

Children again given freedom to draw in their own way, which had interesting results.



Child M

Child C

Child M drew a repeating pattern for each, representing the ratio exactly. By the 10 bottles example it was becoming error prone! She needed to continue drawing and explain in words before making the link with multiplication.

Child C used the more efficient symbolic approach he'd used in the previous task and noticed the multiplicative relationship. He quickly began to generalise and didn't really need to continue drawing, although he did!

Session 5

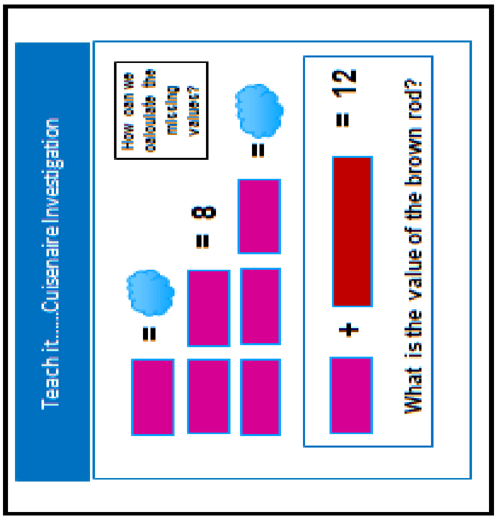
Compare different work – what was the same, what was different?

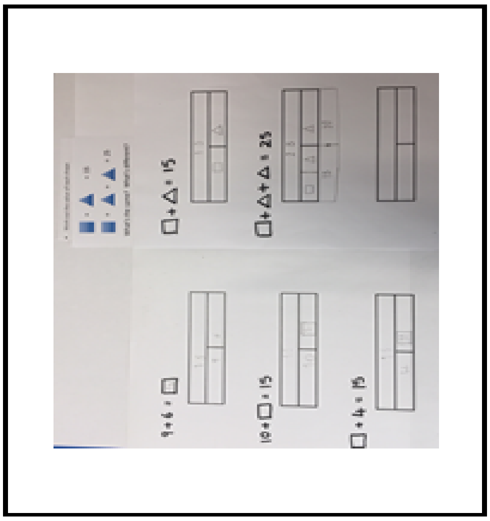
Could they apply this to a different context now?

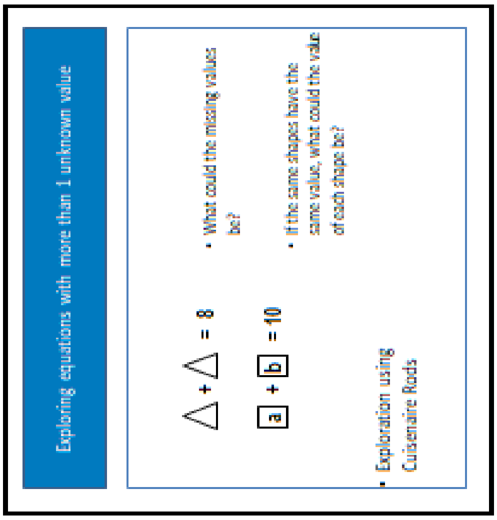
The sequence shows progression from pad 1 to pad 4 from an initial problem with variation of representations, through variations of parameters, to the beginnings of generalisation and symbolic representation.

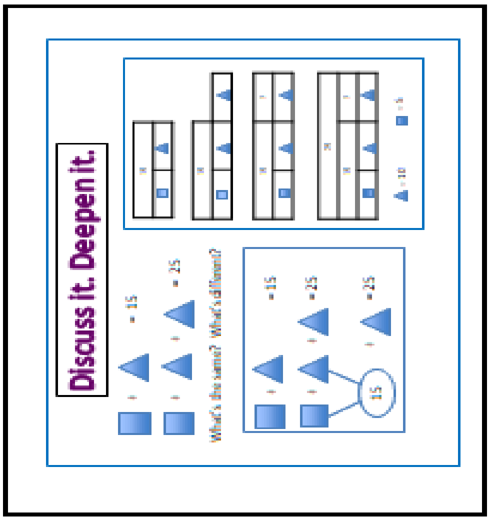
19

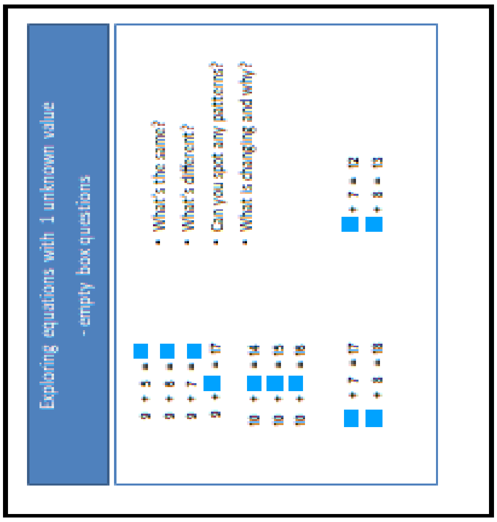
Use of Variation	Mathematical Focus
Controlling variation to see relationships through pattern	Finding unknowns in number sentences

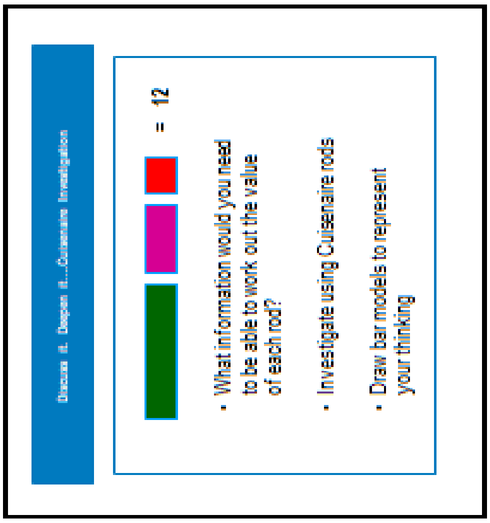












Use of Variation	Mathematical Focus
Controlling variation to see relationships through pattern	Finding unknowns in number sentences

These slides are a trimmed-down version of a sequence of slides used in a lesson sequence (more than one lesson) about finding unknown numbers in equations. The pupils had Cuisenaire rods available and were also familiar with drawing bar models for part-part-whole relationships. They have met missing number problems with one unknown throughout primary school in various formats. The slides start with a reminder about finding missing numbers. There are suggested questions that draw attention to the relationships between the numbers when one value changes. This leads into thinking about the relationship between two unknown numbers in an additive number sentence, that is how they vary together (covariation). The question of whether the same shape always represents the same number or not is raised as well. Different textbooks have different policies about this, but it needs to be discussed as a precursor to algebraic representation.

The next slide uses Cuisenaire rods to 'unpick' equalities to find out what particular rods represent. In the rod sequence, pupils are not told the value of the pink rod but use reasoning to deduce it. They could be asked 'how do you know?' Then they use this to deduce the value of the brown rod. They are then asked to make up examples of their own with what ever rods they like and get friends to work them out. Some of these are discussed using a visualiser.

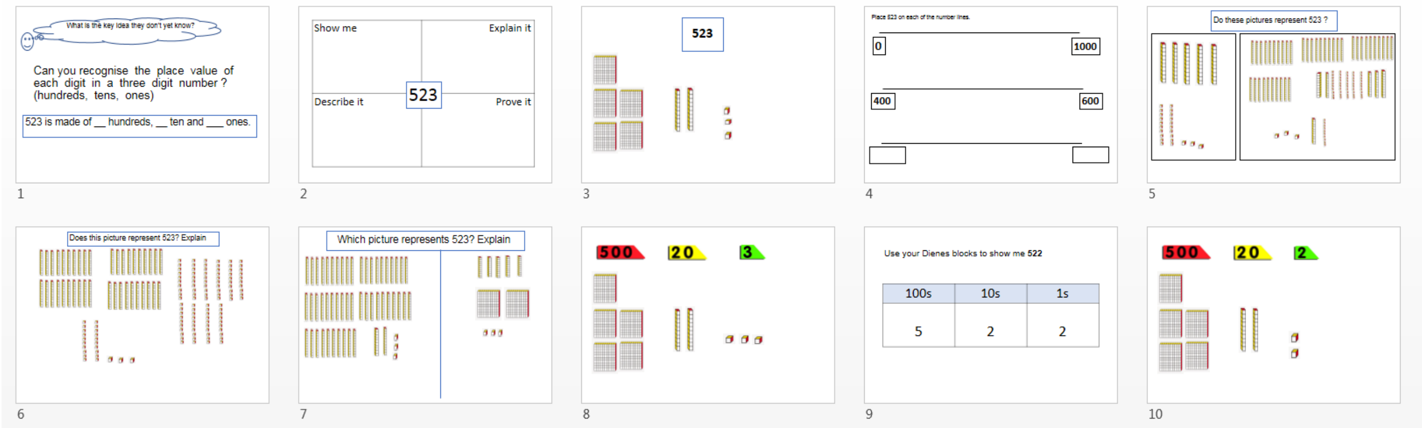
In the fourth slide there are three 'unknowns' and to deduce their value you need to know the relationships between them. It is suggested that bar models are drawn to help with this. Again they are asked to make up their own examples using Cuisenaire rods and bar models.

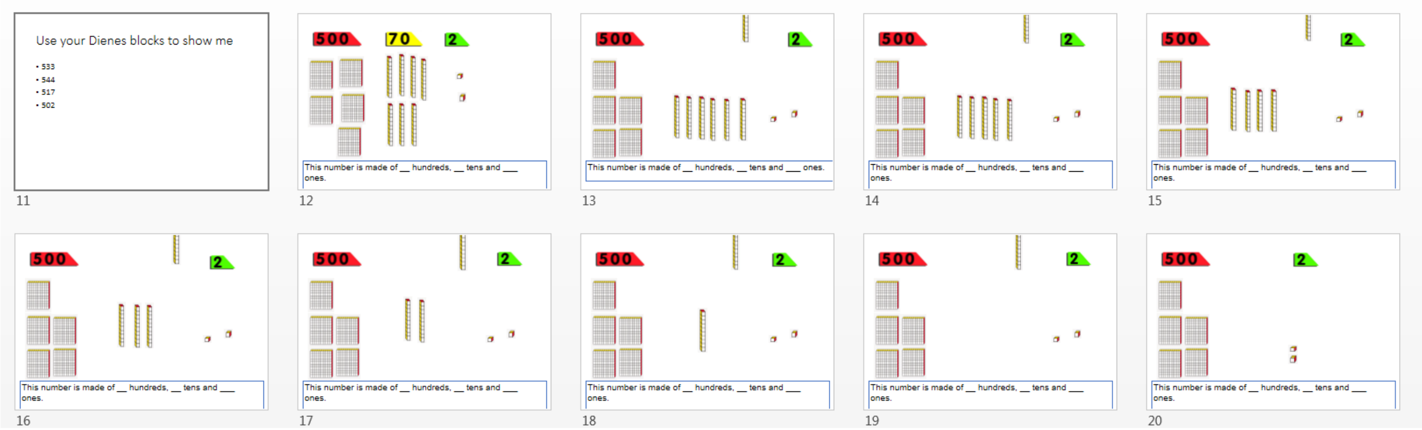
Note the variation of representation so far in the sequence. Only the numbers and the Cuisenaire lengths *have* actual values; shapes and bars represent them. The number of unknowns has increased from one to three. The invariant task is to deduce value from the equality. This requires transformations of the additive relationship – sometimes you have to subtract numbers, sometimes add them, to find what is missing and to check answers.

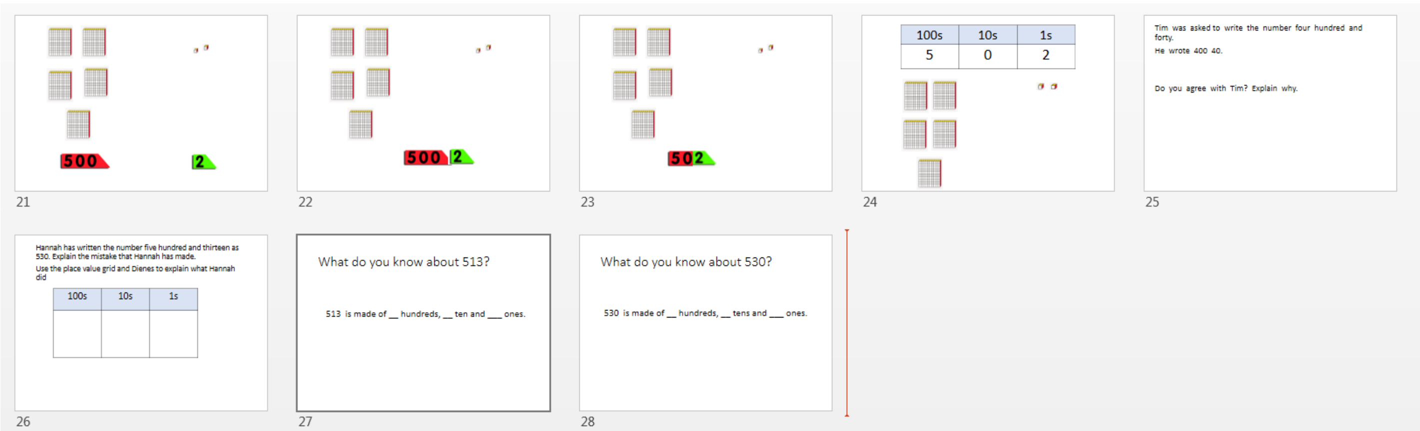
In the fifth slide only the top left section is visible at first. This extends the idea of unknowns to include the fact that unknowns might be used more than once, but still depends on deduction from known relationships as its method. This kind of task bridges between algebra in KS2 and algebra in KS3. A way of representing using bar models is revealed. Finally, pupils get to do some examples by labelling bars in given models.

The slides indicate a progression by controlling what varies for each step forward.

Designing a task sequence using our overlapping lilypad model (full version on Glow Hub website)







This booklet has been produced by the GLOW hub Variation Work Group, by teachers for teachers. We started from pupils' work and then looked for ways to describe the variation involved and its effects. This is what variation does for us – it is a tool to understand learning, or learning problems, and to help design more helpful approaches.

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