Division – the sleeping dragon

I was recently in a year 7 class who were telling their teacher what they understood by the word ‘ratio’. There was an interesting range of suggestions such as: ‘I have 13 pens and you have 3 so the ratio is 13 to 3’, some suggestions which were more about what you ‘do’ when you do ratios such as ‘you write a number and two dots and then another number’, and - most interestingly – one boy who said: ‘it is something to do with division but it is hard to explain the connection’.

I chuckled to myself as I have spent many hours in teachers’ workshops trying to explain such a connection to myself and with others. This work has led me to be completely convinced that division is the sleeping dragon of the primary/secondary interface in mathematics. That is, it is quietly lurking in students’ experience, and if it wakes up it roars, either to destroy their confidence or to lay tripwires in their path or, if they are fortunate, to imbue them with its power to become strong mathematicians.

As I write this a new national curriculum is being written, but whatever it contains, division will continue to weave its ambiguous spell over whether children will understand secondary mathematics.

Before I even try to explain its connection with ratio, I need to say what division is for learners, and, as many aspects of number, it depends on the associated actions how it comes to be understood. My list of associated actions gets longer and longer – you could probably add to it:

1. Sharing out by counting, as we do with chocolate buttons (and eating the spares)
2. Sharing out by cutting up congruent shapes, as we do with a cake or pizza
3. Sharing out by counting and cutting, as we do if sharing three cup cakes between five people
4. Sharing by pouring, as with wine
5. Folding and cutting, as with ribbon
6. Folding and cutting, as with a piece of paper
7. Finding how many of X ‘go into’ Y with physical objects by fitting
8. Finding how many of X ‘go into’ Y with linear measures (e.g. how many centimetres in a metre?)
9. Finding how many Xs ‘go into’ Y with numbers by counting, such as counting in 2s, 3s, 10s and so on
10. Grouping objects in 2s, 3s, 5s and so on.

... there are probably many more. Each of these triggers a different action, and very few of them are described by counting.

Yet meeting division as the inverse of multiplying, when all you know about multiplying is your times tables, repeated addition, and arrays limits your understanding of division to using learnt number facts. If 48 = 6 x 8 then 48 ÷ 6 = 8 and 48 ÷ 8 = 6. And you can ‘do’ division by repeated subtraction, if you hit on the right thing to subtract repeatedly, and if you can keep track of the tally. Even with this model, there are two ways you can imagine the action. Using Cuisenaire rods, the length of an orange rod divided by 5 gives you the length of a red rod: a ‘cutting into equal lengths’ action a bit like folding a ribbon, if you could fold a rod. The question you have answered is ‘what length of rod is worth one fifth of the length of an orange rod?’ However, the length of an orange rod divided by the length of a red rod gives an answer of ‘5’ because the question is ‘how many red rods are equivalent in length to an orange rod’ – the answer is a number. Research tells us that often the first kind of question is easier to answer because of our familiarity with sharing out food between a given number of people: stuff divided by a number gives you stuff. The second question is ‘stuff divided by stuff’.

So the repeated addition model for multiplication does not give easy access to understanding when to apply, and what sense to make, or how to act out, division as its inverse. The algorithm does not yield many clues about what it means either. It depends on identifying suitable subtractions to make until there is nothing left, or a bit left over.

Interestingly, the layout of short division matters. If we use the bus stop method, the dividend goes under the line and a mathematician from Hungary asked me why we do this. She pointed out that putting the dividend under the line can confuse learners, as when writing a division as a fraction, we put the dividend above the line. I shall get back to fractions in a minute, but let’s dwell on this idea for a while. If you lay out short division in another classic way, you get bonus information:

13

222222

 17094

The bonus information you get, directly from the layout, is that the fraction $\frac{222222}{17094}$ equals 13 and that by swapping the 13 and the 17094 you also get another fraction and its value.

The problem, and presumably why this layout is not really taught much now, is that you cannot extend this to long division, should you wish to do so, and that business with fractions does not mean much if you have remainders.

By the way, a good game to play with long division enthusiasts is to then ask them if 2222223 is divisible by 13. It isn’t, and the long division method means you don’t have to start again from the beginning, but it is amazing how many do start from scratch. The division algorithm should tell you that when you have nothing left to ‘carry’ after the 222222 you only need to think about the ‘3’.

So far all I have considered is whole number issues, but the sleeping dragon has stirred and reminded me, through thinking about remainders, that all is not well. Division is not just the application of remembered times tables, it also has to cope when the dividend is not a multiple of the divisor. So if your only experience of multiplication has been through tables and exact multiples of integers you have no idea where these inexact answers come from. If, on the other hand, you have experience of multiplication where one of the numbers is not an integer, than inexact answers are not a surprise. So it makes little sense to think about inexact division until you have gone beyond whole number arithmetic and begun to be familiar with the rest of the real numberline, or with some comfortable, imaginable, families of fractions. If you don’t have these models to think about, then you are limited to the ‘counting out and having stuff left over’ idea of division.

Some educators have shown that very young children who regularly share out liquids, or lengths of ribbon, when answers are inexact become adept at dealing with these new non-whole numbers. Not only that, they get bonus knowledge because they may have learnt the need for smaller units of measure for doing such sharing out. For example, in workshops I have used shot glasses as measures to help teachers share water out fairly. The equivalent tool for ribbon is the tape measure, and the use of centimetres to help the fair share out. So the bonus knowledge is that when you are dividing inexactly, you can make it exact by using smaller units. The sleeping dragon goes back to sleep – smaller units means fractions, particularly tenths and hundredths, and hence decimals. For some reason we expect children to learn decimal place value at a much faster rate than learning whole number place value and it is often said that they can do this through money. Well money is money and you cannot ‘see’ the tenths. A ten pence coin does not look like a tenth of a pound. You still have to learn the place value of the symbolic form for the number. On the other hand a centimetre looks like a tenth of a decimetre, which looks like a tenth of a metre. The dragon folds its tail and snores contentedly.

Am I getting too far away from division? I think not. There is a further model for division as inverse of multiplication which I have not yet described and that is to do with scaling.

Multiplication by anything other than whole numbers and friendly, easy to imagine, fractions does not yield to repeated addition models. It is scaling. To multiply by 3/5 means we have to imagine a fifth, and then enlarge it by scale factor of 3. Alternatively we can enlarge by three and then imagine a fifth. John Mason and I have recently been playing with elastic to express this relation. Imagine stretching a piece of elastic so that it is double its length – how do you have to alter its length to return to the start? You have to halve it. Similarly, if you stretch it to half its length again, i.e. multiply by 3/2 you can ‘see’ that to return to the start you have to reduce it to 2/3. The reciprocal relation pops out, and again and again children have to look at the elastic and think about shopping it into equal lengths, the fraction denominator, and the counting off the number of these they need. This is where the relation with ratio and proportion comes in. No wonder the year 7 student could not explain it. To divide a quantity into unequal portions we have to imagine that we can divide it into smaller equal portions (smaller units, like shot glasses or centimetres) and then count off the number of these smaller units we require. To describe unequal portions we have to do the same. If all we have is counting, then we are limited to discrete situations where the numbers are already related multiplicatively – they work. If we have measurement, which is continuous, then not only can we divide in many different ways, with many different models and applications, but we also have the bonus of ‘seeing’ ratio and proportion.

Finally, to return to the earlier example of 222222 ÷ 13. If instead of diving into a division algorithm we write the dividend and divisor as fractions, we get further bonuses. We get the division meaning of fractions; the simplifying action of finding common factors; and an instant answer. The answer to 36 divided by 7 is 36/7. Sometimes this is all you need. If it isn’t and you have to apply your knowledge of multiplication facts, then the bonus is that you already know how to express remainders as fractions and ‘see’ what they mean.

There is something either supremely elegant or terribly messy about how all this ties together. It could be elegant if time and care is taken over the way learners make these connections and enrich their knowledge; but it is often so messy and rushed that learners suffer and concede defeat, the dragon overpowers them. Division, too often, divides those who are going to be able to achieve in mathematics from those who may not. But it is not long division that grants the mathematical power, it is the multiple meanings (pun intended) that gives the tamed dragon a mighty roar.