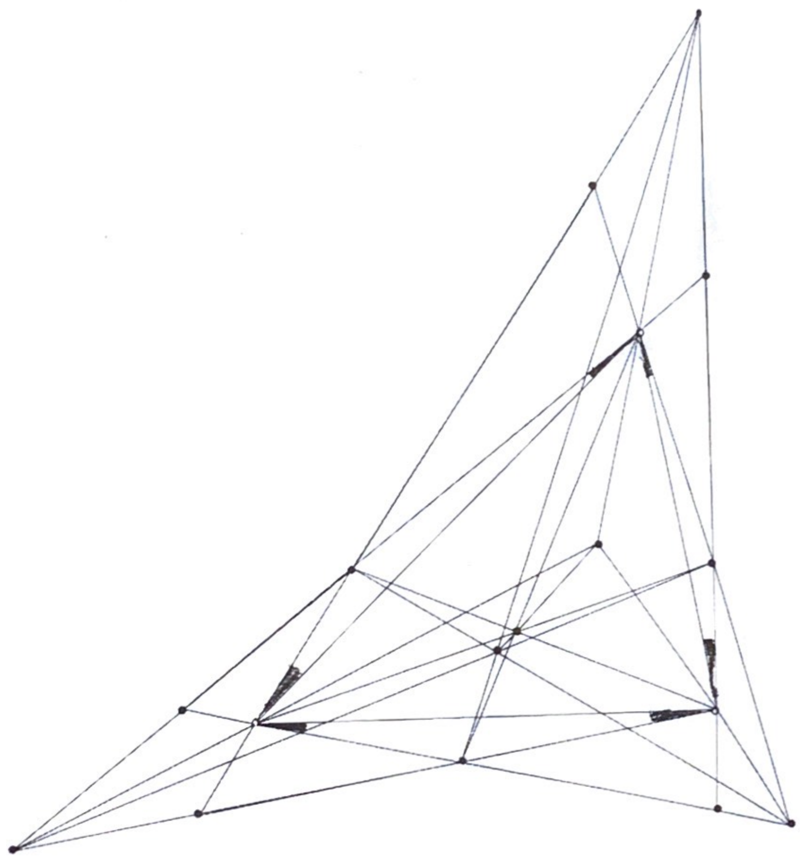
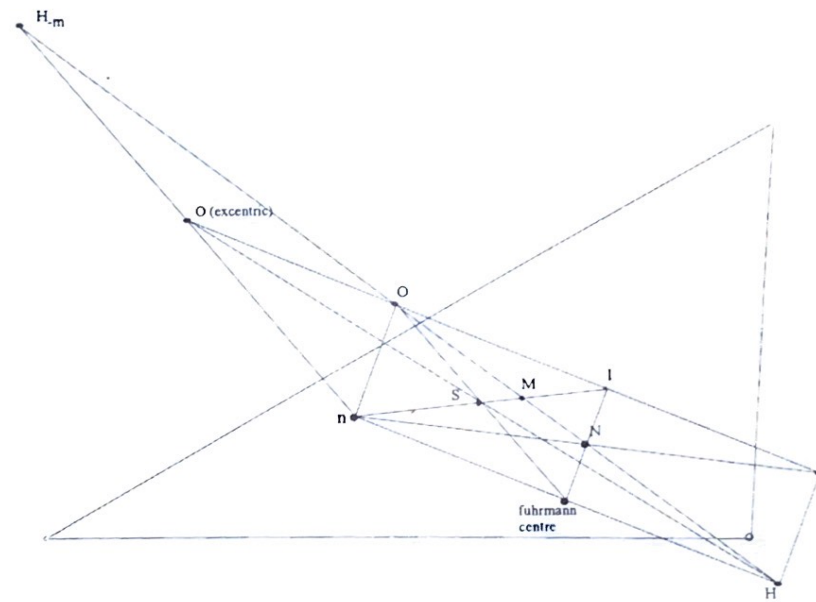
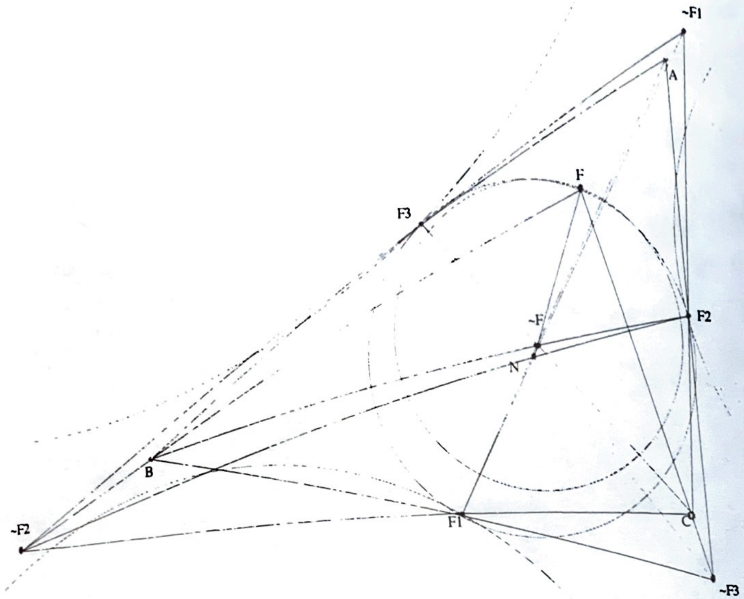
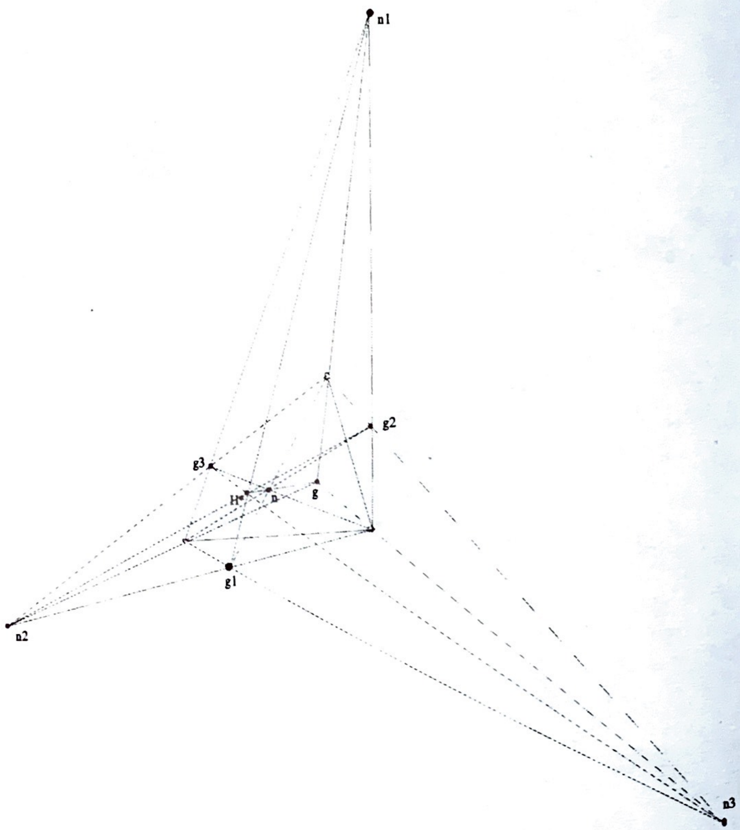


RECASTING TRIANGLE GEOMETRY







We load the sacrificial stands
Of wood and earthenware,
The smell of burning southernwood
Is heavy in the air.

It was our father's sacrifice,
It may be they were eased.
We know no harm to come of it;
It may be God is pleased

distinguished from the classical euclidean methods. The latter were then still described by the word 'synthetic', whatever the order of presentation involved.

People have always, it seems, felt strongly in favour of one or other approach. Newton, for instance, was particularly concerned to present his new discoveries in the more familiar classical form, and was critical of algebraic treatments of geometry: according to him, "equations are expressions of arithmetical computation and properly have no place in geometry".

It is inevitably tempting to see strongly expressed preferences for older or newer methods in terms of reactionaries on the one hand, and radicals on the other. For example, there was a bitter controversy in post-napoleonic Naples in the early 19th century between the university's synthetic geometers, who were Bourbon sympathisers opposed to the analytic geometers from the institute of civil engineers, who were Jacobin supporters. For the former, mathematics was a spiritual science, a powerful resource against atheism and materialism. Their mentor, Nicola Fergola, was a staunch Catholic, who - it was said- saw God behind the circle and the triangle, and who claimed that his opponents saw only the nothingness behind their formulas.

The opposition, however, held that the classical tradition was holding back progress. They thought that the synthetic approach was too specific, requiring skill, knowledge and intuition. This meant it became an elitist activity, whereas the analytic method was general - as the slogan went, "every problem could be put as an equation" - and solutions could then be relatively automatic. A favourite example supporting this was the particular geometrical construction of a tangent to a circle (by taking a perpendicular to a radius through the proposed point of contact), compared with the analytic solution which applied generally to all curves.

Both sides could agree - to some extent - about the nature of their own preference. Pure geometry was held to be natural, concrete and particular, whereas algebraic activity was artificial, abstract and general. The problem was that what was seen as a virtue by one party was seen as a vice by the other.

The same disagreement arose at about the same time in Germany, where the influential geometer, Jacob Steiner, was bitterly opposed to the algebraic techniques being brilliantly employed by his contemporary, Julius Plucker. Steiner threatened to stop contributing to a journal if it continued to publish anything by Plucker. He held that geometry stimulated thought, whereas 'calculating' replaced it. He also objected to the use of diagrams and models, and would give his lectures in darkened rooms to encourage his students to make their own mental images of whatever he was invoking.

Steiner's approach suggests that the opposing views could be seen as a disagreement about the nature of geometrical objects. Classically, these were platonic forms which were approximated to by drawn diagrams; they are often now interpreted, with Steiner, as mental images. The opposing view is that geometric objects are literally the written signs that denote point, lines, curves and so on. In the one case, a circle is the perfect shape you may conjure up in your mind, in the other it is the particular quadratic equation. And it continues to be a crucial question how these different interpretations intersect.

Meanwhile, computer programs like Cabri have enabled people to explore triangle geometry and to discover more and more possible theorems. These can be quite complex, so that if it is felt that Cabri verification is not enough, it often turns out to be easiest to thrash out an algebraic proof. Such a proof is in some sense still a verification - indeed the algebra can involve some quite complicated manipulations and these might then be handed over to some more software, the algebra program then may be said to be verifying the geometrical hypothesis. This, of course, is in a way what Cabri is already doing: the images on the screen are being calculated algebraically by the program.

Cabri continues to be a wonderful tool for generating more and more interesting theorems of triangle geometry. But, as with the proliferation of such results in the nineteenth century, the field often seems to be a collection of apparently unrelated themes . In recent years, however a small band of algebra enthusiasts, inspired by John Conway, have managed to unify many scattered themes. The main aspect of this work has been the use of barycentric coordinates, which has offered a re-interpretation of such books as Roger Johnson's *Advanced Euclidean Geometry* (Dover reprint, 1960) or - more recently - Clark Kimberling's *Triangle centers and central triangles* (Manitoba University, 1998).

I have enjoyed much of this work and have felt like re-casting my booklet in algebraic terms. I still get a special delight in landing upon a synthetic solution, but I have also come to be a bit wary of this thrill, seeing "the quick sharp spurt of the lighted match" as yet another minor addiction, perhaps like crosswords. As the chinese poem has it, "we know no harm to come of it, it may be god is pleased". Getting the algebra right is certainly also satisfying (even if at times addictive as well) but more so, I feel, is getting the structural view that it can provide.

The following pages present, without proofs, a condensed account, of some elementary triangle geometry. In particular, I introduce briefly the unifying theme of *extraversion*, and the structural richness of the many *desmic systems* of points associated with a triangle.

Final appendices provide 1) some algebraic preliminaries invoked in the main text, 2) a comparison of different proofs (geometric, algebraic, computer verification) of some well-known triangle concurrences (you take your pick .... and reflect on the nature of proof), and 3) a partly repeated review of desmic systems .

In the spirit of Steiner's wanting his students to conjure up their own images, there are no diagrams - apart from what I could not resist including on the covers.

The reader is invited to *verify* the theorems with Cabri and/or with barycentric manipulations.

Dick Tahta

SOME TRIANGLE GEOMETRY

Euler

The *medians* (the lines joining vertices to midpoints of opposite sides) are the lines [0:1:1] .... and these are concurrent at the *median point* (or centroid) M (1:1:1). The midpoints of the sides are vertices of the *medial triangle*. The triangle whose midpoints are A,B,C is the *anti-medial triangle*, with vertices (-1:1:1) ....

The *mediators* (or perpendicular bisectors of the sides) are the lines [b^2-c^2:a^2:-a^2] .... and these are concurrent at the *circumcentre* O (a^2S_1 : :), which is the centre of the *circumcircle* {0:0:0}, namely a^2yz+....=0.

The *altitudes* (lines through vertices perpendicular to opposite sides) are the lines [0:S_3:S_2] ....and these are concurrent at the orthocentre H (S_2 S_3 : :). The feet of the altitudes are vertices of the *orthic triangle*.

The midpoint of OH is the *nine-points-centre* N (a^2S_1+ $2S_2S_3$: :) which is the centre of the *medial circle* {S_1/2: : }. This circumcircle of the medial triangle, also known as the nine-points-circle or feuerbach circle, passes through the midpoints of the sides and the feet of the altitudes. It also passes through the midpoints of PH for any point P (p:q:r) on the circumcircle, for then a^2qr+ .... =0, and the midpoint (Kp+S^2S^3(p+q+r): :), where K=S_2S_3+...., lies on the medial circle. In particular, the medial circle passes through the midpoint of AH ....

The *euler line* OH [(b^2-c^2)S_1 : :] also contains M which is the (1:2) trisection point of OH.

The euler line passes through a number of other special points: for example, the point ($2a^2S_1$+S_2S_3: :) which is the nine-point centre of the medial triangle and midpoint of ON; the *de-longchamps point* (a^2S_1-s_2s_3: :) which is the orthocentre of the anti-medial triangle; and the *schiffler point* (as_1/(b+c): :), which is the point of concurrence of the euler lines of IBC .... (I being the incentre, see below).

Feuerbach

The *internal angle bisectors* are the lines [0:-c:b] .... and these are concurrent at the *incentre* I (a:b:c). This is the centre of the incircle {$s_1{}^2$: : }, also known as *a tritangent circle*, touching the three sides of ABC.

The *external angle bisectors* are the lines [0:c:b] .... and these form the *excentric triangle* with vertices at the *excentres* I_1 (-a:b:c) .... These are the centres of the tritangent excircles {s^2:$s_3{}^2$:$s_2{}^2$} ....

The medial circle touches the four tritangent circles, the point of contact with the incircle being the so-called *feuerbach point* f (S_1(b-c)^2: :).

[The pedal circle (see later) of P touches the medial circle if and only if P, its isogonal conjugate and H, are collinear. This will be the case when P is the incentre or an excentre.]

Extraversion

Note that the excentres may be described (with John Conway) as 'extra versions' of the incentre.

The *extraversion* of a point may be defined algebraically by a change of sign of one of the constants a,b,c in the coordinates of a point . For example, an a-extraversion of the incentre (a:b:c) yields the ex-centre I_1 (-a:b:c).

Moreover, the excircles are extraversions of the incircle, and they will touch the medial circle at extraversions of the feuerbach point, namely f_1 (S_1(b-c)^2:S_2(c+a)^2:S_3(a+b)^2) .... Note that in this case the S_i were unchanged, for these expressions contain only terms in a^2,b^2,c^2, so that extraversion leaves them unaffected; thus, for example, the medial circle {S_1: : } is invariant under extraversion.

Gergonne/Nagel

The incircle touches BC at X $(0:s_3:s_2)$ .... and the lines AX,BY,CZ are concurrent at the *gergonne point* g $(s_2s_3:s_3s_1:s_1s_2)$.

The excircle opposite A touches BC at $X_1(0:s_2:s_3)$, CA at Y_1 $(-s_2:0:s)$, AB at $Z_1(-s_3:s:0)$. The lines AX_1,BY_1,C_1 are concurrent at an extraverted gergonne point g_1 $(-s_2s_3:ss_2:ss_3)$. Similarly for the other extraverted gergonne points g_2,g_3.

The lines AX, BY_2,CZ_3 are concurrent at the *nagel point* n $(s_1:s_2:s_3)$.

Moreover, the lines AX,BY_3,CZ_2 concur at an a-extraverted nagel point n_1 $(-s:s_3:s_2)$, and similarly for the other extraverted nagel points n_2 (concurrence of BY,CZ_1,AX_3) and n_3 (concurrence of CZ,AX_3,BY_1).

Each nagel point is the isotomic conjugate of its corresponding gergonne point - for example, n_1 $(-s:s_3:s_2) = (-1/s_2s_3:1/ss_2:1/ss_3)$ which has the inverted coordinates of g_1.

The median point is a trisection (1:2) point of the incentre and nagel point, and, indeed, also a trisection point of each ex-centre I_i and corresponding extraverted nagel point n_i (i=1,2,3).

Desmic tetrads The gergonne and nagel points, with each of their extraversions, form two tetrads of points that are in perspective in four different ways, the perspectors being the points A,B,C and H^*, the isotomic conjugate of H. Thus (writing <A> to indicate a perspector A):

g $g_1g_2g_3$ <H^*> n $n_1n_2n_3$, and <A> n_1n n_3n_2, and <B> n_2n_3n n_1, and <C> $n_3n_2n_1$n

This configuration of 12 points and 16 lines is an example of a *desmic system.*

An alternative description is that of three triangle T_i (i=0,1,2), with vertices $A_iB_iC_i$, which are perspective in pairs, where, furthermore, the perspectors D_k $(i \neq j \neq k)$ are collinear, as are the 12 triples $A_iB_jC_k$ $(i \neq j \neq k)$.

When T_o is taken as triangle of reference , any three collinear points P_i $(x_i:y_i:z_i)$ (i=0,1,2) will define a desmic system $[P_0\ P_1\ P_2]$ with *desmon* P_0, the vertices of the other two associated triangles T_1,T_2 (known as *desmic mates)* being defined as A_1 $(x_1U:y_2:z_2)$ and A_2 $(x_2/U:y_1:z_1)$ .... where $U= (y_2z_0-y_0z_2)/(y_0z_1-y_1z_0)$, with corresponding expressions for V,W.

For example, in the case of the collinear perspectors H^* (S1: :), g $(s_2s_3: :)$ and n $(s_1: :)$, U=-1/s, and hence the vertex A_1 is $(-s_2s_3/s:s_2:s_3) = (-s_2s_3:ss_2:ss_3)$ which is g_1 ...., vertex A_2 is $(-ss_1:s_3s_1:s_1s_2) = (-s:s_3:s_2)$ namely n_1 .... , so that the desmic tetrads in this case are the desmon H^* and ABC, g and its extraversions, n and its extraversions.

Any three collinear points will define a desmic system, but the interesting ones are those that generate other known points. Some further examples of such desmic systems are:

- [M m m′] where the desmon is the medial point, and the tetrads are MABC, the mittenpoint m $(as_1: :)$ and its extraversions, the isogonal conjugate of m, namely m′ $(a/s_1: :)$, and its extraversions. (The mittenpoint is the symmedian point of the ex-centric triangle - see below.)
- [M O H] where the tetrads are MABC, $OA_1B_2C_3$ where A_1 is the midpoint of altitude through A, namely $(a^2:S_3:S_2)$ ...., and $HA_2B_2C_2$ where A_2 is the isogonal conjugate of A_1, namely $(S_2S_3:b^2S_2:c^2S_3)$ ....
- [N f ~f] where N is the centre of the medial circle, and the tetrads are NABC, the feuerbach point f and its extraversions, the harmonic conjugate of f with regard to I and n, namely ~f $(bc(b+c)^2s_2s_3: :)$, and its extraversions.
- [K_m Q R] where K_m is the symmedian of the medial triangle, namely $(b^2+c^2: :)$, Q is the point $((b-c)^2 : :)$, R is the point $((b+c)^2: :)$, and the tetrads are K_mABC, Q and its extraversions, R and its extraversions.

Spieker

The incentre of the medial triangle is the *spieker point* (b+c: :) and this is the midpoint of In, n being the nagel point of ABC, namely (s_1: :).

The incircle of the medial triangle is the *spieker circle*, namely {$s^2/4-s_2s_3$: : }, with radius half that of the incircle of ABC. It touches the sides of the medial triangle at X (a:s_2:s_3) .... , the lines AX,BY,CZ being concurrent at the nagel point n.

Note that the spieker circle may be compared to the medial circle: a spiral similarity about the medial point M (through angle π, and with stretch x1/2) transforms the medial circle into the spieker circle, the the euler line OMNH into a *spieker line* IMn, and the midpoint N of OH into the speiker point, which is the midpoint of In.

The extraversions of the spieker point are (b-c:c+a:-a+b) ...., and these three points are the midpoints ofthe corresponding I_in_i. The corresponding extraverted spieker lines have equations [b-c:c+a:-a-b] ....

The spieker point is the incentre of the medial triangle and may be labelled as I_m. In general, for any triangle centre P (x:y:z) there is a corresponding centre of the medial triangle P_m (y+z: :). Note that for any point P on the circumcircle, P_m lies on the medial circle.

Conversely, the *anti-medial triangle* (whose medial triangle is ABC) yields a corresponding P_{m-} (-x+y+z: :); for example, the point I_{m-} (-a+b+c: :) is the nagel point n.

Note that M_m= M_{m-} = M, so that the medial point is the same in each case, and is collinear with P, P_m, P_{m-} for any P. When P is the circumcentre this leads to the euler line OMNH, and when P is the incentre, to the spieker line IMI_mn.

Lemoine

The isogonal conjugate of the median through A is the *symmedian* through A, namely [0:$-c^2$:b^2] .... The three symmedians are concurrent at the *symmedian point* (or lemoine point) K (a^2:b^2:c^2).

The tangent at A to the circumcircle is the *ex-symmedian* [0:c^2:b^2] .... The three ex-symmedians intersect at K_1 ($-a^2$:b^2:c^2) .... The line AK_1 is [0:$-c^2$:b^2] .... and the three such lines passes through the symmedian point K.

The ex-symmedian through A meets BC at X (0:$-b^2$:c^2) .... and X,Y,Z are collinear on the *lemoine line* [$1/a^2$: :]. This line is the polar of K with regard to the circumcircle.

The line joining the midpoint of BC (0:1:1) and the midpoint of the altitude through A is [S_1-S_2: c^2: $-c^2$] .... and the three such lines concur at the symmedian point.

The line joining A and the midpoint of the feet of the altitudes through B and C is [$b^2S_2+c^2S_3$:$b^2(S_1+c^2)$:$c^2(S_1+b^2)$] .... and the three such lines concur at the symmedian point.

Various lines through K include the following:
- the line MK [u: :], where $u=b^2-c^2$ ....., which also passes through the isotomic conjugate of the orthocentre;
- the line IK [(b-c)/a: :], which also passes through the *mittenpoint* (aS_1: :), namely the symmedian point of the excentric triangle;
- the *van- aubel line* HK [uS_1/a^2: :], which also passes through the symmedian point of the orthic triangle;
- the *tucker line* OK [u/a^2: :] (also known as the brocard diameter - see below), which also passes through a number of other special points such as the isodynamic points (defined below), the isogonal conjugate of the nagel point, and the orthocentre of the orthic triangle.

Brocard

The circle touching BC at A, and passing through B is the *adjoint circle* (12) with equation $\{0:0: a^2\}$, and similarly for other adjoint circles (ij), i,j=1,2,3.

This and the two corresponding circles (23), (31) meet at the *first brocard point* W $(1/b^2:1/c^2:1/a^2)$. It follows that the angles WBC,WCA,WAB are equal and that the brocard point may also be defined in terms of this *brocard angle* ω (where $\cot\omega=\cot A + ...$).

The other three adjoint circles, namely (13),(21),(3 2) meet at the *second brocard point* W' $(1/c^2:1/a^2:1/b^2)$, which is the isogonal conjugate of W. This may also be defined in terms of a brocard angle ω' equal to angles $W'CB$, $W'BA$, $W'AC$.

The *first brocard triangle* is $A_1B_1C_1$, where A_1 is the intersection of BW and CW', namely $(a^2: c^2: b^2)$ .... The point A_1 lies on the mediator of BC, namely $[b^2-c^2:a^2:-a^2]$, and it is also on the line $[-b^2+c^2: a^2: a^2]$ which passes through the symmedian point K and is parallel to BC .... The median point of the first brocard triangle is M.

The line AA_1 $[0:b^2:c^2]$ passes through the isotome of K, namely $(1/a^2: :)$ so that the first brocard triangle is in perspective with ABC.

The *second brocard triangle* is $A_2B_2C_2$ where A_2 is the intersection of the adjoint circles (21), (31), namely the point $(2S_1: b^2: c^2)$ ....

The line A_1A_2 $[b^2-c^2:a^2-b^2:c^2-a^2]$ passes through (1: :) .... so that the two brocard triangles are in perspective, the perspector being the median point of ABC.

The common circumcircle of the two brocard triangles is the *brocard circle* $\{b^2c^2/2S : : \}$, namely $b^2c^2x^2-2a^2Syz + ... =0$. This circle also passes through the brocard points W, W', the symmedian point K and the circumcentre O.

The centre of the brocard circle is the midpoint of the *brocard diameter* OK $[u/a^2: :]$, where $u=b^2-c^2$ .....

Steiner/Tarry

The line through A, parallel to the side B_1C_1 of the first brocard triangle is [0:-v:w] .... and the three such lines are concurrent at the *steiner point* (1/u:1/v:1/w), where $u=b^2-c^2$ ....

The medians of the first brocard triangle, namely [u:w:v] meet BC in X_1 (0:-v:w), X_2 (0:-w:u), X_3 (0:-u:v) .... The nine such points lie in threes on three lines, namely $X_1Y_3Z_2$ [1/u:1/v:1/w] .... which form a triangle with vertices (1/u: :) .... the first of these being the steiner point.

The image of A in the mediator of BC is P $(a^2:-u:u)$ .... , and the line QR meets BC at X (0:w:v) .... The lines AX,BY,CZ are concurrent at the steiner point.

The perpendicular from A to the side B_1C_1 of the first brocard triangle is [0:-V:W] where $U=b^2w-c^2v$ .... and the three such lines are concurrent at the *tarry point* T (1/U:1/V:1/W), this being diametrically opposite the steiner point.

The perpendiculars from the tarry point to the sides of ABC meet BC at X_1 (0:-vW:wV), X_2 (0:-uW:vV), X_3 (0:-wW:uV) .... The nine such points lie in threes on three lines, one of which is the pedal line of the tarry point.

Apollonius

The circle through A and the points X (0:b:c) and X′ (0:b:-c), where the angle bisectors at A meet BC, is the *apollonius circle* $\{0:-b^2k:c^2k\}$ where $k=a^2b^2c^2/(b^2-c^2)$ ....

The centre of the apollonius circle through A is L $(0:b^2:-c^2)$ .... and the tangent at A to the circumcircle is LA ...., so that each apollonius circle is orthogonal to the circumcircle. The three centres L,M,N lie on the lemoine line $[1/a^2:1/b^2:1/c^2]$.

The three apollonius circles are coaxal, through the *isodynamic* points U_+ $(a^2S_1+a^2f: :)$ and U_- $(a^2S_1-a^2f: :)$, where $f= 2\Delta/\sqrt{3}$, which lie on the brocard diameter OK

Fermat

The vertex of an (outer) equilateral triangle, which is externally constructed on BC, is the point X $(-a^2: S_3+f:S_2+f)$, where $f = 2\Delta/\sqrt{3}$ as above, ... The lines AX,BY,CZ are concurrent at the outer *isogonic point* (also known as the fermat point) F_+ $(1/(S_1+f): :)$. The corresponding point when the triangles are internally constructed is F_- $(1/(S_1-f): :)$. The isogonic points are isogonal conjugates of the isodynamic points.

The circumcircle of the outer equilateral triangle BCX is $\{S_1+f:0:0\}$.... and the three such circles pass through the fermat point F_+ The centres of the circumcircles are the points $(-a^2:S_3+3f:S_2+3f)$ .... and these form an outer *Napoleon triangle*, which is in perspective with ABC, the perspector being the outer *napoleon point* N_+ $(1/(S1+3f: :)$. The inner triangle constructions yield corresponding results.

Miquel/Wallace

Three points on the sides of ABC, say X $(0:u:u')$, where $u' =1-u$, and corresponding coordinates for Y, Z, determine three *miquel circles* AYZ $\{0:c^2w:b^2v'\}$.... These are concurrent at the *miquel point* of the triple XYZ, namely $(-a^4uu'+a^2b^2u'cc^2uw: :)$.

When X,Y,Z are collinear, $uvw+u'v'w'=0$ and the miquel point lies on the circumcircle. It also lies on a circle through the circumcentre and the centres of the miquel circles.

Taking a slightly different point of view, consider the quadrangle BCYZ. The midpoints of the diagonals of this quadrangle, namely AX,BY,CZ, are $(1:u:u')$ .... and when X,Y,Z are collinear these midpoints are collinear (a result ascribed to Newton). The orthocentres of AYZ .... are collinear with the orthocentre H of ABC.

The foot of the perpendicular to BC from a point P (p:q:r) is X $(0:u:u')$ where $u=(a^2q+S_3p)/a^2(p+q+r)$ .... The triangle XYZ is the *pedal triangle* of P, which is the miquel point of the vertices of its pedal triangle.

When P lies on the circumcircle, so that $a^2qr+ .... =0$, then X,Y,Z are collinear and the pedal triangle collapses into the *pedal line* (or wallace/simson line).

The pedal line of P is YZ $[-vw':vw: v'w']$, or the two equivalent forms for ZX,XY. This passes through the midpoint of PH, namely $(Kp+S_2S_3(p+q+r): :)$, where $K=S_2S_3+....$, (which lies on the medial circle). Note that the pedal line of A is the altitude through A.

Tucker

When P is not on the circumcircle of ABC, the *pedal circle* of P is the circumcircle of its pedal triangle. This has a complicated equation, namely {l:m:n} where $l=K.vwp/a^2$, with $K=4\Delta^2/(uvw+u'v'w')(p+q+r$ and u,v,w as above .... This is also the pedal circle of the isogonal conjugate of P - its centre being the midpoint of PP'.

A spiral similarity about P, through angle θ and with stretch secθ, takes the pedal trangle to a θ-pedal triangle, its circumcircle being a θ-pedal circle. In particular, when P is the Brocard point W, the θ-pedal circle is known as a *tucker circle*, with centre T on OK, (where angle TWW'=θ) and radius Rsinω.secθ (ω being the Brocard angle).

Some example of tucker circles are:

- the circumcircle, with $\theta=\pi/2-\omega$ and centre O;

- the *first lemoine circle*, $\theta=-2\omega$ and centre at midpoint of OK, is $\{b^2c^2(b^2+c^2)/4S^2: : \}$; this meets BC in points X_1,X_2 ...., and Y_1Z_2 is parallel to BC ....;

- the *second lemoine circle*, $\theta=-\omega$ and centre K; this meets BC in points X_1,X_2 ...., and in this case Y_1Z_2 is anti-parallel to BC ....

APPENDIX 1 algebraic preliminaries

Triangle notation

For a triangle ABC with sides a,b,c, set $s=(a+b+c)/2$ and $s_1=(-a+b+c)/2$, with corresponding expressions for s_2,s_3. Thus, for instance, Heron's formula for the square of the area of a triangle is $ss_1s_2s_3$.

Also set $S=(a^2+b^2+c^2)/2$ and $S_1=(-a^2+b^2+c^2)/2$, with corresponding expressions for S_2, S_3, so that, for instance, $\cos A= S_1/bc$, and so on. Note the following useful identities: $a^2S_1/2 + ... = S_2.S_3 + ... = a^2 S_1 + S_2.S_3 = ... = b^2c^2/2 - a^4/4 + ...$ (where the dots indicate corresponding terms), and these are each $4\Delta^2$, namely four times the square of the area of triangle ABC.

Barycentrics

Point coordinates A point in the plane of a triangle of reference ABC may be represented by point coordinates (x:y:z) which are homogenous in the sense that (kx:ky:kz)=(x:y:z). The coordinates of a point P may be calculated as x=areaPBC ..... (where the dots indicate symmetrically corresponding y,z coordinates). The coordinates are said to be *normalised* when x+y+z=1; thus the normalised coordinates of the vertices are A(1:0:0) .... (where the dots indicate corresponding expressions for B,C).

To find the point on a segment P_1P_2, where P_i is $(x_i:y_i:z_i)$, $i=1,2$, dividing it in the ratio k:1-k, it is more convenient to start with the coordinates P_i in normalised form, in which case the required point is $(k.x_1 + (1-k).x_2 : :)$, where the repeated colons represent the symmetrically corresponding y,z coordinates. Thus the midpoint of BC, say X, is (0:1/2:1/2), or equivalently (0:1:1), and the point dividing the segment AX in the ratio 2:1 is (1/3: :) which is (1: :).

The line AP, where P is the point (p:q:r), meets BC at X (0:q:r) = (0:bh:ck) where h,k are the perpendicular distances of X from AC,AB. Similarly, the reflection of AP in the angle bisector through A meets BC at X′ (0:bh′:ck′). But hh′=kk′ (by the construction of X′) so that X′ is (0:bk:ch) = $(0:b^2/q:c^2/r)$. Hence AX′, BY′,CZ′ are concurrent at the *isogonal conjugate* of P, namely P′ $(a^2/p:b^2/q:c^2/r)$, with equal angles BAP,CAP′ ....

The line AP, where P is the point (p:q:r) meets BC at X (0:q:r); so that a point X^* on BC such that CX^* =BX will be (0:r:q) .... (the dots again indicating similat expressions for Y^*, Z^*). Then AX^*,BY^*,CZ^* are concurrent at the *isotomic conjugate* of P, namely P^* (1/p:1/q:1/r).

Line coordinates A line lx+my+nz=0 may be represented by homogeous line coordinates [l:m:n], the square brackets distinguishing these from point coordinates. For example, the sides of the triangle of reference ABC are BC[1:0:0] ....

The line through the point (u:v:w) parallel to the line [l:m:n] is [l(v+w)-mv-nw : :].

The line through the point (u:v:w) perpendicular to the line [l:m:n] is $[l(S_3.v-S_2w-m(b^2v+S_1w)+n(S_1v+c^2w) : :]$.

Circle coordinates A circle will have an equation of the form $(px+qy+rz)(x+y+z) = k(a^2yz+b^2zx+c^2xy)$. This circle may be represented by circle cooordinates {p:q:r}, the curly brackets distinguishing these from other coordinates. These circle coordinates are not homogenous and the representation will be always taken to be in normalised form with k=1.

The centre of the circle is $(a^2S_1-a^2p+S_3q+S_2r : :)$. Thus the circumcircle of ABC is {0:0:0} with centre $(a^2S_1: :)$.

APPENDIX 2: proofs compared

Median point

1: Medians AX, BY intersect at M, BC is parallel to and twice XY; so that from similar triangles MBC, MXY, it follows that BX,CY are twice MX,MY, so that M is a (2:1) trisection point of both medians AX, BY. Similarly for medians AX,CZ; hence the three medians concur at the common trisection point M.

2: The midpoint of BC is X (0:1:1), the median AX is [0:1:-1] .... and the three such lines concur at M (1:1:1).

3: Plot three points, and the join of each point to the midpoint of other two. These three lines will be concurrent, and continue so be so for any variation of any of the initial points

Circumcentre

1: The mediators of AB, AC intersect at O so that OB=OA=OC, so that O lies on the mediator of BC; hence the three mediators concur at O.

2: The perpendicular to BC through the midpoint of BC is the mediator [b^2-c^2: a^2:-a^2] .... and the three such lines concur at O (a^2S_1:b^2S_2:c^2S_3).

3: Plot three points and the mediators of each pair. These three lines will be concurrent and continue to be so for any variation of the initial points.

Orthocentre

1: The altitudes BY,CZ intersect at H and AH meets BC at X, so that AYHZ, BCYZ are each concylic and angles BAX = HYZ = 90°-AYZ = 90°-B; hence AXC = ABC+BAX = 90°, so that AX is an altitude, ie the three altitudes concur at H.

2: The perpendicular through A to BC is [0:-S_2:S_3].... and the three such lines concur at H [1/S_1:1/S_2:1/S_3).

3: Plot three points and the perpendiculars through each point to the join of the other two. These three lines will be concurrent and continue to be so for any variation of the initial points.

Incentre

1: The angle bisectors through B,C meet at I, which is equidistant from BA, BC, and also from CA,CB, and so from all three sides; hence the three angle bisectors concur at I.

2: The angle bisector through A is [0:-c:b] .... and the three such lines concur at I (a:b:c).

3: Plot three points and through each plot the bisector of the angle between the joins of this point with each of the other two. These three lines will be concurrent and continue to be so for any variation of the initial points.

APPENDIX 3: desmic systems review

A *desmic system* consists of three triads of points $A_iB_iC_iD_i$ (i=0,1,2) such that any two triads are in perspective in four different ways, the perspectors being the points of the third triad. Thus (writing <P> to indicate a perspector P)

$A_1B_1C_1D_1$ $\langle D_0\rangle$ $A_2B_2C_2D_2$ $\langle C_0\rangle$ $B_2A_2D_2C_2$ $\langle B_0\rangle$ $C_2D_2A_2B_2$ $\langle A_0\rangle$ $D_2C_2B_2A_1$.

An alternative description is that of three triangle T_i (i=0,1,2), with vertices $A_iB_iC_i$, which are perspective in pairs, and where, furthermore, the perspectors D_k $(i \neq j \neq k)$ are collinear, as are the 12 triples $A_iB_jC_k$ $(i \neq j \neq k)$.

When T_o is taken as triangle of reference, any three collinear points D_i $(x_i:y_i:z_i)$ (i=0,1,2) will define a desmic system [D_0 D_1 D_2], with the vertices of the other two associated triangles, or *desmic mates*, T_i being defined as $A_i = (x_iX_i:y_jX_j:z_jX_j)$,where $X_i=y_jz_k-y_kz_j$, and similarly for B_i,C_i. The first perspector D_o is known as the desmon.

These coordinates may be *standardised* by setting $x_i=u_i/X_0$, $y_i=v_i/Y_0$, $z_i=w_i/Z_0$. Since the perspectors are collinear, $x_oX_o+x_1X_1+x_2X_2=0$; it follows that $-x_0=u_1+u_2$, and similarly for y_o,z_o. Hence the standard form with D_i $(u_i:v_i:w_i)$ the associated triangles A_i $(u_i:v_j:w_j)$ ...., and where the desmon is the point D_o $(u_i+u_2:v_1+v_2:w_1+w_2)$.

A general example of a desmic system is to be found in the intersections of the sides of a hexagon that touch a conic. Let the sides be x,y´,z,x´,y, z´ where the opposite sides x,x´, meet at A_0, and similarly for B_0,C_0. Let the adjacent sides y´,z meet at A_1, and the sides y,z´ meet at A_2, and similarly. Three collinear brianchon points are defined as follows: D_0 is the meet of A_1A_2,B_1B_2,C_1C_2; D_1 is the meet of AA_1,BB_1,CC_1; and D_2 is the meet of AA_2, BB_2, CC_2. Note the special case when the underlying conic is a point-pair.

There is also a dual construction, namely of joins of the vertices of a hexagon lying on a conic. In this case the desmic system consists of four triads of lines, any two being in perspective in four different ways.

A particular example of a desmic system of points can be found in the intersections of pairs of lines through each vertex of a triangle ABC where these make equal angles with the sides meeting at each vertex. With the intersections labelled as shown, angle BAC_2= angle CAB_2 ...., and the sides of the hexagon $A_1B_2C_1A_2B_1C_2$ touch a conic, so that ABC and the triangles $A_iB_iC_i$ (i=1,2) form a desmic system. Note that the vertices of these triangles may lie inside or outside triangle ABC. The equal angle condition is satisfied in the case when the lines through each vertex are the trisectors of the angle at that point.

Any triangle $A_1B_1C_1$ in perspective with triangle ABC has a desmic mate $A_2B_2C_2$ where A_2 is the intersection of BC_1, B_1C .... In this case, the three triangles are perspective in pairs and form a desmic system with the three perspectors. For example, the desmic mate of the midpoints of the altitudes of ABC is the triangle whose vertices are the isogonal conjugates of the midpoints; the perspectors in this case being M,O,H.

Any three collinear points will define a desmic system, but the interesting ones are those that generate other known points. Some examples of such desmic systems are:

- [M m m′] where the desmon is the medial point, and the tetrads are MABC, the mittenpoint m $(as_1: :)$ and its extraversions, the isogonal conjugate of m, namely m′ $(a/s_1: :)$, and its extraversions.
- [M O H] where the tetrads are MABC, $OA_1B_2C_3$ where A_1 is the midpoint of altitude through A, namely $(a^2:S_3:S_2)$ ...., and $HA_2B_2C_2$ where A_2 is the isogonal conjugate of A_1, namely $(S_2S_3:b^2S_2:c^2S_3)$ ....
- [N f ~f] where N is the centre of the medial circle, and the tetrads are NABC, the feuerbach point f and its extraversions, the harmonic conjugate of f with regard to I and n, namely ~f $(bc(b+c)^2s_2s_3: :)$, and its extraversions.

Transformations

A harmonic transformation may be defined algebraically by a change of sign of one of the three coordinates: thus the *a-harmonian* will change the sign of the first cooordinate. Denoting this by by a prefix 1, the a-harmonian of a point P (x:y:z) is $_1$P (-x:y:z), with corresponding coordinates for the b,c-harmonians.

The *harmonic triangle* ~T of a triangle T (PQR) is the triangle with vertices $_1$P, $_2$Q, $_3$R, which are the a,b,c- harmonians of the vertices P,Q,R respectively.

Note that the a-harmonian of the a-extraversion of P, say $(-x_1:y_1:z_1)$, is the same as the a-extraversion of the a-harmonian of P.

There is a corresponding geometric transformation: with AP meeting BC at X, and similarly, then $_1$P is the meet of YZ,BC, or the harmonic conjugate of X with regard to B,C. The points $_1$P, $_2$P, $_3$P are collinear on what is sometimes called the *polar line* of P.

Any desmic system has a harmonic image with the desmon replaced by its harmonic conjugate with regard to D_1, D_2. In standardised coordinates this point, known as the *harmon*, is ~D_0 $(u_1-u_2:v_1-v_2:w_1-w_2)$, so that the harmonic system changes the sign of the coordinates of D_2, and the associated triangles ~T_i are then the harmonians of the original ones.

For example, when the collinear perspectors are the isotome of the orthocentre, the gergonne point g and the nagel point n, the standardised coordinates of the associated triangles are A_1 $(-s_2s_3:-s_3s_1:-s_1s_2)$ .... and A_2 $(ss_1:ss_2:ss_3)$ .... , so that the triangles are $g_1g_2g_3$ and $n_1n_2n_3$, namely the extraversions of g and n. Here the desmon is H* $(S_1: :) = (ss_1-s_2s_3: :)$, so that the harmon is $(ss_1+s_2s_3: :)$ which is (bc: :), namely the isotome of the incentre I. The associated triangles in this case are the harmonians of the extraversions.

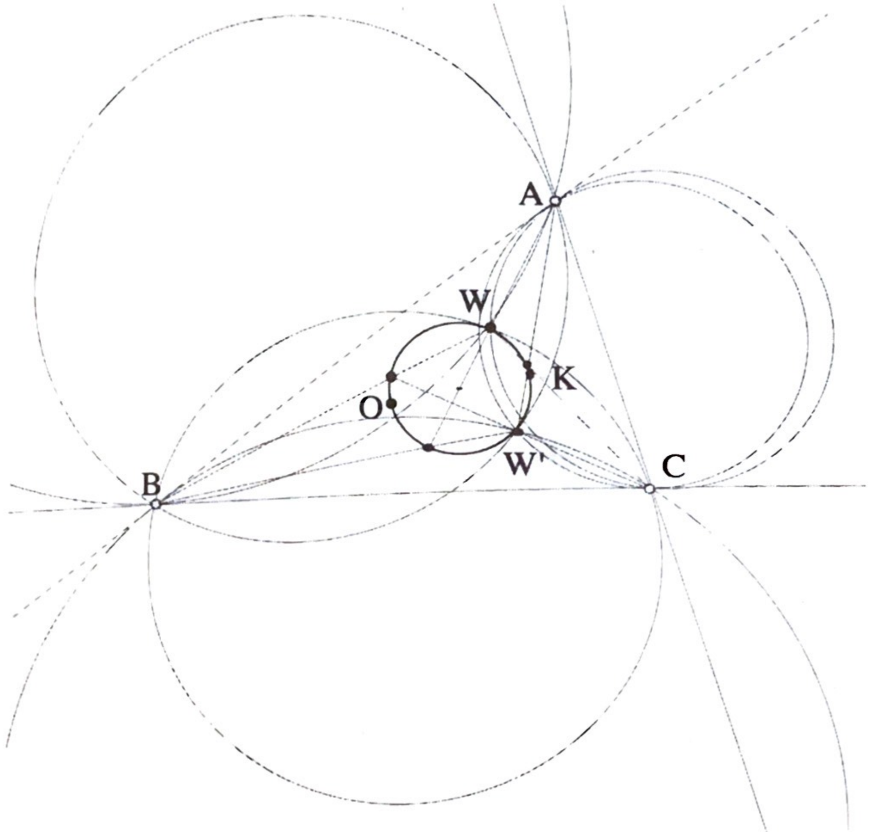
Another transformation of a desmic system maps each P(x:y:z) to its image (ax:by:cz); this may be seen as a multiplication by the incentre I and so represented by I.P. The transformation is linearity-preserving in that collinear points P,Q,R transform into collinear images I.P,I.Q,I.R. Hence the *I-product* of a desmic system <D_i> will be a desmic system <I.D_i > .

Note, however, that the associated triangles of the latter system will be the harmonians of I.D_1 and I.D_2. For example, the I product of the desmic system <H* n g> will be <I.H* m m´> where m = I.n = $(as_1: :)$ namely the mittenpoint, and m´= I.g = $(as_2s_3: :)$ namely the isogonal conjugate of m. The associated triangles are now the harmonians of the extraversions of n and g.

The harmonic and I-product transformations commute and, moreover, when the original associated triangles are extraversions of D_1 and D_2, then the product of the two transformations will yield a new desmic system whose associated triangles are also extraversions.

For example, the desmic system <H* n g> whose associated triangles are the extraversions of n,g yields a desmic system <M m m´> with T_i being the extraversions of m, m´; furthermore, another application of the double transformation yields the further system <O g´ n´>.

Many important desmic systems have associated triangles that are extraversions, like those already mentioned. When the associated triangles are extraversions, the associated vertex A_1 $(u_1:v_2:w_2)$ will be the a-extraversion, say α, of D_1 $(u_1: :)$ .... so that $\alpha u_1=k.u_1$, $\alpha v_1=k.v_2$, $\alpha w_1=k.\ w_2$, for some factor k. Then, $u_1 = \alpha(\alpha((u_1)) = \alpha(k.u_1) = (\alpha k).\ \alpha(u_1) = (\alpha k).k.u_1$ so that $\alpha k=1/k$. Hence k must be 1 and $\alpha u_1=u_1$. Since u_1 is unchanged by a-extraversion it can contain no odd powers of a. Similarly for the other standardised coordinates, so that this is a necessary condition for the associated triangles to be extraversions. [??? ... is this argument valid?]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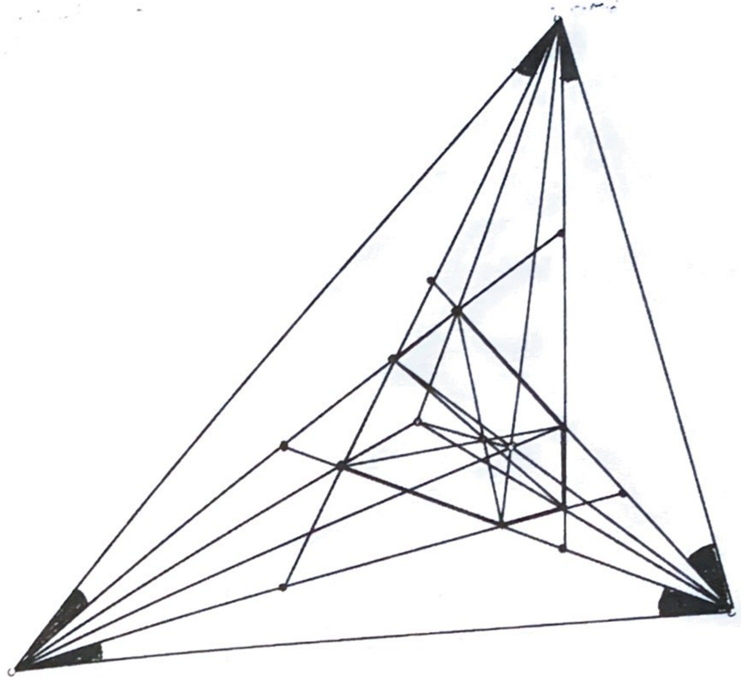

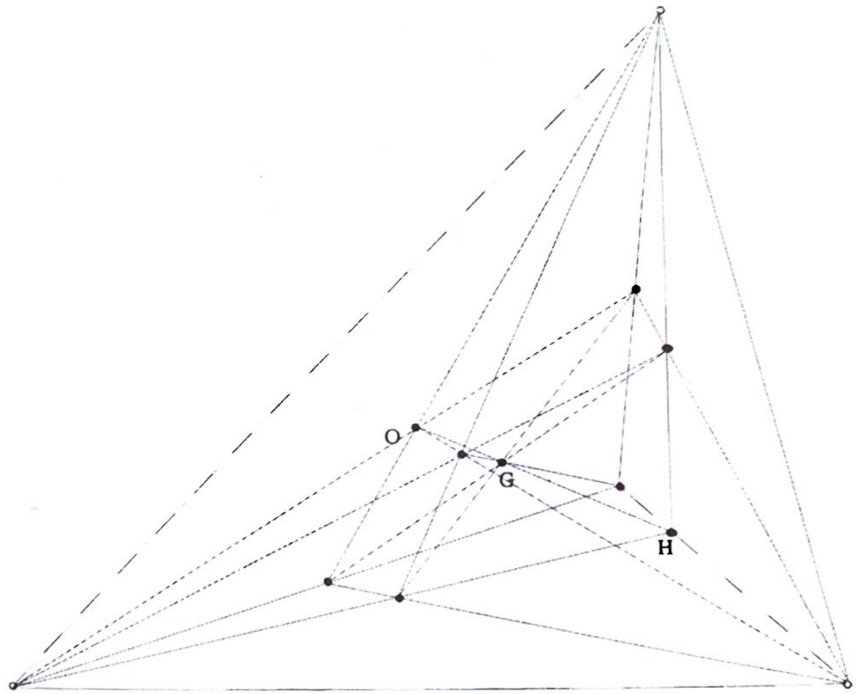
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