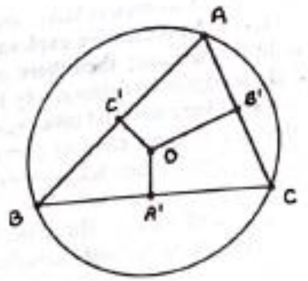
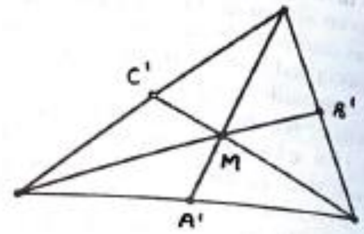
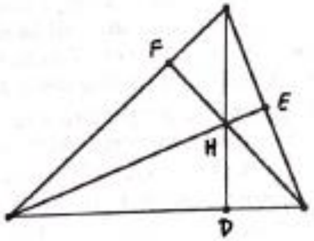
Excerpts from:

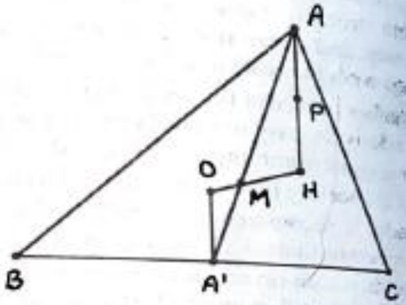
Some Triangle Geometry

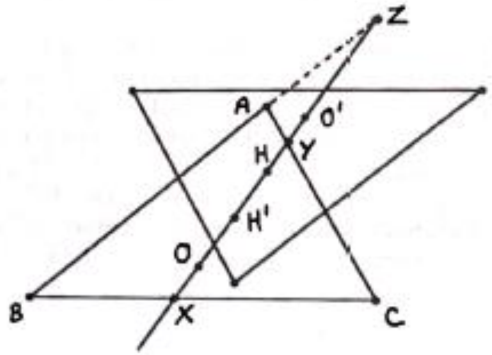
Dick Tahta

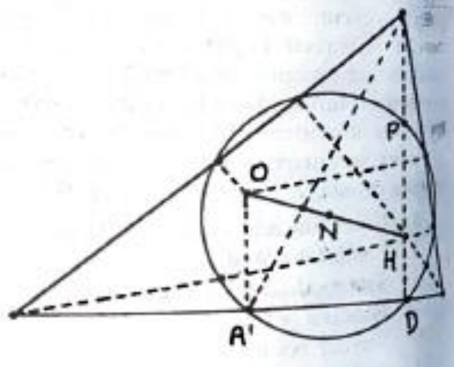












1: Euler line and medial circle

• The perpendicular bisectors of the sides of a triangle are concurrent at a point which is equidistant from the vertices.

The perpendicular bisectors are also called mediators; they concur at a point O, called the circumcentre, which is the centre of the circumcircle through the vertices of the triangle.

• The lines joining vertices to midpoints of opposite sides are concurrent at a point that trisects the lines.

The lines are called medians and they concur at a point M called the median point. The median point is the centre of gravity of equal weights at the vertices and is often also called the centroid.

• The perpendiculars from the vertices of a triangle to the opposite sides are concurrent, at a point which is isogonal to the circumcentre.

The perpendiculars are called altitudes; they concur at a point H called the orthocentre.

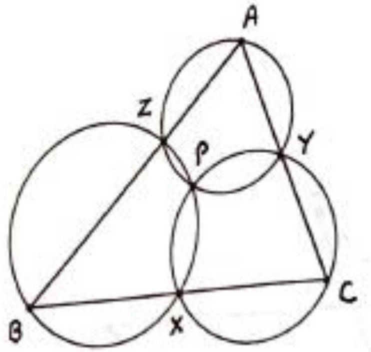
• The circumcentre O, the median point M, and the orthocentre H, are collinear, with M trisecting OH.

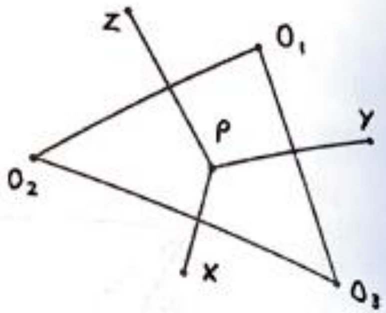
The line OMH is called the Euler line.

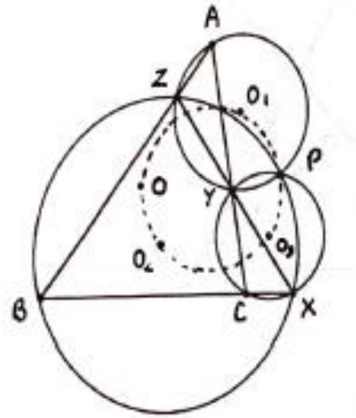
• The Euler line meets the sides of the triangle in points X, Y, Z: the Euler lines of triangles AYZ, BZX, CXY form a triangle congruent to ABC and with the same Euler line.

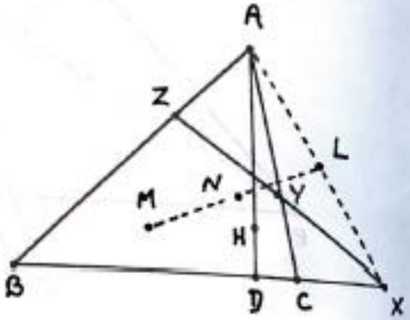
• The midpoint of OH is the centre of the circle through the midpoints of the sides; this circle also passes through the feet of the altitudes and the midpoints of the lines joining H to the vertices.

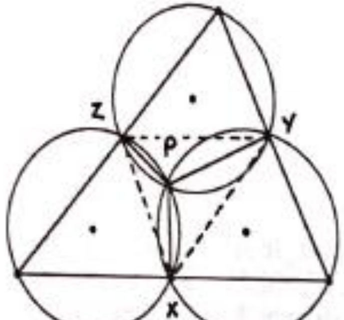
The circle through the midpoints of the sides is called the medial circle, or the nine-points circle after the nine points it passes through, as described in the previous paragraph.

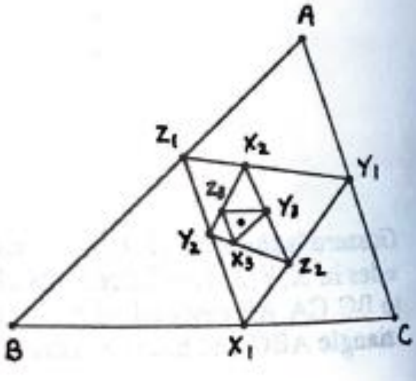












2: Miquel circles

• With any three points X, Y, Z on the sides of the triangle ABC, the circles AYZ, BZX, CXY meet in a point.

The circle are called the Miquel circles for the triple X, Y, Z, and the point common to them is called the Miquel point.

• The centres of a set of Miquel circles form a triangle that is similar to the original triangle.

• The Miquel point of a collinear triple of points lies on the circumcircle.

• The centres of the Miquel circles of a collinear triple and the circumcentre of the triangle lie on a further circle also passing through the Miqel point.

The previous result can be restated as follows: four lines form four triangles whose circumcircles meet at a point, and the circumcentres lie on the a circle also passing through the point. This can then be generalised: five lines form ten triangles whose circumcentres meet in fours at five points which lie on a circle.

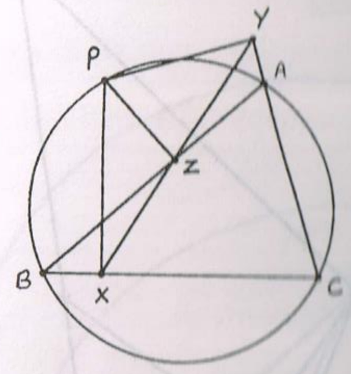
• The circumcentres of the four triangles formed by four lines are concyclic, and the orthocentres are collinear.

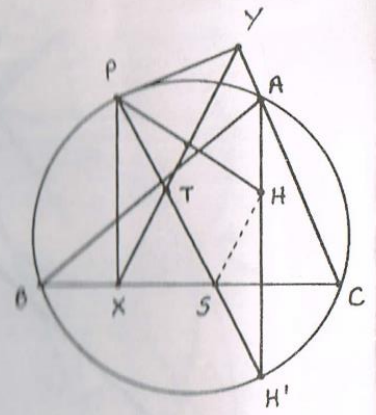
• The midpoints of the diagonals of a quadrilateral are collinear.

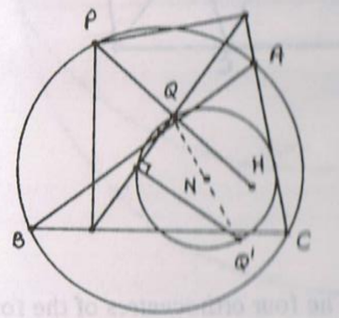
• Any point P is the Miquel point of the triple formed by the feet of the perpendiculars from a point to the sides of a triangle; the Miquel circles in this case have diameters PA, PB, PC.

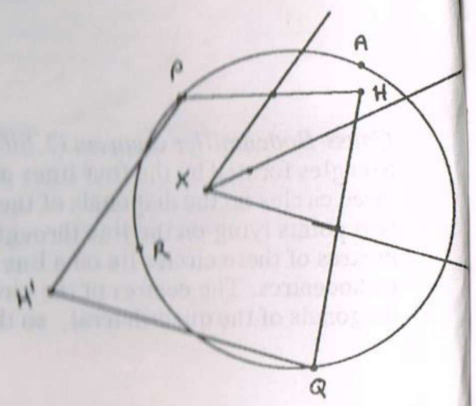
The triangle formed by the triple is known as the pedal triangle of the point P with regard to the triangle ABC. For example, the medial circle is the circumcircle of the pedal triangle of the orthocentre as well as the pedal triangle of the circumcentre.

• The pedal triangle of the pedal triangle of the pedal triangle of a point is similar to the original triangle.









3: Pedal (Simson) lines

• The feet of the perpendiculars from a point P on the circumcircle are collinear, and conversely,

> *In this case, the pedal triangle of P collapses into a straight line called the pedal line of P with regard to the triangle ABC. The pedal line is also often called the Simson line, through an erroneous ascription, and sometimes the Wallace line after the actual discoverer.*

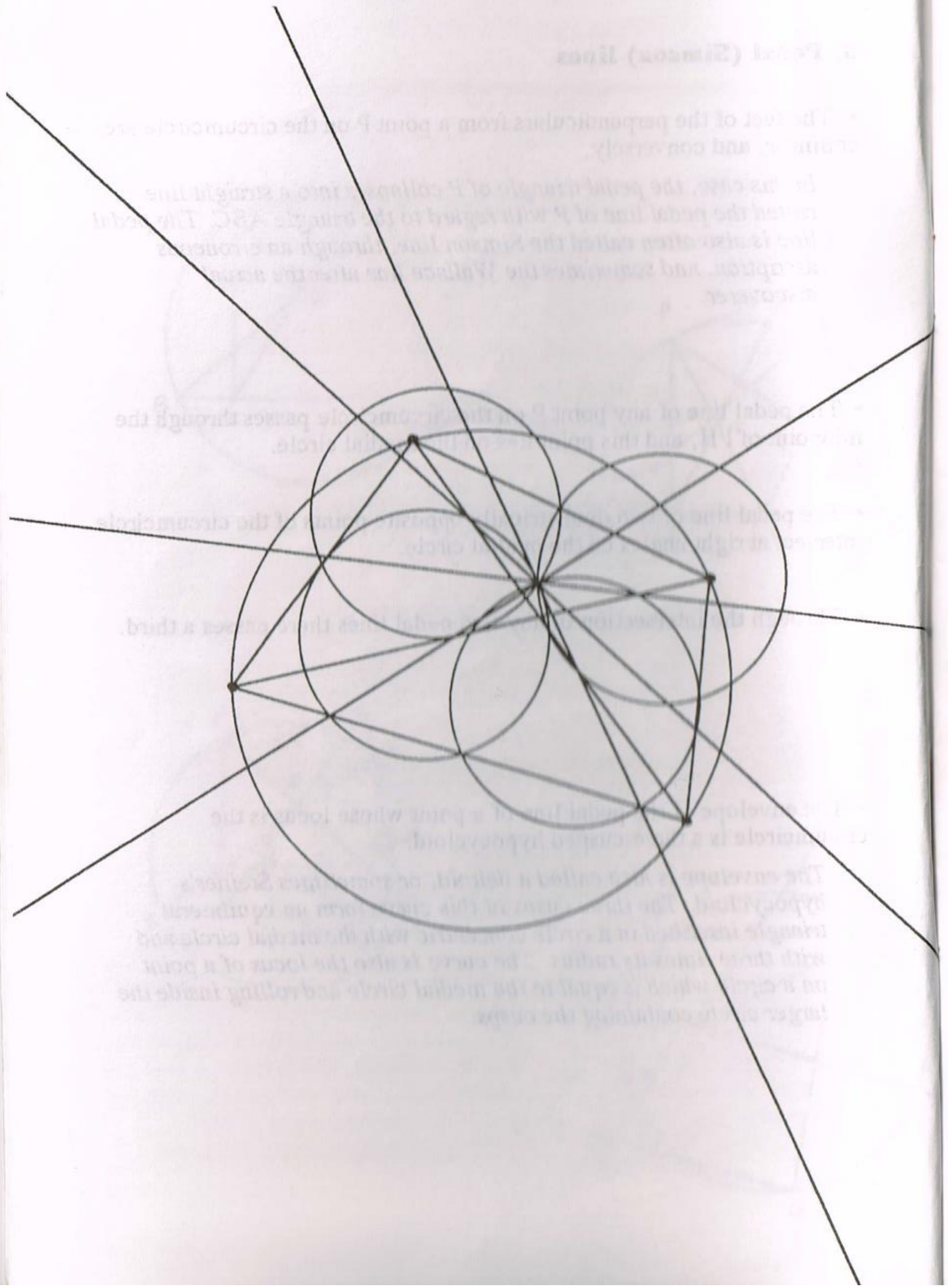
• The pedal line of any point P on the circumcircle passes through the midpoint of PH, and this point lies on the medial circle.

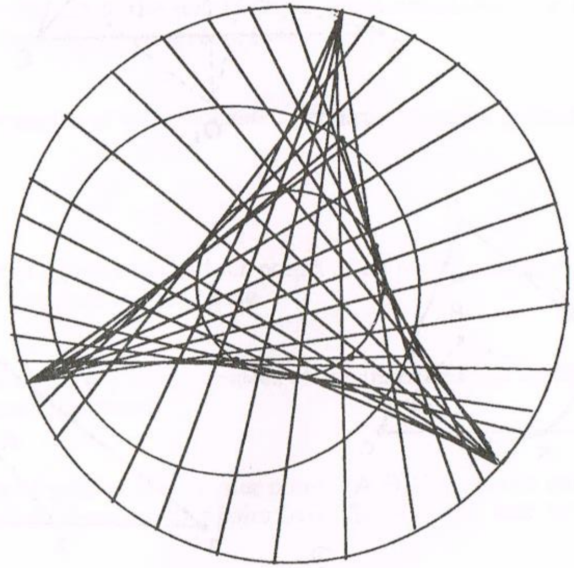
• The pedal line of two diametrically opposite points of the circumcircle intersect at right angles on the medial circle.

• Through the intersection of any two pedal lines there passes a third.

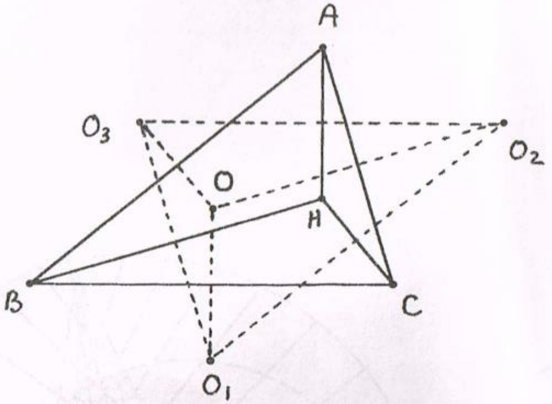
• The envelope of the pedal line of a point whose locus is the circumcircle is a three-cusped hypocycloid.

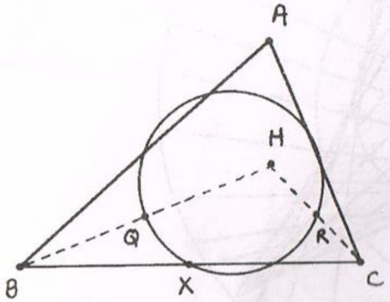
> *The envelope is also called a deltoid, or sometimes Steiner's hypocycloid. The three cusps of this curve form an equilateral triangle inscribed in a circle concentric with the medial circle and with three times its radius. The curve is also the locus of a point on a circle which is equal to the medial circle and rolling inside the larger circle containing the cusps.*

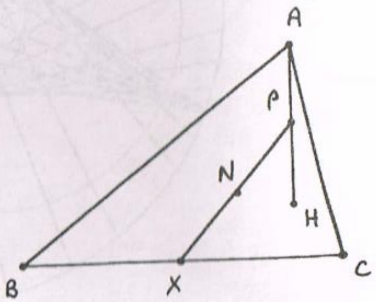


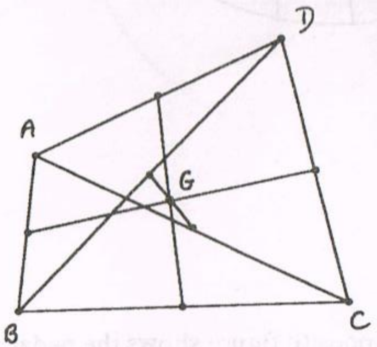


Some pedal line properties : the opposite figure shows the pedal lines of each of four concyclic points with regard to the triangle formed by the other three - these are collinear at the common point of the four medial circles. The above figure shows the envelope (Steiner's hypocycloid) of the pedal line of a point on the circumcircle.









4: Orthocentric sets

• Each vertex of a triangle is the orthocentre of the triangle formed by H and the other two vertices.

> *A set of four points each the orthocentre of the other three is known as an orthocentric set.*

• The circumcentres of the four triangles of an orthocentric set form another orthocentric set congruent to the first.

• The four triangles of an orthocentric set have the same medial circle.

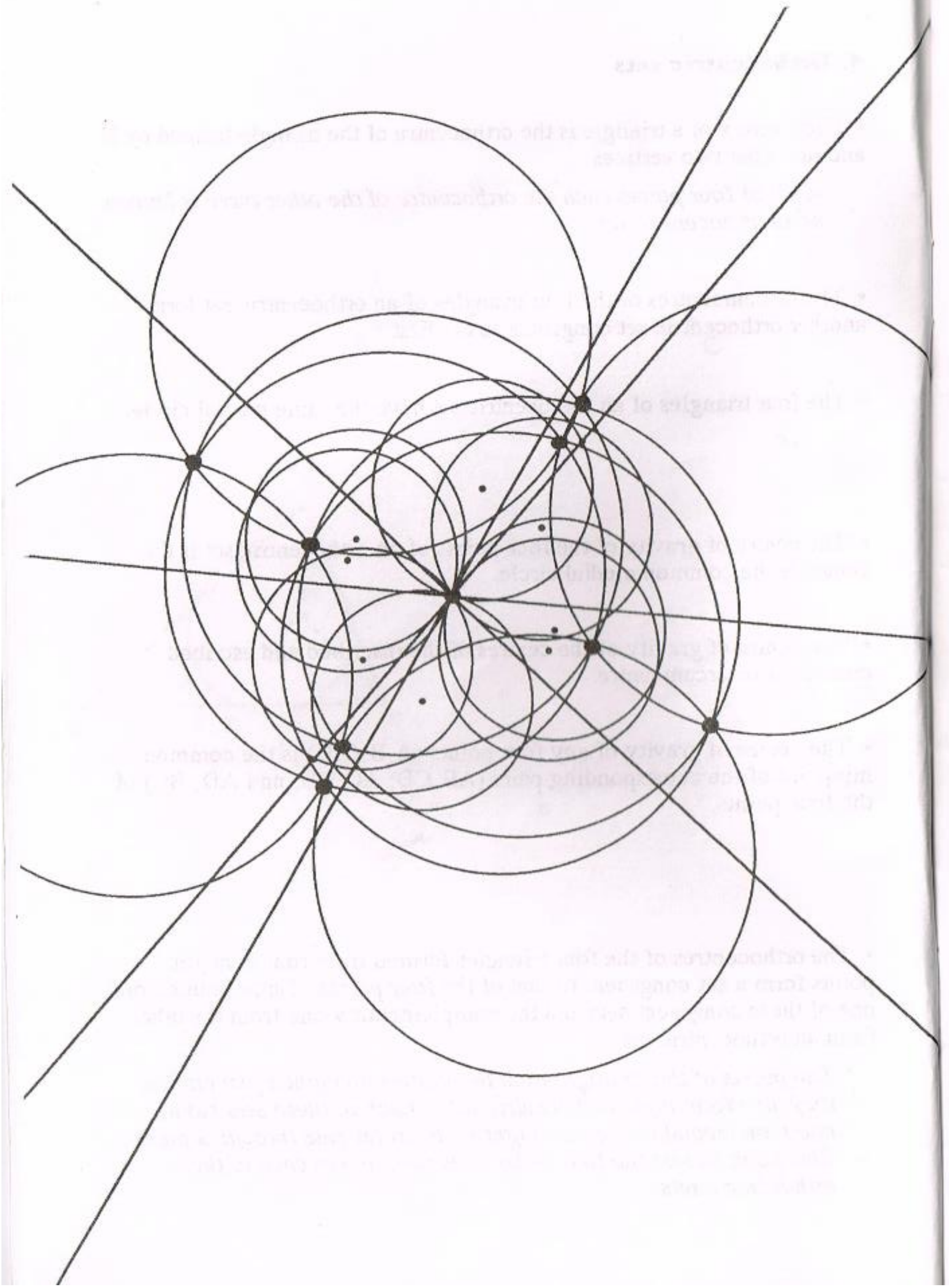
• The centre of gravity of the four points of an orthocentric set is the centre of the common medial circle.

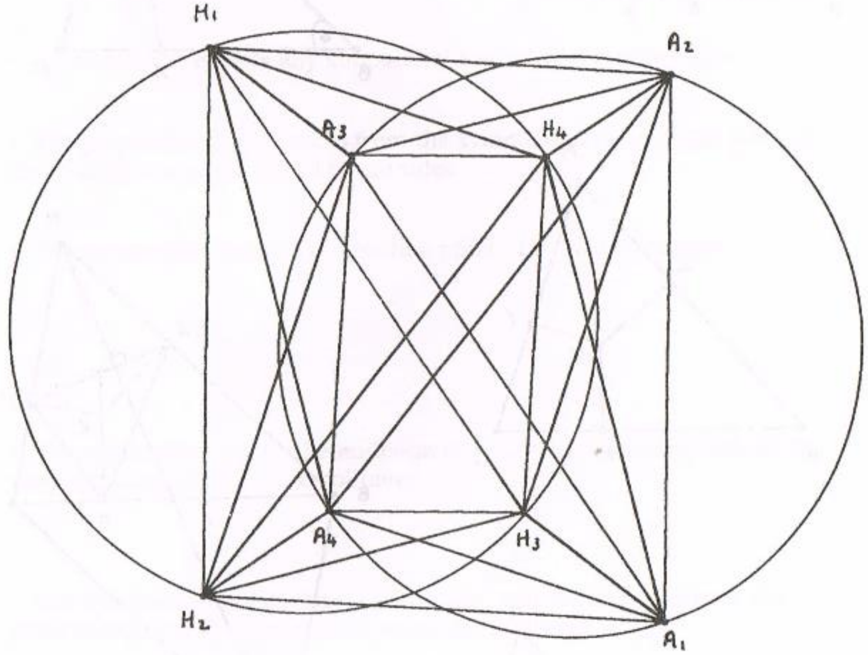
• The centre of gravity of the centres of the inscribed and escribed circles is the circumcentre.

• The centre of gravity of any four points (A,B,C,D) is the common midpoint of the corresponding pairs (AB,CD; AC,BD; and AD, BC) of the four points.

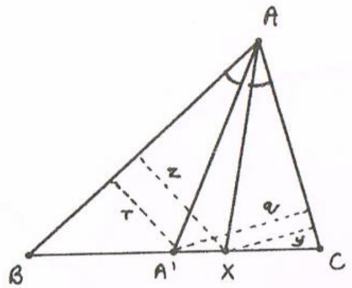
• The orthocentres of the four triangles formed from four concylic points form a set congruent to that of the four points. Three points from one of these congruent sets and the complementary one from the other form an orthocentric set.

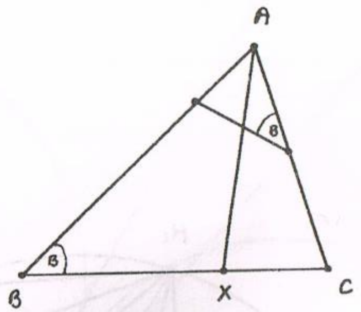
> *The points of this configuration lie in fours on eight equal circles; they also form eight orthocentric sets. Each of these sets has a common medial circle : the eight such circles pass through a point.. This point lies on the four pedal lines common to each of the orthocentric sets.*

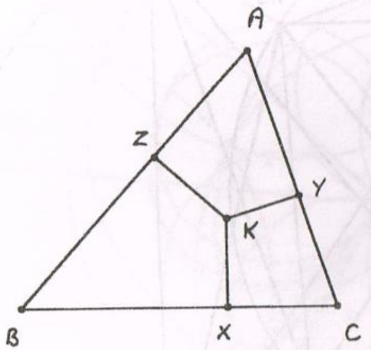


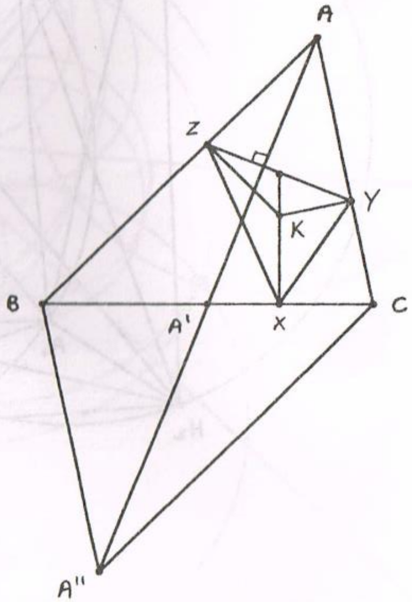


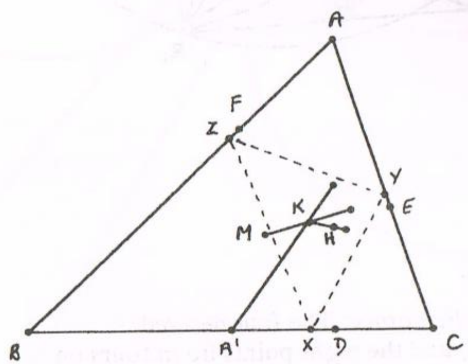
A configuration from four concyclic points : The four derived orthocentres form a congruent set; and the eight points lie in fours on eight equal circles with the various further properties given in 4.7.











5: Symmedian (Lemoine) point

- The isogonal conjugate of the median through A meets BC in X where $BX/CX=c^2/b^2$, etc, and AX,BY,CZ are concurrent.

  The isogonal conjugates of the medians are called symmedians and they concur at a point K called the symmedian point, or sometimes the Lemoine point.

- A symmedian bisects any antiparallel to its corresponding side.

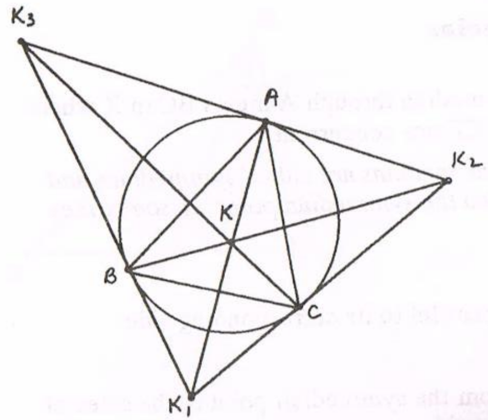
- The perpendicular distances from the symmedian point to the sides of the triangle are proportional to the sides.

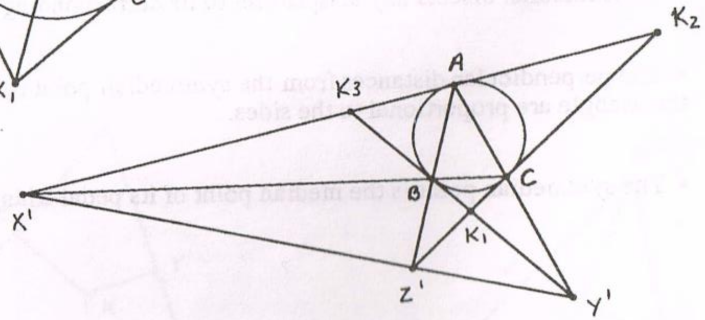
- The symmedian point is the median point of its pedal triangle.

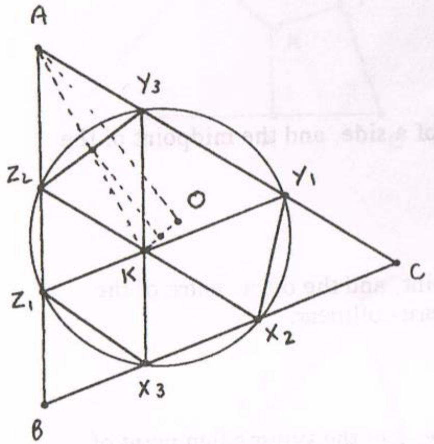
- The symmedian point, the midpoint of a side, and the midpoint of the corresponding altitude, are collinear.

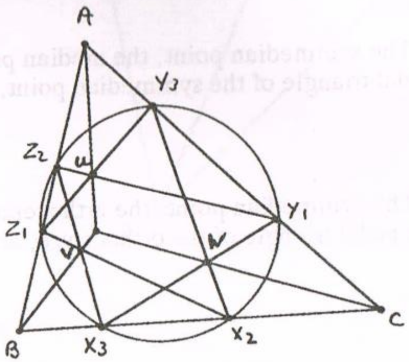
- The symmedian point, the median point, and the orthocentre of the pedal triangle of the symmedian point, are collinear.

- The symmedian point, the orthocentre, and the symmedian point of the pedal triangle of the orthocentre, are collinear.









6: Lemoine line and circle

• The tangents to the circumcircle at A, B, C meet at K_1, K_2, K_3: the lines AK_1, BK_2, CK_3 are concurrent at the symmedian point.

The tangents are sometimes called exsymmedians, and their intersections are the exsymmedian points K_1, K_2, K_3.

• The exsymmedians meet the opposite sides in X', Y', Z' respectively: these points are collinear, lying on the polar of the symmedian point with regard to the circumcircle.

The polar of the symmedian (Lemoine) point is called the Lemoine line.

• The lines through the symmedian point parallel to the sides of the triangle meet the sides in six points that lie on a circle,

The circle containing the six points is called the Lemoine circle.

• The centre of the Lemoine circle is the midpoint of the symmedian point and the circumcentre.

• The Lemoine circle meets BC at X_2, X_3, etc; the chords Y_1Z_2, Y_3Z_1 meet at U, etc, : then AU, BV, CW are concurrent.

• The chords Y_1Z_1, Y_3Z_2 meet at U', etc: then U', V', W' are collinear on the polar of the previous point of concurrence with regard to the Lemoine circle.

• The chords X_2Y_1, X_3Z_1 meet at U'', etc: then AU'', BV'', CW'' are concurrent at the symmedian point.

PROOFS

1.1 Mediators of AB,AC meet at O: OB=OA=OC, so O lies on mediator of BC.

1.2 Medians meet opposite sides at A',B',C': BA'/CA'=1,etc, so (Ceva) AA',BB',CC', concurrent. Moreover, triangles AB'M, etc, are equal in area $\Rightarrow$ M trisects each median.

1.3 Altitudes meet opposite sides at D,E,F: BD/CD=cotB/cotC, etc, so that (Ceva) AD, BE, CF concurrent. (Note $\angle$DAC=90°–C=$\angle$BAO $\Rightarrow$ AO isogonal conjugate of AH, etc.)

1.4 AP=1/2.c.cosA/sinC=1/2.a.cotA=OA'$\Rightarrow$ OA' = and ‖ 1/2.AH $\Rightarrow$ AA',OH trisect each other, and so at M with OM=1/2.MH.

1.5 *Gossard's theorem* - suitable for Cabri confirmation.

1.6 N midpoint of OH: NA'=ND=NP=1/2.OA, etc.

2.1 Circles BZX,CXY intersect in P: $\angle$YMZ=B+C=180°-A $\Rightarrow$ circle AYZ through P.

2.2 Centres O_1,O_2,O_3 of Miquel circles: $\angle O_2O_1O_3$=180°–$\angle$ZPY=A, etc.

2.3 $\angle$BPC=$\angle$BPX–$\angle$XPC=$\angle$BZX–$\angle$XYC=A, so P lies on circumcircle of ABC.

2.4 $\angle O_2PO_3=\angle O_2PX+\angle XPO_3$=90°–$\angle$PBX–90°+$\angle$PCX=$\angle$BPC=A =$\angle O_2O_1O_3$, etc, $\Rightarrow$ circumcentres concyclic though P.

2.5 See 2.6, or using properties of pedal line (cf 3.1) note that the common pedal line of P with regard to the four triangles bisects line joining P to the four orthocentres, so these are collinear.

2.6 AD is altitude from A, through orthocentre H, and L is centre of circle through H, on AX as diameter, etc: the product AH.HD = $4R^2$.cosA.cosB.cosC=a^2/4-LH^2, and similarly for M,N: $\Rightarrow$ H lies on line joining common points of the circles on AX,BY, CZ as diameters, so that the centres of these circles are collinear. (*This is known as the Gauss-Bodenmiller theorem.*)

2.7 $\angle$PYA=$\angle$PZA=90° $\Rightarrow$ PA is diameter of circle AYZ, etc.

2.8 Successive pedal triangles are $X_iY_iZ_i$: $\angle B_3A_3P=\angle B_2C_2P$ = $\angle A_2B_1P$ =$\angle A_2B_1P$=$\angle$BAP, similarly $\angle PA_3C_3$=$\angle$PAC, so that $\angle B_3A_3C_3$=$\angle$BAC.

3.1 $\angle$BZX=$\angle$BPX=90°–$\angle$PBC=90°–$\angle$PAY=$\angle$APY, so XYZ a line.

3.2 AH meets circumcircle again at H', PH' meets BC at S and pedal line at T: $\angle$YXP=YCP=$\angle$AH'P=$\angle$XPH' $\Rightarrow$ TP=TX=TS, ie T midpoint of PS $\Rightarrow$ XY‖ ZHS, so XYZ bisects PH.

3.3 Pedal lines of diametrically opposite P,P' are perpendicular and meet medial circle at Q,Q': Q,Q' are diametrically opposite so pedal lines through Q,Q' meet on the medial circle.

3.4 Pedal lines of PQ meet at X, H' image of H in X, orthocentre of PQH' is R: pedal lines of P,Q bisects PH,QH and so parallel to PH',QH', $\Rightarrow$ $\angle$PRQ=180°-$\angle$PAQ, $\Rightarrow$ R on circumcircle and pedal line of R‖ RH' and hence the third pedal through X.

3.5 *Steiner's theorem* - suitable for Cabri confirmation.

4.1 For A is on altitudes of triangle HBC, etc.

4.2 O_2O_3 is mediator of AH, so perpendicular to OO_1, etc $\Rightarrow$ O is orthocentre of $O_1O_2O_3$; moreover OO_1=AH so that circumcentres form a set congruent to A,B,C,H.

4.3 Medial circle of BHC passes through the midpoints of BC,CH, BH, and so is medial circle of triangle ABC.

4.4 Equal weights at the vertices A,B,C and orthocentre H may be replaced two at the midpoint A' of BC and two at the midpoint P of AH; and the centre of gravity of these at the midpoint of A'P, namely the centre of the medial circle.

4.5 *Beltrami's theorem*: an application of 4.4.

4.6 Equal weights at the vertices A,B,C,D may be paired in three ways each of which yields the common centre of gavity.

4.7 Concyclic points A_i ; orthocentre of A_2,A_3,A_4 is H_1, etc: $A_1H_4H_1A_4$ is a parallelogram, $H_1H_4=A_1A_4$, etc, ie the H_i are congruent to the A_i , and $A_1A_2A_3H_4$, etc, or $A_1H_2H_3H_4$, etc, form orthocentric sets.

5.1 Distances of X from AC,AB are y,z, of A' from AC,AB are q,r: y/AX=r/AA' and z/AX=q/AA' $\Rightarrow$ y/z=r/q=b/c $\Rightarrow$ BX/CX =c^2/b^2, etc, so that (Ceva) AX,BY,CZ concurrent.

5.2 Reflecting in an angle bisector, the corresponding symmedian and antiparallel become median and parallel.

5.3 Distances of K from sides are x,y,z: y/z=b/c,etc, (cf 5.1) $\Rightarrow$ x/a=y/b=z/c=2abc/R($a^2+b^2+c^2$).

5.4 $\angle$MAB=$\angle$KAY=$\angle$KZY=90°-$\angle$AZY, so that AM is perpendicular to YZ; with A'' the image of A in A', triangles CAA'', KYZ are similar, and since these are rotated through 90° and CA' bisects AA'', then XK bisects YZ.

5.5 The centres of rectangles inscribed in the triangle with a side on BC are collinear; and the midpoint of BC, the symmedian point K and the midpoint of the altitude through A are such centres. (*Brocard's theorem*)

5.6 Alias, the median point lies on the Euler line of the pedal triangle of the symmedian point; for the pedal triangles of the isogonal conjugates M and K have the same circumcircle, with its centre at the midpoint of MK. (*Tucker's theorem*)

5.7 Alias, the symmedian point, the incentre, and the symmedian point of the ex-centres. are collinear. (*Van Aubel's theorem*) The isotomic conjugate of the orthocentre also lies on the line.

6.1 $A'K_1, K_2K_3$ meet BC at X,X': BX'/sinC=AX'/sinB and CX'/sinB=AX'/sinC $\Rightarrow$ BX'/CX'=c^2/b^2=BX/CX $\Rightarrow$ AK_1 passes through symmedian point K.

6.2 The polar of X' is AX through K $\Rightarrow$ polar of K through X'.

6.3 AY_3KZ_2 is a parallelogram, so AK bisects Y_3Z_2; hence Y_3Z_2 is antiparallel to BC, $\angle AY_3Z_2=\angle AZ_1Y_1$, etc, ie the six points are concyclic, with centre at midpoint of OK.

6.4 See previous note.

6.5 BV,CW meet at T, distant x from BC, etc: so $x:y=X_3X_2:Y_1Y_3$ and $x:z=X_3X_2:Z_2Z_1 \Rightarrow y:z=Y_1Y_2:Z_1Z_2$, so that T lies on AU.

6.6 For U'V'W' is Pascal line of the six points on the circle.

6.7 For KA bisects Y_3Z_2, etc.

7.1 Cot w=1/2.a/KK_1=$(a^2+b^2+c^2)/4D$=cotA+cotB+cotC.

7.2 $\cot^2 w-3=1/2.\Sigma(\cot B-\cot C)^2 \Rightarrow \cot w \geq \sqrt{3} \Rightarrow w \leq 30°$.

7.3 BX/sinw=AX/sinB and CX/sin(A-w)=AX/sinC $\Rightarrow$ BX/CX $=c^2/a^2$, etc, so that (Ceva) AR,BP,CQ concurrent.

7.4 $\angle$WBC=$\angle$BAW=$\angle$WCA $\Rightarrow$ AC tangent to circle BWC, etc.

7.5 AW,BK,CM meet sides at X,Q,C': BX/CX=c^2/a^2 as above;

Contact annewatson1089.gmail.com if you want the rest of this booklet. There are 16 sections in all.