Development of Students' Understanding of Functions throughout School Years

Michal Ayalon[1], Stephen Lerman[2] and Anne Watson[1]
University of Oxford[1], London South Bank University[2]

Despite a plethora of research about misconceptions and the teaching of functions, little is known about the overall growth of students' understanding of functions throughout schooling. We aim to map the development of students' understanding of concepts which contribute to understanding functions throughout school in two different curriculum systems: in the UK and in Israel. The research uses a survey instrument that was developed in collaboration with a group of teachers.

Keywords: functions; variables; covariation; mathematics curriculum; graphing

Introduction

The function concept is both an explicit and implicit foundation for advanced study in mathematics itself and as a tool in other subjects. The ways in which functions are understood and used in advanced study vary, for example: in pure mathematics it is important to understand the roots of the function; in physics, that a wave function might represent a phenomenon; in statistics, the effect on a distribution of a change in a parameter; in economics, that a function is a particular relationship. The roots of these understandings can be found in most school mathematics curricula. Study of pure mathematics encompasses all of these meanings and purposes, but neither the routes to understanding functions, nor the school-level definitions, consist of a single hierarchical pathway (Schwindgendorf, Hawks, & Beineke, 1992). There are multiple branching curriculum decisions to be made about how functions develop for learners through school. It would be helpful to have an overall map of the development of pathways towards the function concept and thus to understand how students' concept images develop. This is what we have set out to do. We report our research approach and demonstrate its application to one task in one national context. The research has four stages and we are currently in stage three:

1. Development of hypothetical conceptual map for functions.
2. Design cycles with teachers to devise the survey instrument.
3. Implementation of the survey instrument and analysis in two countries.
4. Comparison between countries.

Development of a hypothetical map

We synthesized research about functions (Watson, 2013) consisting mainly of evaluations of learning in particular instructional contexts, and records of misconceptions in particular manifestations of functions. There are several distinct routes for development through school: generalisation of sequences; graphical representation of realistic data; sets of points generated from equations and formulae; input/output models; relations and covariation between variables; mappings between sets. All these could have the word 'function' attached. Some relevant research offers ways in which distinct routes become connected, such as through multi-representational software (e.g. Yerushlamy, 1990). Misconception research usually points to over-reliance on one route or lack of connection between routes.

Learning does not only depend on the written curriculum, it also depends on school and classroom context, teaching, and possibly on the level to which teachers are 'functions aware' (Watson & Harel, 2013) and national expectations through assessment regimes. We are therefore working in two countries: UK and Israel. The curriculum in the UK has an informal approach to functions, not requiring a formal treatment until year 12[1] for those who continue to advanced study, but younger students will generalize sequences and meet input-output models as 'function machines'. In the curriculum in Israel approaches to functions are more explicit for younger students and the word and the notation are introduced in year 7. This difference forms the main justification for the comparison across the two countries. Project teachers are 'functions aware' due their qualifications.

Designing a survey instrument

We developed a survey instrument over several design cycles, starting with a test of college-readiness in functions developed in the US (Wilmot et al., 2011). This consisted of 26 questions which were used randomly with students across ages. We analysed the mathematical affordances of the questions and presented this to four teachers from two schools in the UK, and a similar sample in Israel[2]. We took teachers' comments about the questions and length into account for the next stage. We then selected an optimal set of questions that addressed the distinct routes we had identified from the literature. There were some omissions for which we sought well established tasks to add to the survey (Swan, 1980). We asked teachers for their working definitions of functions to ensure that these were included. This second version was trialled and further meetings held with teachers. They indicated language and comprehension difficulties, and we analysed the students' work with them. We were not interested merely in task completion, nor in well-known misconceptions. Instead we were looking for fidelity between our intentions and students' responses by analysing what could legitimately be said about the understanding behind students' answers. This iterative process led to further modification of the survey. The questions had to be accessible for students in years 7 to 13, and had to be completed in one lesson. Also, apart from task six which was only accessible for students who had heard the word 'function', there was no implied progression through the paper.

Implementation of the survey instrument

The third stage of the research was implementation of the survey in schools. This process was constrained by the time and classes available. We required, as a minimum, data from every school year, and asked each school to provide data from alternate years. We also wanted data from a suitable spread of students in terms of their past attainment. We therefore asked each school to provide data from their highest achieving class (called A) and a middle achieving class (called B) in each of their contributing years. The teachers would provide us with random anonymised samples of 10 scripts from each class. In this way we received 20 scripts from each UK year 7 to 11 inclusive, and 10 scripts from the first and second years of post-16 mathematical study. The purpose of this data and analysis is to learn about progress towards functions in secondary school, while being aware that grouping, teaching, curriculum, prior attainment, and other variables make a difference.

[1] we are using UK years in this paper.
[2] the paper was translated into Hebrew and appropriate modifications of language applied.

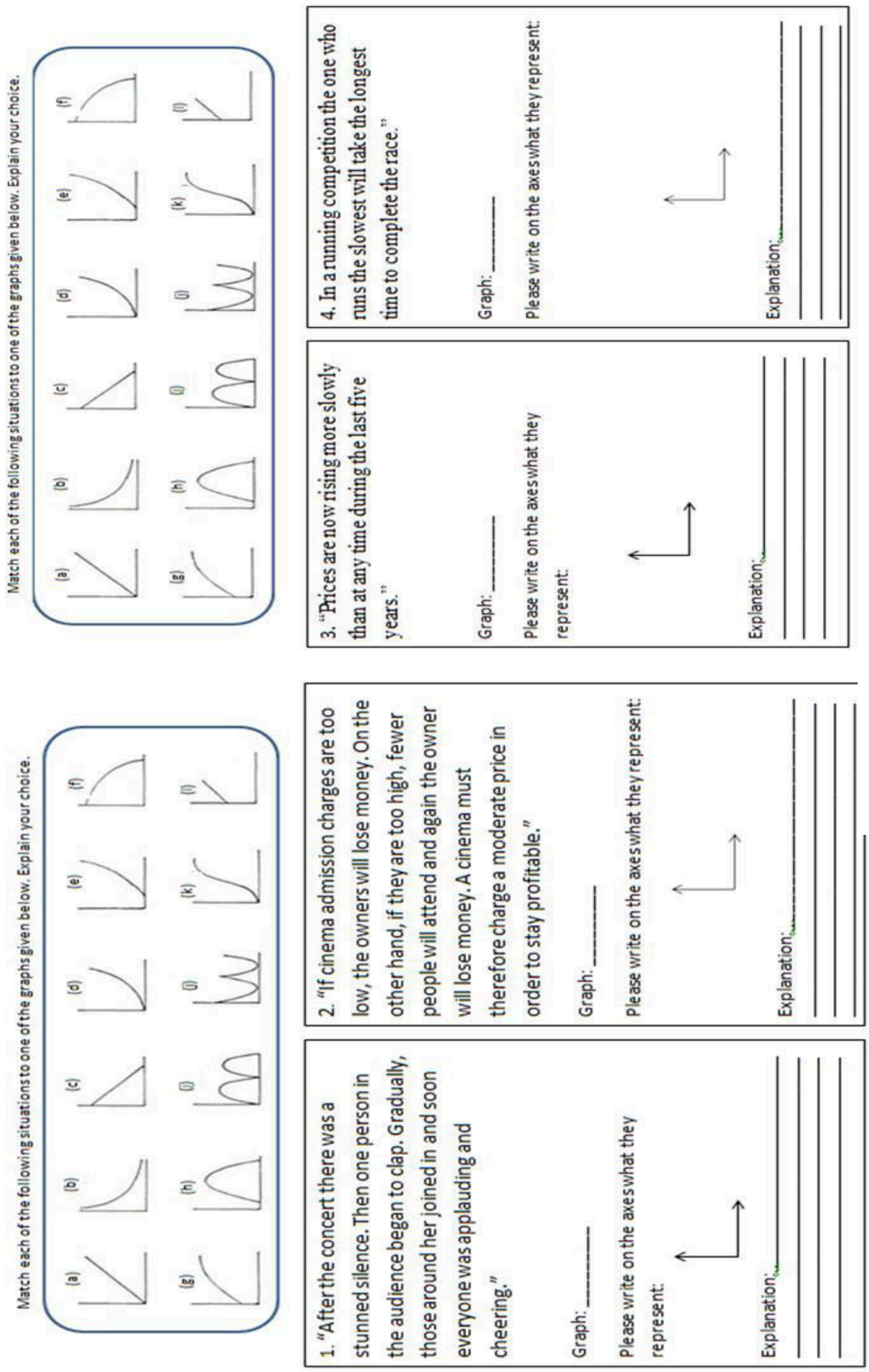

Figure 1. A task given in the survey; derived from Swan (1980)

In our UK sample one important difference between the two schools was in grouping. In one school students were placed in a single hierarchy of sets based on prior attainment, so that the 'top' set in year 10 was 1/11th of the whole cohort, and the available 'middle' set was ranked 4 in the hierarchy. In the other school students were split into two comparable halves, so there were two parallel 'top' sets, and the 'middle' sets used for the study were of average and slightly below average prior attainment for the year group. The schools were similar in many ways (size, socio economic factors, ethnicity, stability, qualifications) but differences in grouping are likely to have an impact on learning. Some students in the B group in one school might have similar prior attainment to some in the A group in the other school.

A sample task

We now present one task derived from Swan (1980) (Figure 1), and the data from UK. Students had to match 4 realistic situations to graphs, selected according to an expected increase of difficulty: (i) straightforward identification of unidimensional variables; (ii) choice of variables; (iii) the use of compound variables; (iv) compound variables and an inverse relationship. In all the situations students need some grasp of covariation to complete their analysis.

We hoped students would think analytically about the graphs rather than visually, identify variables and covariation in the situations, and match these. An iterative and comparative process of analyzing students' responses led to the following foci: variables, covariation, zeroes, and contextual features. For each situation responses were coded: 1: Lack of analysis; 2: Incomplete analysis; 3: Full analysis. In Table 1 we align examples of responses and codes related to situation 3: the use of compound variables.

Categories	Lack of analysis	Incomplete analysis	Full analysis
Example of response	Graph chosen: (e). Explanation: B*ecause it is going up slowly.*	Graph chosen: (k). Explanation: *Before rapid increase now the rise is slow* (wrote price on the y-axis and time on the x-axis)	Graph chosen: (k). Explanation: *At the start there couldn't be no price and the gradient drops towards the right so prices are rising slower* (wrote price on the y-axis and time on the x-axis)
Interpretation	Misinterpretation, probably due to the word "slowly".	Missing the idea that price does not start with zero.	Explicitly refers to the idea that price cannot be zero

Table 1: Examples of responses and codes related to situation 3.

Results

Figure 2 presents the distribution of codes for the A and B classes for each situation. If students generally make progress towards understanding functions in multiple ways through school, we would expect the darker shaded areas to increase towards the right of the graphs in Figure 2, i.e. their understanding becomes more formal/analytical/complete as they grow older. In this task expect the same kind of display of progression for each separate question. With a small sample, we also expect differences between teachers and schools. Allowing for these variations we observe a progression towards analytical and complete interpretation in task 5 as a whole (not shown), and also in the separate features on which we focused in the A groups. In other words, in 'top' sets students progress throughout secondary school to being better able to select graphs to represent situations, to choose variables, to deal with compound variables, and to recognise inverse relationships. For the B groups the story is more variable. Some of the variation is undoubtedly to do with different approaches to grouping, particularly the results of 10B, which was the class ranked 4/11 and whose survey results are similar to those of 10A. Other aspects of the variation could be to do with what has been taught recently. For instance in 8B, it looks as if plotting data from realistic situations with unidimensional variables might have been a recent focus.

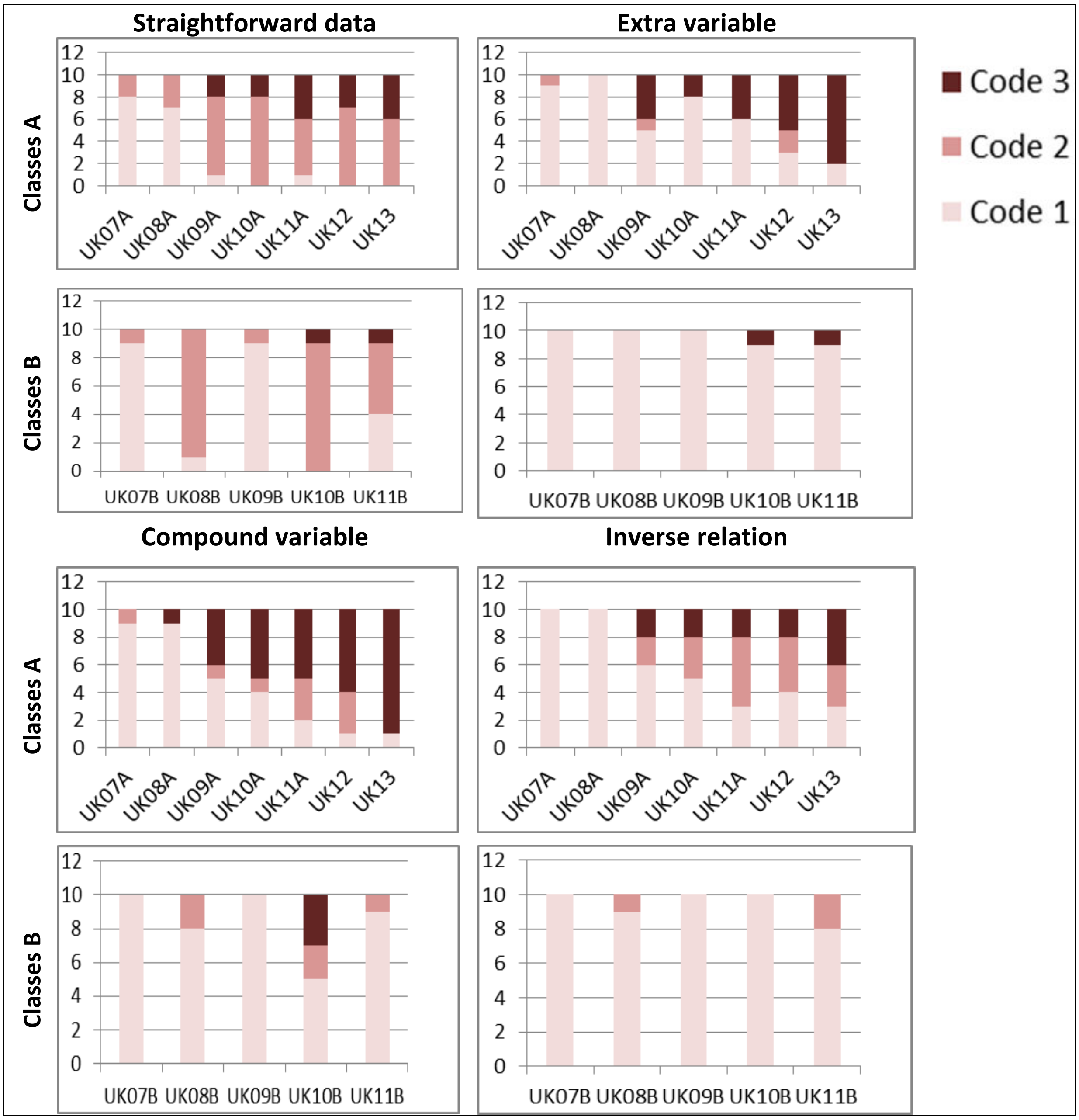

Figure 2. The distribution of the codes for each situation within A and B classes.

We can get some idea of students' curriculum experience from the teachers' expectations, which we interviewed them about after the survey. All teachers thought that their students would be able to handle situations 1 and 3, but that situation 4 would create problems in defining variables. They did not think there would be much variation in response throughout years 7 to 13. Situational graphs of different kinds would be introduced from year 7 and in the other school this kind of graph-matching task was done in year 9 but without ambiguity about variables. In the survey, situations 1 and 3 were done better than the other two, but there was variation and progression across the years, which the teachers did not expect, and numbers handling any situation completely and analytically were small and did not support their expectations of what younger students would do.

Our position on the B groups is that for employment, higher study and critical citizenship, *all* students might need to relate situations to plausible graphs. Whereas people might become able to do this when immersed in meaningful situations, one of the purposes of formal education is to prepare students with the formal thinking that is available to others, i.e. the A groups. Another line of argument is that students who have been chosen to be in the A groups are those whose ways of thinking already fit well with the demands of formal schooling, in which case the students in the B groups need special teaching which enables them to develop formal thinking. For these reasons we intend to look further at the data from this task, using a framework from Leinhardt et al. (1990) composed of four constructs – the action of the student, the situation, the variables and their nature, and the focus – to identify what else the students might be bringing to it, as well as the knowledge we anticipated. Furthermore, data from other tasks included in our survey which address function-related aspects linked to those addressed the task reported here, along with data from the Israeli classes, are expected to provide us with further information on students' understandings alongside possible contextual explanations for the results.

References

Leinhardt, G., Zaslavsky, O., & Stein, M. S. (1990). Functions, graphs and graphing: Tasks, learning, and teaching, *Review of Educational Research, 1* (1), 1-64.

Schwindgendorf, K., Hawks, J. & Beineke, J. (1992). Horizontal and Vertical Growth of the Students' Conception of Function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp.133–52). Washington, D.C.: Mathematical Association of America.

Swan, M. (1980). *The language of functions and graphs.* Nottingham, UK: Shell Centre for Mathematical Education. University of Nottingham.

Watson, A. (2013). Functional relations between variables. In A., Watson, K., Jones, & D., Pratt (Eds.), *Key Ideas in Teaching Mathematics* (pp. 172-199). Oxford: Oxford University Press.

Watson, A. & Harel, G. (2013). The role of teachers' knowledge of functions in their teaching: A conceptual approach with examples from two cases. *Canadian Journal of Science, Mathematics and Technology Education, 13* (2) 154-168.

Wilmot, D. B., Schoenfeld, A. H., Wilson, M., Champney, D., & Zahner, W. (2011) Validating a learning progression in mathematical functions for college readiness. *Mathematical Thinking and Learning*, *13* (4), 259-291.

Yerushalmy, M. (1991). Students' perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning*, *7* (1), 42-57.