TWENTY-SIX YEARS OF PROBLEM POSING:

MATHEMATICAL INVESTIGATIONS USED IN OPEN UNIVERSITY MATHEMATICS FOUNDATION COURSE SUMMER SCHOOLS

1971 – 1996

Dedicated to the memory of
Lynne Burrell

who worked so tirelessly
to promote and support
mathematical thinking
at summer school

Compiled by
John Mason
to mark the end of
Mathematics Foundation Course Summer Schools

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INTRODUCTION

Since the first Mathematics Foundation Summer School in 1971, we have tried to stimulate and support students in tackling unfamiliar mathematical problems. This booklet offers as complete a record as was possible of the problems used, and thus provides an opportunity to look at the transformation in the sorts of problems offered during that time. One can detect in the changes, for example, indications of what might not have been happening , in an effort to make it happen. In some instances extensions have been added, together with some new problems at the end which might have been used but perhaps were not.

In 1971 it was a novel notion that students should work on anything other than routine tasks to revise topics and techniques from the course, or to have an introduction to ideas yet to come. Polya's book *How To Solve It* had been taken as a set text, yet was hardly referred to, so at each week of the summer school we showed Polya's film *Let Us Teach Guessing*. This was followed by a tutorial in which students were guided in reconstructing the film and resolving outstanding questions, mostly to do with proving the final conjecture made in the film. We then offered a follow-up session with other problems to work on called *Active Problem Solving*. It was assumed that the film would remind tutors that mathematical thinking functions best in a conjecturing atmosphere, and that tutors would naturally invoke the processes of specialising and generalising with their students. However it turned out that many tutors were unfamiliar with guiding a group of students in problem solving, and so there were very mixed reports of the success of these sessions. There was a close correlation between students' enthusiasm and that of their tutors.

Over the years, we changed the sorts of problems proposed, searching for a format and a structure which tutors and students would both find comfortable and instructive. We also conducted some in-service sessions, both nationally and at summer school, to try to demonstrate ways of working which had been found to be effective. In the process, tutors and students alike have come to appreciate the effectiveness of working on ways of thinking mathematically.

Eventually we found that what was most successful were short tasks in which students naturally drew diagrams, and naturally specialized, and generalized, so that tutors could draw students' attention to these, and students could see that they already knew what to do. When students get stuck it is often because they are not using these very skills. Students were then in better shape to tackle more challenging problems, including the Summer School Assessment option of a report on an investigation.

Some of the problems included here are original, but most are adaptations and variations of standard problems culled from many sources while preparing the Summer School materials. Since this collection is taken from old notes which did not record sources, no attempt has been made to track down those sources. Furthermore, no resolutions are provided, because I believe that the pleasure and value lies in the struggle and the reaching of a personal or collective conjecture, not in reading someone else's solution.

I am very grateful to Tracy Johns for her help in assembling this collection.

John Mason
Milton Keynes, August 1996

ACTIVE PROBLEM SOLVING

INTRODUCTION

The title APS has been chosen to indicate that the students are to be actively involved in a collective problem solving session. It is the tutor's responsibility to try to involve as many students as possible, and to help *guide* the discussion, but not *direct* it. Certainly, except in exceptional circumstances, we recommend that the tutor does *not* give answers. It is much better to develop one or two problems than to tackle several without development.

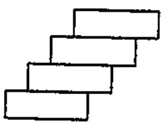
APS 1: Prime Numbers

Let $S = \{n: n$ an integer whose remainder on dividing by 3 is $1\}$. Show that S is closed under multiplication, and interpret the notion of prime within this set. Is there unique factorisation? In analogous situations, is there a meaning for prime if there is no number playing the role of 1?

Try the same task with $S = \{a + b\sqrt{2}, a$ and b integers$\}$. Try changing $\sqrt{2}$ to other surds.

APS 2: Leaning Towers

Given an unlimited supply of identical bricks with uniform density, how great a horizontal displacement can be achieved by piling the bricks one on top of the other as shown?



APS 3: Max Vol

What is the largest volume of a box with fixed surface area? What is the largest volume of a box with fixed perimeter? Extend to cones, cylinders etc.

APS 4: Integer Part

Investigate the function $x \longrightarrow [x]$.

Is it one-to-one? Is it continuous? Is it differentiable? Is it integrable?

Is $[x+1] = [x] + 1$? Is $[nx] \geq n[x]$?

Is $[x+y] = [x] + [y]$?

Is $[xy] = [x]\,[y]$?

When are they true?

What is the effect of the function

$x \longrightarrow 10^{-t}[10^{t}x + 0.5]$ for a fixed t?

Sketch the graphs of

$y = [x] + [x + 0.5]$ & $y = [2x]$;

$y = [x] + [x + \frac{1}{3}] + [x + \frac{2}{3}]$ & $y = [3x]$.

Generalize, and justify your conjecture.

Show that $[x] + \left[\frac{1}{x}\right] \geq 2$.

Which if either of

$$[x] = \left[n\left[\frac{x}{n}\right]\right] \text{ and } \left[\frac{[nx]}{n}\right] = [x]$$

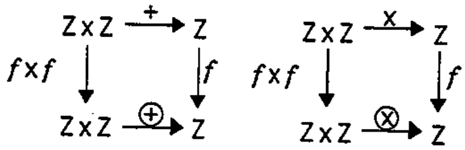
is always true?

Show that

$$[x] + \left[x + \frac{1}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx]$$

APS 5: Commutative Diagrams

Find all the functions f: Z —> Z for which both of the following diagrams commute.



where

$a \oplus b = ab - 1$ and $a \otimes b = a + b - ab$.

Can f be extended to the rationals? Continuously to the reals?

APS 6: Minimal Cross Sections

Find a surface for which the curve formed by each vertical cross-section through a given point has a minimum at that point, and yet the surface does not have a minimum at that point.

Polya's Problem from the film Let us Teach Guessing

Into how many regions do p planes in general position divide space?

Suggestion: specialize to lines dividing the plane, and to points dividing a line.

M100 1972
ACTIVE PROBLEM SOLVING
(EASIER)

APS 1: Tiled Path

Given a supply of 1 x 2 tiles, in how many distinct ways is it possible to tile an $n \times 2$ path?

Extension: given a supply of a by b tiles, what sized rectangles n by m can be tiled? What sizes of rectangle can be tiled so there are no fault lines (the tiling does not break up into tilings of two sub rectangles)?

APS 2: Coin Tossing

A coin is tossed until two heads appear consecutively. Find the probability that it takes n tosses.

APS 3: Die-Hard

You have a cube. In how many ways can you make a die? (You could choose to make a commercial die, or merely require that the sum of the opposite faces by 7, or relax the restrictions and consider how many different ways there are to paint a cube with two or three or ... different colours.)

APS 4: Leaning Towers (APS 2 of 1971 p2)

APS 5: Integer Part (APS 4 of 1971 p2)

APS 6: Commutative Diagrams (APS 5 of 1971 p2)

M100 1972
ACTIVE PROBLEM SOLVING
(HARD)

APS 1: Finite Configuration

For any configuration of a finite number of points in the plane, is it possible that the straight line specified by any two of them passes through a third?

APS 2: Partition Product

Find all integers n for which it is possible to partition the set $\{n, n+1, \ldots n+5\}$ into two subsets such that the product of the members of each subset is the same.

APS 3: Base Deletion

Two numbers x and y have the same representation in both base A and in base B. The leading digit is deleted to give the new numbers x' and y' in the respective bases. Show that $A > B$ if and only if

$$\frac{x'}{x} < \frac{y'}{y}.$$

APS 4: Sequence Limit

The sequence $\{a_n\}$ is specified by $a_{n+1} = a_n(1 - a_n)$ with $0 < a_1 < 1$.

Show that $\lim a_n = 0$ and $\lim na_n = 1$ as n tends to infinity.

M100 1971-1974
ADDITIONAL PROBLEMS

SS Chat

An OU student and a part-time tutor are eating lunch at summer school. "I'm a student" says the tall one. "I'm a tutor" says the short one. If at least one of them is lying, who is which?

Right-Angled Tetrahedra

How many faces of a tetrahedron can be right angled?

For a tetrahedron with the maximum possible number of right-angles, show that the sum of the squares of the areas of some of the faces is equal to the sum of the squares of the areas of the remaining faces.

What can you deduce about a tetrahedron all of whose faces have the same perimeter?

Grid Squares

Which sizes of squares have all their vertices on the points of an integral grid? Which triangles?

Possible Rational Roots

What are the possible rational roots of $x^3 + 2x^2 + 3x + 1$? Generalize.

Some Sums

Let R be a subset of $A \times B$, and S a subset of $B \times C$. Show that the sum

$$\sum_{a \in A} \; \sum_{b \in B \; (b,a) \in R} \; \sum_{c \in C(b,c) \in S} x_{abc}$$

is independent of the order in which the indices are summed.

Digit Transforms

Take four digits, arrange them in ascending and descending order, and treat these as numbers. Take the smaller number from the larger. Rearrange the digits of the answer and repeat. What happens in the long run? What about other numbers of digits?

Join the Dots

Take a rectangular array of dots with an even number of rows and of columns. Colour the dots either red or blue in such a way that every row has the same number of red and blue dots, and likewise every column. Whenever two dots of the same colour are adjacent in the same row or column, connect them with a segment of that colour. Show that the total number of blue segments equals the total number of red segments.

Productive Sequence

For any three-digit number, form the product of the digits, then the product of its digits, and so on. If a number has only two digits, then the product is 0. Show that at most ten products are necessary to obtain a 0. Is ten best possible?

Penny Placement

Can you place a number of pennies flat on a table so that if any two pennies which are touching are to be assigned different colours, 4 colours are required?

Ice-Cream Cone

What shape is the ice-cream cone having a given volume but using the least biscuit?

Rocks in the Boat

A fisherman sits in a boat in a stagnant mountain pool from which no water flows. In the boat is a rock which is sometimes used as an anchor. The rock is thrown overboard. The anchor rope holds for a while, and then the rock slips free and sinks to the bottom. What this process?

Inequality

$a_1, a_2, a_3, \ldots a_n$ are positive numbers, and $b_1, b_2, b_3, \ldots b_n$ are a permutation of them.

Show that $\displaystyle\sum_1^n \frac{a_i}{b_i} \geq n$

Divisibility

Show that $11^{n+2} + 12^{2n+1}$ is always divisible by 133.

Show that $4^{2n+1} + 5^{2n+1}$ is always divisible by 9

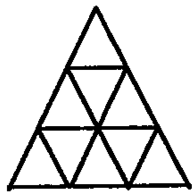
Surd Inequality

Show that $\frac{1}{\sqrt{1}} + \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}$

Integrity

Show that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7}{15}n$ is always an integer for n an integer.

Triangular Count



The edges of a triangle are divided into k uniform segments by the insertion of k–1 points. Lines are drawn through each of those points parallel to each of the three edges. How many triangles are formed with in the original triangle?

Random Integral Division

If $n + 1$ distinct numbers are chosen at random from $\{1, 2, \ldots, 2n\}$, show that one of them divides another integrally.

Tangentially Equal

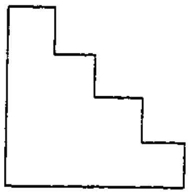
If $\frac{a-b}{1+ab} = \frac{c-d}{1+cd}$, prove that

$\frac{a-c}{1+ac} = \frac{b-d}{1+bd}$ and suggest why.

Growing Ratios

If a, b and $x > 0$, then what relationship must there be between $\frac{a}{b}$ and $\frac{a+x}{b+x}$?

Boxing



I have a staircase which rises in steps of 1 unit, and whose steps are of size 1 unit. I also have a collection of boxes one each of size 1 by k for $k = 1, \dots,$ which just fit underneath the stairs. In how many ways can I arrange the boxes under the staircase?

Ferries

Two ferryboats start from opposite banks of a river at the same time, and proceed across the river at right angles to the opposite banks. Each travels at a constant speed, but one is faster than the other. They pass at a point p metres from the nearest shore. Both boats dock and remain there for w minutes before starting back. On the return trip they meet m metres from the other shore. How wide is the river?

Extend to starting at different times and staying in dock different times.

Irrational Logs

Show that it is not possible to express

$$\log(x) = \frac{f(x)}{g(x)}$$

when f and g are polynomials.

Find Them All

Find all continuous positive functions on $[0, 1]$ such that $\int_0^1 f(x)\,dx = 1$, $\int_0^1 xf(x)\,dx = \alpha$ and $\int_0^1 x^2 f(x)\,dx = \alpha^2$.

Never Integral

Show that the sum of the reciprocals of the first n integers ($n > 1$) cannot be an integer.

Probably More

A tosses $n+1$ coins while B tosses n. Find the probability that A gets more heads than B. Generalize.

Integral Products

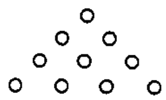
What is the probability that given two numbers chosen at random from [0, 1], the integer part of their product will be the product of their integer parts?

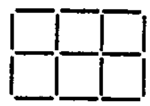
The Argument

Three men agree to share a sum of money in the parts 1/2, 1/3, and 1/6. respectively. While making the division they are surprised by the enemy, and snatching what they can, they run off. The first gave up 1/2 what he had snatched, the second 1/3, and the third 1/6. They shared this equally and each then had his proper share. What was the total sum? John of Palermo is said to have posed this to Fibonacci.

Rearranging Coins

Given ten coins arranged in a triangle as shown, how few can be moved so as to invert the triangle?





Matchless

Given an array of n by m squares outlined by matches as shown, how few matches can be removed so that no square of any size is fully outlined in matches?

Fished Out

In a certain large lake the proportion of desirable fish is p. If someone fishes until they catch a desirable fish (of course putting the others back) what is the probability that they have to catch f fish, and what is the expected value of f?

If two people fish in the same lake under the same conditions, what is the probability that one of them has to catch f fish, and the other at least f fish, and what is the expected value of f in this case?

False Equalities

For each of the following purported proofs, find the error(s).

A: To prove that 9 = 0:

Solving $1 + x + x^2 + x^3 + ... + x^8 = 0$ by expressing it as the sum of a geometric progression, the equation is $\frac{1-x^9}{1-x} = 0$, whence $1 = x^9$, so $x = 1$, and substituting in the original, $0 = 1 + 1 + 1^2 + ... + 1^8 = 9$, so $0 = 9$.

B: To prove that $0 = -1$:

Consider $I = \int \tan x \, dx = \int \sec x \sin x \, dx =$

$-\int \sec x \, d(\cos x)$. Integrating by parts gives

$$I = -\sec x \cos x + \int \cos x \, d(\sec x)$$

$$= -1 + \int \cos x \sec x \tan x$$

$$= -1 + \int \tan x.$$ Thus $0 = -1$.

Morphisms

1: T is a set, S is the set of subsets of T and f: $S \rightarrow S$ is specified by $A \rightarrow A'$ (the complement of A in T). The binary operation AoB is $A \cup B$. Is f compatible with AoB? What is the induced operation?

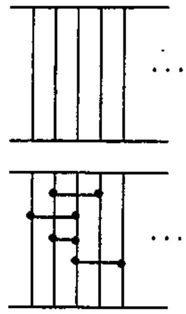
2: E is a fixed subset of T, and g: $S \rightarrow S$ is specified by $A \rightarrow A \cap E$. The binary operation is $AoB = A \cup B$. *Is* g compatible with o? What is the induced operation?

3: S is a set of cubes and spheres with distinct volumes, and f is specified by f(cube) = 0 and f(sphere)=1. The binary operation is

$$aob = \begin{Bmatrix} a \text{ if } a \text{ has the smaller volume} \\ b \text{ if } b \text{ has the smaller volume} \end{Bmatrix}$$

Is f compatible with o? What is the induced operation on the image set {0, 1}?

Matching



Imagine two horizontal lines. Consider n vertical line segments stretching from one to the other. Now draw horizontal line segments between arbitrary pairs of the vertical lines subject only to the condition that no two horizontal segments share an end point.

Specify a mapping from the top points to the bottom points as follows: start at a given top point and proceed downwards on its vertical segment; if the end of a horizontal segment is encountered, traverse it to the other end, then continue down the vertical segment reached. The image point is the bottom point reached.

Show that the mapping is one-to-one, and that every vertical line segment is traced exactly once, while every horizontal segment is traversed twice, once in each direction.

Wool-winding

You have a number of large cones of wool, with varying amounts of wool on each one. You wish to wind off k stranded wool (k no larger than the number of cones), maximising the amount of wool to be used without rewinding surplus wool onto further cones. Find necessary and sufficient conditions for all the wool to be used, and a strategy for maximising wool use.

Surface as Hyper-Perimeter

To be the sides of a triangle, three non-negative lengths a_1, a_2, a_3 have to satisfy

$$\sum_1^3 a_i \geq 2a_j \text{ for } j = 1, 2, 3.$$

What is the corresponding statement for the areas of a tetrahedron? What about higher dimensions?

SUM PATTERNS

$1 = \frac{1.2}{2}$	$1 + 2 = \frac{2.3}{2}$;	$1 + 2 + 3 = \frac{3.4}{2}$	... ?
$1.2 = \frac{1.2.3}{3}$	$1.2 + 2.3 = \frac{2.3.4}{3}$;	$1.2 + 2.3 + 3.4 = \frac{3.4.5}{3}$	... ?
$1.2.3 = \frac{1.2.3.4}{4}$;	$1.2.3 + 2.3.4 = \frac{2.3.4.5}{4}$;	$1.2.3 + 2.3.4 + 3.4.5 = \frac{3.4.5.6}{4}$	... ?
... ?	... ?	... ?	... ?
$\frac{1}{1.3} = \frac{1}{3}$	$\frac{1}{1.3} + \frac{1}{3.5} = \frac{2}{5}$	$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} = \frac{3}{7}$	... ?
$\frac{1}{1.4} = \frac{1}{4}$	$\frac{1}{1.4} + \frac{1}{4.7} = \frac{2}{7}$	$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} = \frac{3}{10}$	... ?
$\frac{1}{1.5} = \frac{1}{5}$	$\frac{1}{1.5} + \frac{1}{5.9} = \frac{2}{9}$	$\frac{1}{1.5} + \frac{1}{5.9} + \frac{1}{9.13} = \frac{3}{13}$	... ?
... ?	... ?	... ?	... ?
$\frac{1^2}{1.3} = \frac{1.2}{2.3}$	$\frac{1^2}{1.3} + \frac{2^2}{3.5} = \frac{2.3}{2.5}$	$\frac{1^2}{1.3} + \frac{2^2}{3.5} + \frac{3^2}{5.7} = \frac{3.4}{2.7}$	... ?
... ?	... ?	... ?	... ?
$1.1! = 2! - 1$	$1.1! + 2.2! = 3! - 1$	$1.1! + 2.2! + 3.3! = 4! - 1$	... ?

Chocolate bars

What is the best strategy for breaking a chocolate bar into its pieces (perhaps you can pile the pieces up and break several at the same time, perhaps you cannot).

Minimise the amount of silver paper required to wrap a rectangular cuboid chocolate bar.

Digit Transfer

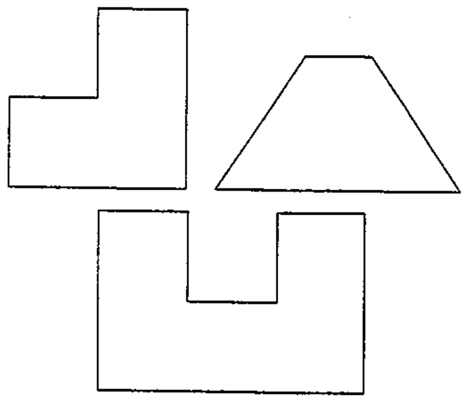
For which numbers (if any) does multiplying by 9 have the effect of sending the units digit up to the front and sliding the others one place to the right? What about transferring the leading digit to the units place? What about digits other than 9?

Acute Division

Partition an obtuse angled triangle into acute angled triangles. How many acute angled triangles are required?

Area Division

Divide each of the following regions into four congruent regions:



Make up your own!

Fair Shares

How might two or more people share a can or cans of beer, bottles of wine, pie or cake, so that everyone gets what they consider to be their fair share of the whole?

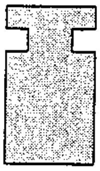
How might a cake, iced on top and on the sides, be shared so that each person gets a fair share of the icing as well as the cake?

Buckets of Bounce

Two buckets are connected by a rope over a pulley so that they just balance. A ball is dropped in one bucket and bounces several times. As it bounces, will the distance it travels be the same, shorter, or longer than the distance it would travel when bouncing in a stationary bucket?

Shadows of an Object

On a dark night after a Summer School Chez Angelique Mathematical Cabaret, I thought I saw the shadow of an object off to my right. I went a quarter of the way round it, and saw a second shadow. What was I looking at?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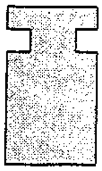
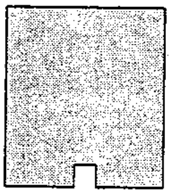


The same thing happened on a second night, but with a different object. What was I looking at?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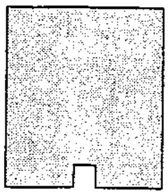
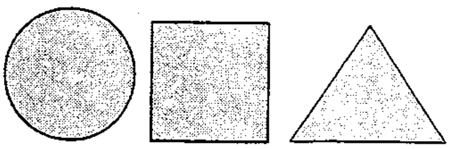


On a third evening I got three views from three different directions:



Was I dreaming?

A piece of wire can be shaped so as to cast a shadow of a square in three mutually perpendicular directions. Can the wire be arranged to cast circular shadows in three mutually perpendicular directions?

Stable and Level

I have a delicate instrument that must be absolutely level in one dimension in order to work properly. My table is steady, but the floor is uneven. Can I arrange the object so as to work on the table?

The table has four legs but the floor is uneven though continuous. Can I arrange for it to be stable?

The edges of my sitting room are level (being on solid foundations), but the floor has heaved (continuously) in places. Can I place a 3-legged stool so that it is perfectly stable? Perfectly level?

Beware of the Bull

Phyllis (P) and Quentin (Q) are dallying in a field. Phyllis has stopped to tie her shoe, and Quentin has gone ahead chasing butterflies. Having failed to notice the sign at the gate, Quentin is arrested by the sight of a bull charging straight for him. Quentin dodges out of the way (but essentially remains on the spot, while the bull goes rushing past him to the edge of the field before turning.

By this time Quentin is on the ground and Phyllis has stood up. The bull charges at her, and she does the same as Quentin, dodging the bull but returning to her same spot. The bull gets as far as the edge of the field before turning around. He carries on, alternately heading for Phyllis and Quentin, but not turning until he gets to the edge of the field.

What effect does the shape of the field have on what happens in the long run (for the bull)? Suppose the bull acts in the same manner but one or more of Quentin and Phyllis are outside the field; what happens in the long run? What if Rosemary joins them?

Moon Struck

The moon repeats its position in the sky above the earth every 29.53 days. A full moon last Wednesday shone in my window at 10 pm. When will it next be in the same position at the same time? and on a Wednesday? and on the same day of the same month?

Black Friday

How many Friday the 13ths are there in a typical year?

Snowplough

At some time before noon, it starts snowing. At noon, a snow plough sets out. In the first hour it manages to plough 2 miles of road. In the second hour, 1 mile. When did it start snowing?

Rectangles

How many rectangles are there on a chessboard?

Odd Subsets

How many subsets of $\{1, ..., n\}$ have an odd number of elements?

Satellite Dish

What shape is the circumference of a household satellite dish, and why?

Substitution Rule

The OU had a rule for a while which meant that a student's worst score out of 6 TMA scores was replaced by the average of those 6. What is the maximum benefit a student can obtain, and what happens if the process is repeated indefinitely?

M100 1973 WEEK-LONG PROBLEMS

Tower of Hanoi

The 'Tower of Hanoi' consists of three pegs fastened to a stand and a number of circular discs of different diameters each of which has a hole in it through which a peg may pass. Initially, the discs are all on one peg, with the smallest on top and no disc on top of a smaller one. The objective is to transfer the discs to another specified peg by moving the discs one at a time from peg to peg so that no disc ever rests on top of a smaller one. What is the least number of moves if there are n discs?

Recent extensions:

What happens if at the start there are piles of a, b, and c discs on each of the three pegs respectively: can the discs always be transferred to a specified peg? What if the discs on each peg have a specified colour so the task is to get all the discs of a given colour transferred to a specified peg, keeping to the original rules?

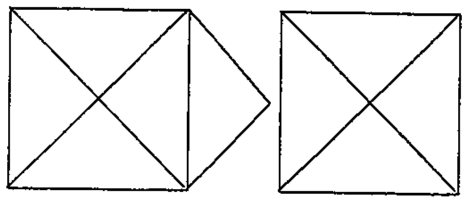
What if the discs start in random position without regard to size, but moves require discs never to be placed on top of a smaller one?

You are given s stacks of carpet tiles, and each tile has a unique number written on it. Your job is to sort the tiles into p stacks each in numerical order, simply by moving the top tile of a stack to another stack. You are subject to the rule that when a tile is placed on top of another tile the number on the lower tile is at least d more than the number on the upper tile.

What relationships must occur between s, p and d so that the sorting is possible?

Tracing

Is it possible to trace the following figures starting and ending at a vertex and covering each line only once without lifting your pencil from the paper?



Two Towns

Two towns A and B lie on the same side of a straight river and they share a common outlet of their sewage pipes at the river. Where on the river should the outlet be, so that the total pipe is as short as possible?

Least Weighing

You are given a pair of scales for which you place the weights in one pan and the object to be weighed in the other. What is the least number of weights which will enable you to weigh in one ounce intervals from 1 to n ounces?

Tiled Path (1971 p2)

Cornered Chessboard

Given a chess board with the top left hand and bottom right hand corner removed, and a supply of rectangular tiles which cover exactly two squares, is it board?

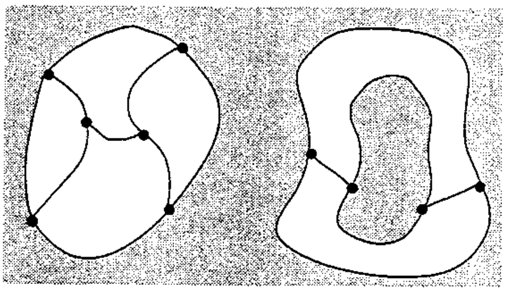
Leaning Towers (1971 p2)

Chess-Board Squares

How many squares are there on a chess board? How many rectangles?

Euler Mapping

An imaginary map has seas, and continents which are divided into countries by boundaries which meet at vertices. Let F be the number of countries and seas, E be the number of boundaries, V be the number of vertices and C be the number of continents. For the example shown (seas are toned) $F = 8$, $E = 15$, $V = 10$ and $C = 2$. Draw a few maps of your own and investigate $F - E + V - C$. Sate and prove a theorem.



Stamped

In a recent announcement the Post Office stated that as from January 1st, 1984, the postal rate for inland letters and cards will be 2p per gram, subject to a minimum charge. From the same date, as an economy measure, postmasters will only be issued with two denominations of stamps: 5p and 7p. Have you any idea what the Post Office contemplates as a minimum charge?

Divisible

Show that $n^2 - 1$ must be divisible by 8 if n is an odd integer?

Fibonacci Divisibility

The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... is defined by the recurrence formula $u_{n+1} = u_n + u_{n-1}$ for $n > 1$ and the fact that $u_1 = u_2 = 1$. Prove that any two consecutive terms of the sequence are relatively prime. (Two integers are *relatively prime* if the only positive integer which divides them both is 1.) Show that u_k divides u_{kn}.

Relatively Prime Sums

Prove that the sum of two consecutive natural numbers and the sum of their squares are relatively prime.

Induction

Prove using mathematical induction that the sum to n terms of the series

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots \text{ is } \frac{1}{2}(-1)^{n-1}\, n(n+1)$$

Deduce that

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$$

(Note: $1^2+2^2+ \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.)

Clog Tables

To find axb using log tables we look up log a and log b, add and then take the antilog of the resulting number,

i.e. axb = antilog (log a + log b).

The recently invented 'clogs' cut down on the work needed because

$a \times b$ = clog a + clog b

We have not yet received your sets of clog tables from the printers so perhaps you would start making up your own.

Right Divisibility

Prove that every right-angled triangle with integral sides has at least one side whose length is a multiple of 5.

Partition Product (1971 p3)

Save the Last Dance ...

At a school dance every boy dances with at least one girl, but no boy dances with all the girls. Similarly every girl dances with at least one boy and no girl dances with each of the boys. Prove that there must be two boys B_1 and B_2, and two girls G_1 and G_2, such that B_1 dances with G_1 and B_2 dances with G_2, but B_1 does not dance with G_2 and B_2 does not dance with G_1.

Perfect Relationship

Let ρ be a relation on Z^+ defined by

> $x \rho y$ if and only if xy is a perfect square.

i) Prove that ρ is an equivalence relation.

ii) The relation σ is defined on $Z^+ \cup \{0\}$ by the same rule. Is σ an equivalence relation?

Not The Pentagon

A plane takes off from each airport of a certain country and later each lands at the airport nearest to its point of departure. Prove that there is no airport at which more than five planes land.

Piled Coins

There are 2^n coins arranged in piles of any size. You are allowed to repeat the following operation as many times as you wish:

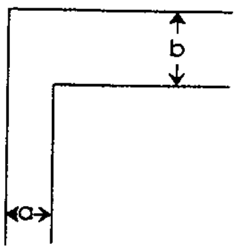
> Take two piles containing p and q coins where $p \le q$ and transfer p coins from the second pile to the first giving piles with $2p$ and $q - p$ coins.

Prove that you can always obtain a single pile containing all the coins. What initial configuration will take the most moves?

Triple Points (Finite Configuration 1971 p 3)

Cornered Pole

Find the length of the longest pole which can be carried horizontally round the corner whose dimensions are indicated in the figure.



Sphere Drill

Find the volume remaining when a cylindrical hole is drilled through the centre of a sphere so that the resulting hole has length *L*.

Find the volume remaining when a cylindrical hole is drilled down the axis of symmetry of a paraboloid or hyperboloid so that the resulting hole has length *L*.

Soft Drink Centroid

The centre of mass of a uniform cylindrical can full of liquid is obviously at the centre. Now imagine that the liquid is gradually removed from the can by means of a straw through a very small hole in the lid. When it has all gone, the centre of mass of the can is once again at the centre, since the can is taken to be symmetrical. Yet between these two situations the centre of mass of the can and liquid was below the centre. When is the centre of mass lowest?

Nullarbor Train

A man lost in the desert hears a train whistle due West of him. Although the track is too far away to be seen, he knows that it is straight and runs in a direction somewhere between South and East. He realizes that his only chance to avoid perishing from thirst is to reach the track before the train has passed. If both he and the train travel at constant velocities, in which direction should he walk to give himself the best chance of survival?

Recursive

For each positive integer x we define a sequence recursively as follows:

i) $T_1 = x$

ii) $T_{n+1} = \frac{T_n}{2}$ if T_n is even

iii) $T_{n+1} = \frac{3T_n + 1}{2}$ if T_n is odd.

NOTES TO TUTORS 1978

On The Nature of Learning Mathematics

Learning mathematics is much like learning a language. There is a basic vocabulary which must be learned thoroughly, and a basic grammar which must be mastered. This takes the form of becoming familiar with basic concepts and the words that describe them, like differentiation, Integration and so on, and ability to manipulate numbers and [illegible]. To a large extent this is an individualist activity, which can be well started at least, by reading course units at home, watching TV and so on. But neither languages nor mathematics are individualist activities. There comes a point at which it becomes necessary to start using the vocabulary yourself.

It is not possible to progress very far with a language unless you begin to have to use it, hearing yourself and others, and receiving immediate feedback. It is the same with mathematics. To some extent, tutorials play this role, but in the limited time available, tutorials are largely an opportunity for students to hear a tutor using the language in explaining the concepts. It is necessary to provide an environment in which plenty of interaction is possible, between students, as well as between student and tutor.

There is another analogy which matches learning with waves on a lake, looked at in terms of energy. A text is like a very shallow lake – it can provide only a relatively low level of background energy, from which a student can experience small waves of interest and motivation. Most texts, ours included, have a deadening effect when read in large chunks. We provide Radio and Television, media which excel in the transmission of emotional energy, which can provide a deeper lake on which larger waves are possible. But these media are essentially passive as far as the student is concerned, and to a larger extent, so is reading. This is why reading mathematics is hard. It demands the activity of pencil and paper, and the forming of quite precise mental images to cope with the various concepts, and this requires real discipline.

One other important aspect of learning to accumulate concepts is the need to return again and again over old ground. In mathematics, with its strong concept building, one on top of another, it is important to return again to first principles, and experience them again, over and over. This is why it is a common experience to begin to feel confident about new material one, two or three years after first meeting it. Thus every opportunity to go over basic ideas, in a slightly different form or context, is of tremendous value to the student, but not really measurable in terms of instant changes in behaviour. Learning mathematics is much more like water seeping into a semiporous material, than like a steeple chase, leaping barrier after barrier.

It seems that an essential component of mathematics learning is an opportunity to experience real activity, language practice and concept accumulation all at the same time. With our present resources and teaching methods, this demands a summer school.

M101 1978 INVESTIGATIONS

Arising from *practical* / *abstract* *situations*

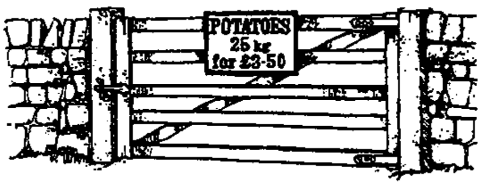
Each situation confronts you with an initial question indicating a possible direction of thinking. Your task is to enter into the situation, come to grips with some question such as the proposed one, and try to reach a conclusion.

There are suggestions for getting started and indications of possible generalizations.

It is more important to be able to explain your thinking than to reach some final conclusion

Potatoes

Mr. Smith purchases a bargain pack of potatoes (25 kg for £3.50) at a local farm. The shop price is £4.50. Mrs. Smith points out that the cheaper ones are rather smaller (3 cm mesh size rather than the usual 9 cm). She questions whether they are really a bargain.



What could mesh size mean?

What is the problem?

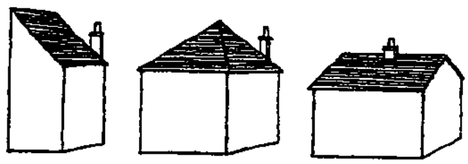
Simplify the shape of a potato.

What if the potatoes need peeling?

How about time considerations?

Ceilings

The following is an extract from the 1976 building regulations.



K8 Height of habitable rooms.

(1) Any habitable room in a building shall be so constructed that (except beneath a beam or beneath the ceiling to a bay window) the height of such room shall be not less than 2.3 m:

Provided that, if such a room is wholly or partly in the roof of the building, its height shall be not less than 2.3 m over an area of the floor of the room equal to not less than one half of the area of that room measured on a plane 1.5 m above the floor.

(2) The height of such room measured beneath any beam in that room and the clear headroom in any bay window in such room shall be not less than 2 m.

(3) For the purposes of this regulation, no account shall be taken of the projection of any joist or rafter in the ceiling of a room.

What do these mean for various types of roof?

—symmetrical sloping
—monopitch
—square based pyramid

Could you improve the specification?

The Stagecoach

In films of stage coaches driving into the sunset or escaping from bandits, the spoked wheels often appear to stand still, or even go backwards. Why?

Note: a film has about 25 frames per second (or 16 for home movies), and television has 625 lines, 312 of which change every 1/30th of a second. The eye tends to interpret any perceived change as the least possible rotation.

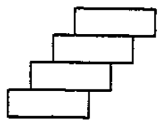
Can you predict the perceived direction and speed of rotation?

What happens for a camera moving past, or panning past a picket fence?

What about films of people and horses running?

What would you see if a television presenter waved a pencil in front of the camera? Would it seem to be straight?

Piling Bricks

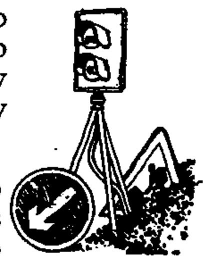


Most people try to pile bricks up high, so that they won't fall over. In the process, the bricks tend to overhang each other.

With ten bricks, how much overhang can you achieve between the top and bottom brick?

There is a surprise lurking here! It is not necessary to put each new brick on top of the pile! Nevertheless, it is worthwhile confining your attention to bricks piled one on top of another.

Temporary Traffic Lights



Road repairs on a two lane road reduce it to single file traffic. How should the temporary traffic lights be timed?

What is your aim—to get the most cars through safely? to minimize the waiting time?

Suggestion: start with very limited aims and add complexity later! For example: how does the number of cars passing through per hour vary with the timings of Green and Red? What conditions are needed for safety?

Possible simplifications:

(i) Cars arrive at a constant rate from either direction.

(ii) The lights have a fixed length of time in each state.

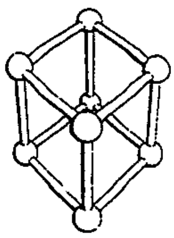
(iii) Cars travel at constant speed (when not at rest) and their length is negligible.

(iv) Cars maintain a separation proportional to speed.

Maximize the number of cars passing through in an hour. Is this an acceptable solution?

Minimize the waiting time for a car unlucky enough to arrive just as the light turns red.

Cubes



If you balance a cube on one vertex, the vertices are seen to lie on layers. For the cube shown there are four layers.

Can you predict the number of layers for any cube?

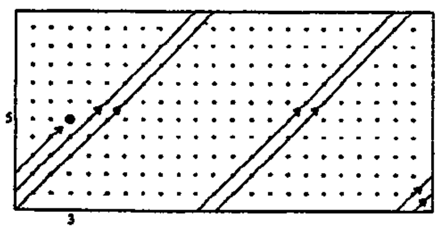
How many vertices will be on each layer?

Try looking at the balanced cube from different angles to find a way to simplify the counting problem. Alternatively, set up a recurrence relation which relates the numbers for an n-cube with numbers for smaller cubes.

Can you deal with an n by m by k cuboid?

Chinese Remainders and Medieval Puzzles

Take a rectangle such as the one illustrated and proceed at 45° from one vertex. When you reach an edge, re-emnter at the same point on the opposite edge. Can you reach every grid point (for example (3,5))? What happens if the sides have other sizes?



Medieval Puzzle: A woman, when asked how many eggs she has, replies with "Taken in groups of 11, 5 remain over, and taken in groups of 23, 3 remain over. What is the least number of eggs she has?

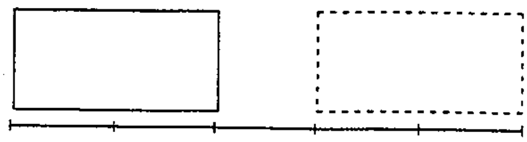
Medieval Puzzle: The same woman on another occasion replied "Taken in groups of 2, 3, 4, 5, 6 and 7 there remain over 1, 2, 3, 4, 5 and no eggs respectively."

The Chinese Remainder Theorem states the required conditions for a number to exist with remainders $r_1, ..., r_n$ when divided by $m_1, ..., m_n$, and also gives a method for finding it.

Having tried to solve the medieval puzzles, try making up some simpler examples to devise an algorithm. That should lead you to the remainder theorem.

Shifting Heavy Objects

Heavy objects like cupboards, armchairs and settees can be moved by rotating them about one corner. Suppose we permit *only* 90° rotations, and we wish to move a settee (twice as long as wide) as shown:

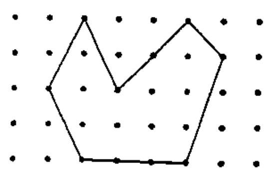


What positions can you reach? Don't be satisfied with a yes-no answer: an explanation is wanted.

What happens if the settee is of different proportions?

Polygons

A polygon is drawn using points of the grid as vertices:



What is the area of the polygon (the grid points are at unit distance horizontally and vertically). Can you relate the area to the number of grid points involved in the polygon? And prove it?

What happens if holes are permitted?

Try lots of examples.

M101 1978 WEEK-LONG PROBLEMS

1. The devil and the fool

The devil has agreed to double the fool's money each time the fool crosses a certain rather narrow and rickety bridge, but extracts a fixed payment of £10 for each trip.

After a number of crossings the fool finds he has no money left. What happened?

Generalize: what deal should the devil arrange for a given initial capital of the fool?

2. Criticise fully:

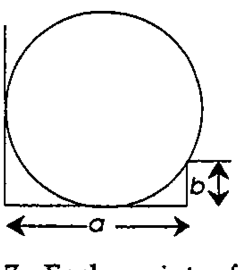
$$x^2 = x + x + \ldots + x$$
$$\therefore \quad 2x = 1 + 1 + \ldots + 1$$
$$\therefore \quad 2x = x$$
$$\therefore \quad 2 = 1$$

3. Beware of the Bull! (1974 p8)

4. Does the curve $y = a^x$ ever fall below the line $y = x$?

5. What is $\frac{d}{dy}\left(\frac{dy}{dx}\right)$ when $y = x, x^2, x^3, e^x, f(x)$?

6.



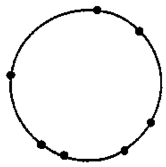
What radius of cylinder will exactly fit the slot shown?

7. Each point of the plane is to be coloured, either red or blue. Show that there must be an equilateral triangles with all three vertices the same colour. Harder—what about squares?

8. Which positive integers can be written as the sum of consecutive positive integers? In how many ways?

9. Fuel Shortage!

A number of cars are parked at various points around a race track. Collectively they have enough petrol to get one car right round the track.



Can one car make it round by choosing a starting pump and picking up supplies along the way? What about two cars going in opposite directions?

10. A affirms that D denies that C confirms that R refutes that T tells the truth. If each of A, C, D and R tell the truth with probability $\frac{1}{3}$, what is the probability that T tells the truth? Generalize.

11. (This referred to a set of posters showing various figurate number configurations)

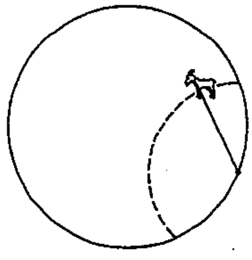
12. The Juke Box

A juke box has two rows of 12 buttons each. A record is selected by pushing one button from each row. For a fixed number of buttons, how can we maximize the number of choices possible? Is this practical?

13. Show that the product of any two numbers, each of which is the sum of two squares of integers, is itself the sum of two squares of integers.

14. In a certain collection of objects there are some which differ in colour and some which differ in shape. Must there be objects differing in both colour and shape?

15. A goat is tethered to the edge of a circular field(!). How long should the lead be so that the goat can graze just half of the field?



16. Compare

$$5+\cfrac{9}{5+\cfrac{9}{5+\cfrac{9}{5+\dots}}}) \text{ and } 16+\cfrac{16+\cfrac{16+\dots}{3}}{3}$$

Construct some more examples for yourself – can you find all integral examples?

17. Evaluate

$$2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}$$

$$4+3\sqrt{4+3\sqrt{4+3\sqrt{4+\dots}}}$$

$$1+.05^{1+\ .05^{1+.05^{1+\dots}}}$$

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\dots}}}$$

18. Consider the following six functions (with domain the real numbers excluding 0 and 1):

$f_1: x \longrightarrow x$ $\qquad f_2: x \longrightarrow \frac{x-1}{x}$

$f_3: x \longrightarrow \frac{1}{1-x}$ $\qquad f_4: x \longrightarrow 1-x$

$f_5: x \longrightarrow \frac{1}{x}$ $\qquad f_6: x \longrightarrow \frac{x}{x-1}$

Is the set closed under composition of functions? Is the inverse of each function in the set?

19. Can you find an integer expressed in powers of 10

(eg, $567 = 5 \times 10^2 + 6 \times 10 + 7$)

such that when multiplied by 2 the right-most digit becomes left-most, and all others move right one position?

Change 2 to other digits, and try making the left-most digit move to the right-most.

M101 1978 EXTRA COMPUTING PROBLEMS

Stamped (1974 p10)

Day of the Week

Given that you know today's date, and what day of the week it is, find an algorithm to work out which day of the week each of the following falls on:

(i) Christmas Day
(ii) St. Patrick's Day (March 17)
(iii) St. David's Day (March 1)
(iv) St. Andrew's Day (Nov 30)
(v) St. George's Day (April 23)

Biorhythms

In the 1890's Dr. Herman Swoboda, professor of psychology at the University of Vienna noticed that dreams, ideas and creative impulses in patients recurred with a regular rhythm. He also noticed a similar pattern of anxiety in new mothers. He kept detailed records of various symptoms, and came to the conclusion that cycles of 23 and 28 days seemed to be involved. At the same time, Dr. Wilhelm Fliess in Berlin studied changes in resistance to disease and related them to date of birth. He also came up with cycles of 23 and 28 days. Another Austrian, Alfred Teltscher came to the conclusion that his students had varying performances on school work which conformed to a 33 day cycle.

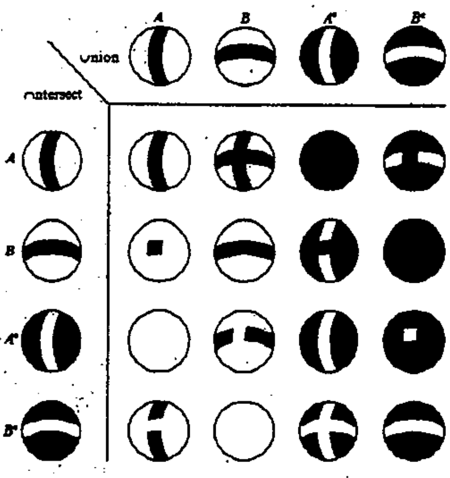
From these roots, the idea of biorhythms has grown. It is suggested that our physical well being, emotional stability and intellectual alertness have a cyclical behaviour. The cycle lengths suggested are:

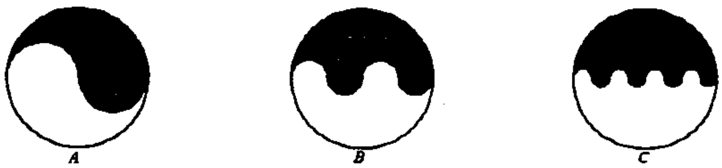
Physical—23 days
Emotional—28 days
Intellectual—33 days

They are described as being sinusoidal, all starting at 0 at your date of birth. A cycle is called critical if it is passing through 0 on that day.

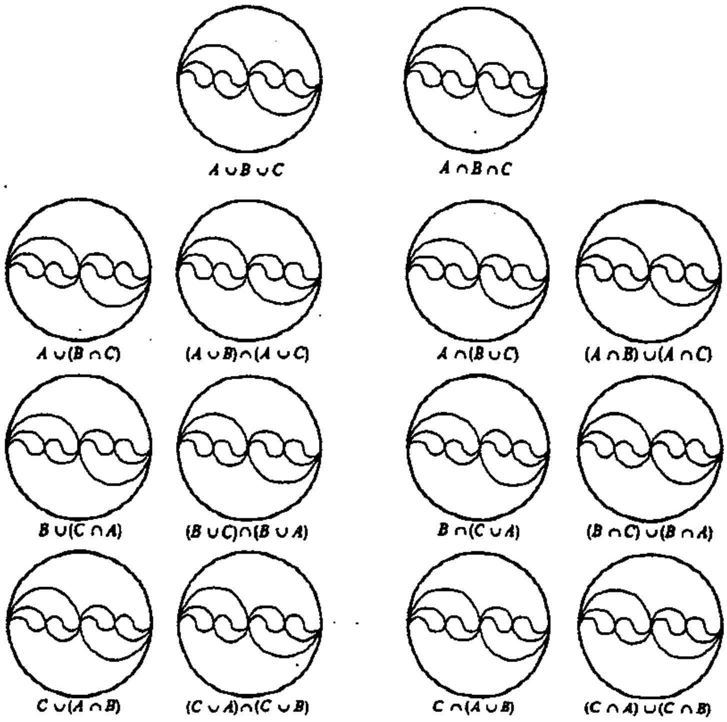
Compute when your biorhythms will next all be critical together.

M101 1978
PSYCHOLOGICAL PERSPECTIVES



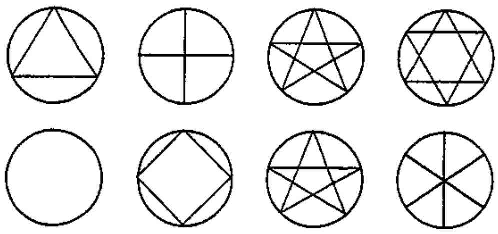


SHADE THE APPROPRIATE DARK AREAS FOR EACH OF THE FOLLOWING SETS AND EXPLAIN ANY COINCIDENCES



Thread patterns

In the following diagram the dots represent pins and the lines represent threads connecting the pins according to a fixed rule.



For the first row, the rule is to join each pin to the next but one in a clockwise direction. For the second, the rule is to connect each pin to the next but two in a clockwise direction.

When will one single thread reach every pin? How many threads will be needed in general?

If the step lengths are alternately k_1 and k_2, what patterns arise and how many threads will be needed for p pins?

If the step length is chosen at random, say by rolling a die, what is the probability that only one thread will be needed to reach every pin?

Irrational Powers

Show, using for example the number $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$, that an irrational number, raised to an irrational power, need not be irrational. Generalize.

Rational Product

By specialising, make conjectures for the values of the following calculations, and then verify that your conjectures hold in general.

$$\frac{3}{4} \times \frac{8}{9} \times \ldots \times \frac{n^2-1}{2n}$$

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n \times (n+1)}$$

Min-Max

Investigate the validity of the following statements, modify where necessary, and then generalize to more complicated calculations with more numbers:

For all real numbers a, b, c, and d,

$\text{Max}(\text{Min}(a, b), \text{Min}(c, d))$
$= \text{Min}(\text{Max}(a, b), \text{Max}(c, d))$

$\text{Max}(a, \text{Min}(b, c))$
$= \text{Min}(\text{Max}(a, b), \text{Max}(a, c))$

Annulus

How few equilateral triangles are required in order to make true annulus of triangles which must meet edge to edge?

Coned and Cylindered

What shape of paper is required in order to make a right-circular cone? What shape is required for a right elliptical cone?

What shape of paper is required in order to make a cylinder which has been sliced by a plane at an angle?

Reel to Reel

Between the two reels of a tape cassette there are marks which can be used to give some measure of the tape still unplayed. The marks are uniformly spaced. Is this sufficiently accurate, or could the marks be improved?

The Barge

A boy at the stern of a canal barge leaps off onto the tow path and while the barge keeps moving, runs along the path until he gets to the bow, where he instantly picks up a thermos flask of coffee and runs back until he gets to the stern. An observer noticed that while the boy was running, the barge moved forward a distance equal to its length. How far does the boy run compared to the length of the barge?

Salad Dressing

A chef is about to make a salad dressing and has before him a glass of oil and a glass of vinegar. She takes a spoonful of the oil and puts it into the vinegar glass and stirs. She then takes a spoonful out of the vinegar glass and puts it into the oil glass. At this point she pauses. Is there more oil in the vinegar glass than vinegar in the oil glass or the other way round?

Population Control

To help control the rapidly expanding population in a country it is suggested that a new law be passed which would prohibit a family from having further children after they had had their first son. Some families would have a son at once and so would have to stop having children; other families might have one or more daughters first, and some might stop having children without ever having a son. What would happen to the composition of the population?

Concern for equal rights led to a proposal that each family should only be forced to stop having children when they had had both a boy and a girl. What would happen to the composition of the population in this case?

True and False

A	B
The statement in box B on this page is true.	*The statement in box A on this page is false.*

C
The statement in box C on this page is false.

D
There are two misstakes in the statement in box D on this page

M101 SS 1979
INVESTIGATIONS

Arising from / *practical* \ *situations*

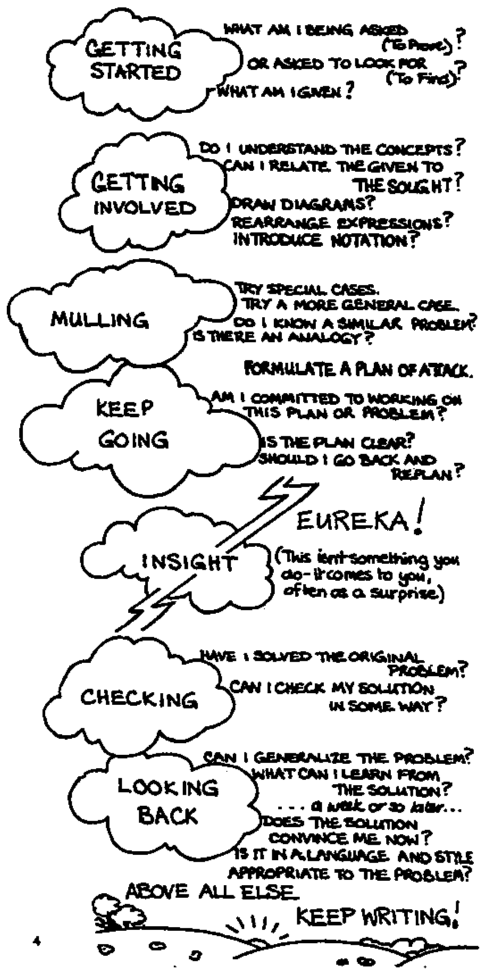
Arising from \ *abstract* / *situations*

Each situation confronts you with an initial question indicating a possible direction of thinking. Your task is to enter into the situation, come to grips with some question such as the proposed one, and try to reach a conclusion.

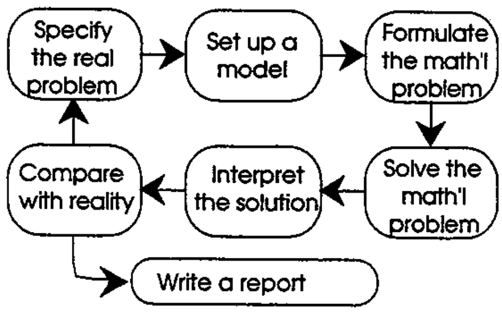
There are suggestions for getting started and indications of possible generalizations.

It is more important to be able to explain your thinking than to reach some final conclusion

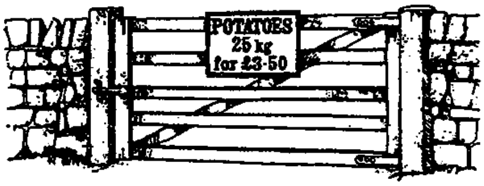
Keep a protocal of your thinking. Whenever you get stuck, make a note that you are, and why you think you are stuck. When you feel a flush of energy or enthusiasm, note that down too, and its source.



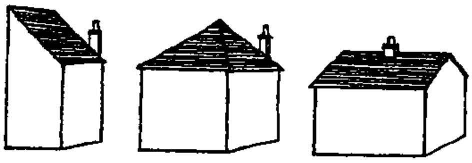
Modelling Diagram



Potatoes (1978 p12)



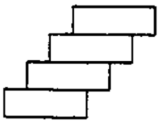
Ceilings (1978 p13)



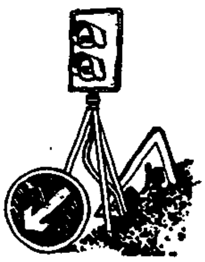
The Stagecoach (1978 p13)

Note: information about television frame speeds were omitted, as were questions about moving cameras and people running.

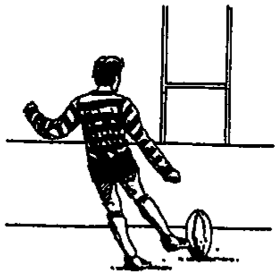
Piling Bricks (1978 p 13)



Temporary Traffic Lights (1978 p32)



Rugby Try



When his team has scored a try by touching the ball once on the ground behind the opponents' goal line, the kicker attempts to kick the ball over the bar, between the uprights.

He may do this from any point on a line perpendicular to the goal line through the point where the ball was touched down. Many kickers walk as far from the goal line as possible subject only to being able to kick it that far. What would you suggest?

Before diving in, be absolutely clear what you are trying to do.

Draw pictures!

Opposing players are permitted to charge at the kicker from behind the goal line, moving as soon as the kicker starts to move. Does this alter your first conclusion?

Cubes (1978 p 14)

Note: The suggested genmeralization was removed.

Medieval Egg Puzzles

Medieval Puzzle: A woman, when asked how many eggs she has, replies with "Taken in groups of 11, 5 remain over, and taken in groups of 23, 3 remain over." What is the least number of eggs she has?

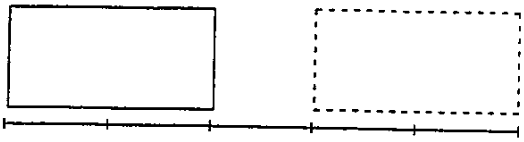
Medieval Puzzle: The same woman on another occasion replied "Taken in groups of 2, 3, 4, 5, 6 and 7 there remain over 1, 2, 3, 4, 5 and no eggs respectively."

Can you draw a helpful picture?

Having tried to solve the puzzles, try making up some simpler examples to devise an algorithm. Can you say when an egg puzzle will be solvable?

Shifting Heavy Objects (see 1978 p14)

Heavy objects like cupboards, armchairs and settees can be moved by rotating them about one corner. Suppose we permit only 90° rotations, and we wish to move an armchair, which is effectively square in shape, so that it ends up beside its original position. Can this be done? What about a settee to be moved as shown:



Can it be done? What if it must face the same direction?

What positions can you reach? Don't be satisfied with a yes-no answer: an explanation is wanted.

Generalize to other proportions.

Polygons (1978 p14)

Reflections

Denote lines in the plane by lower case letters, a, b, c, ... and points by upper case letters P, Q, R, ... To each line, say a, we have a corresponding transformation: reflection in that line, which it is convenient also to denote by a. To each point, say P, we have a corresponding transformation: reflection in point P, also denoted by P.

Reflections can be composed, and sometimes the total effect will be to leave all points fixed: the identity transformation denoted by I. For example aa (written a^2) $= I$ is always true for any line a, but $(PQ)^2 = I$ is only true if P and Q are the same point.

Investigate the geometrical meaning of statements like

$(PQR)^2 = I$
$(abc^2 = I$
$PQRS = I$

$(ab)^2 = I$ but $ab \neq I$
$(abc)^2 = I$

and so on.

Investigate how to express, symbolically, geometrical statements of the form

R is the mid-point of PQ ($P \neq Q$);

a and b are perpendicular lines meeting at P;

and so on.

To investigate the geometrical meaning of a statement like $(PQR)^2 = I$, try various examples for P, Q and R which lead to you to a geometrical relationship between P, Q and R that holds precisely when $(PQR)^2 = I$ and not otherwise.

M101 1982 INVESTIGATIONS

NUMBER PATTERNS

I was using a calculator to subtract numbers from their square, when Pat, looking over my shoulder, turned to me and said

"I can get your answers by adding numbers to their squares."

Is she right? Always?

What sorts of numbers arise from the sequence

$3 \times 5 + 1,\ 4 \times 6 + 1,\ 5 \times 7 + 1,\ \ldots$

Convince me!

What sorts of numbers arise from the sequence

$3(5^2 - 4) + 1,$
$4(6^2 - 5) + 1,$
$5(7^2 - 6) + 1, \ldots$

"Look at this!" said S.P.

$10 \times 1 \times (1-2)^2 + 5 \times (1+1)^2 = 1^4 + 29$

$10 \times 2 \times (2-2)^2 + 5 \times (2+1)^2 = 2^4 + 29$

$10 \times 3 \times (3-2)^2 + 5 \times (3+1)^2 = 3^4 + 29$

"Aha!" said G.E.N., "The general pattern is ... " (he mumbled). Comment! Convince!

What sorts of numbers arise when you add one to the product of four consecutive numbers?

What sorts of numbers do you get from calculations like

$(4^2-1)+1 \qquad (6^2-1)(7^2-1)$

$(8^2-1)(9^2-1)+1, \ldots ?$

Generalize? Convince!

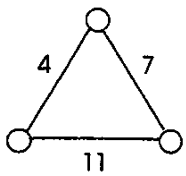
One Sum

Take any two numbers that sum to one. Square the larger and add the smaller. Square the smaller and add the larger. Which result will be bigger? (Conjecture, then convince.) Try using a diagram.

Similitude

What shapes of paper have the property that they can be cut in half by a straight line to yield two pieces each similar to the original.

Arithmagons



Hidden at each vertex of the triangle is a number.

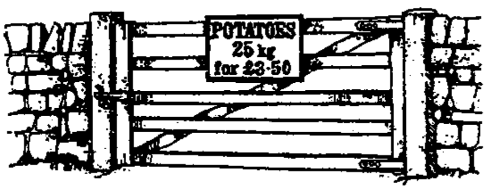
The edge numbers are the sums of the numbers on adjacent vertices.

Can you reconstruct the vertex numbers? Can you find a quick rule of thumb to reveal vertex numbers, that always works? Convince! Generalize!

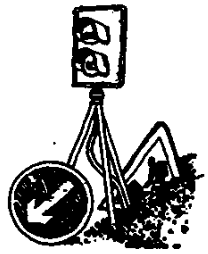
Fare is Fair

I wish to divide 18 identical chocolate bars equally amongst 30 children. How many cuts *must* be made, and how many pieces *must* there be? Generalize!

Potatoes (1978 p12, 1979 p17)



Traffic Flow (1978 p12, 1979 p 17)



Drilled

A soft block of wood 12 cm by 20 cm and variously described as a rectangular parallelepiped or a cuboid, has a very fine (mathematical) hole drilled between one pair of opposite corners. The block is then sliced into centimetre cubes. How many cubes have a hole in them?

Right-Angled

Given the number of sides of a polygon, what is the maximum number of right angles it can have?

M101 1982
ADDITIONAL PROBLEMS

Lifetime

Walking down the street in my home town, I realized that I had worked at the OU for one third of my life. How much longer will I have to work there until I have worked there for half my life? Generalize.

Rolling Coins

Place one of two identical coins on a table (possibly stuck down in some way). Place the second coin on the table beside the first, touching it, with the Queen's head showing. How many times does the Queen's head rotate as the second coin is rolled around the first fixed coin (always staying in contact)? Generalize to different sized coins. Extend to three or more coins with their centres in line all in contact, and extend that to one or more coins having infinite radius.

Sliding Shapes

Place a coin or shape (plastic or card, triangle, rectangle, etc.) on a table, possible stuck down in some way. Now place a second shape (not necessarily the same as the first) on the table beside the first, and in contact with it. Mark a point on the second shape. Slide the second shape (never rotating it at all), so that it remains in contact with the first fixed shape, and slides all the way round. What is the perimeter of the curve followed by the marked point during the sliding?

What happens if you choose a different marked point?

The Clock Problem

When my son was born, my wife decided that if he cried in the night before 5 am she would go and attend to him, otherwise I was to go and get him. In what seemed like the middle of the night he awoke, and she, having looked at the clock and seen it was 5:30 am, sent me to get him. In the morning it transpired that the clock had been upside down. Will an upside down clock ever show a proper clock time? If so, when; if not, how close can it get to being a proper clock time?

Will a clock seen in a mirror ever show a proper clock time? If so, when; if not, how close can it get to a proper clock time?

Coming to Summer School

I reckoned on an average of 40 mph in order to get to Summer School on time. By halfway I'd managed only 25 mph. How fast did I need to go to get here on time? Generalize.

Traffic Flow

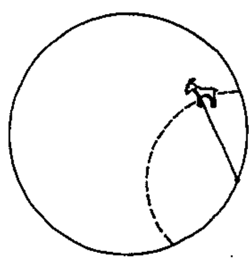
The highway code suggests a stopping distance made up of thinking distance (equal in magnitude in feet to the speed in mph) and a breaking distance of $v^2/20$. A frequently used rule of thumb is one car length for each 10 miles per hour. How well do these rules match?

Prisoner Probabilities

Three prisoners, *A*, *B*, and *C* are informed by a friendly guard that one (and only one) of them is to be pardoned, but that it has been decreed that they are not to know which until an announcement is made at a pardoning ceremony. Each reasons that his probability of being pardoned is 1/3. We may assume that each has been imprisoned for participating equally in the same crime. Prisoner *A* suggests to the guard that since at least one of the other two prisoners has not been pardoned, the guard can safely tell him (*A*) which one, without changing the situation in anyway. The guard agrees and announces to *A* that *B* is not to be pardoned. Prisoner *A* now reasons that his chances have just changed to 1/2, even though the 'situation' has not changed.

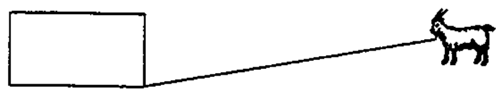
Account for this apparent sudden change of probability. Generalize to *p* prisoners, with *n* not to be pardoned.

Tethered Goat



A goat is tethered to the edge of a circular paddock, with a lead adjusted so that the goat can eat just half of the field. How long is the tether?

Another goat is tethered to the outside of a rectangular shed. What area of grass can it reach?



Yet another is tethered to the outside of a circular silo. What area of grass can it reach?

Compasses Only

You are marooned on a desert island with only a pair of compasses for your mathematical investigations. You adopt a particular length as your unit. Can you draw any integral length? (Careful, you don't have a straight edge!) Can you draw the vertices of a square given two adjacent vertices? What square-root lengths can you draw (using, for example, Pythagoras)?

INVESTIGATIONS

These questions are offered as starting points for investigation. Each one is potentially interesting in its own right, but the primary purpose for working on them is to reveal clearly the fundamental mathematical processes of

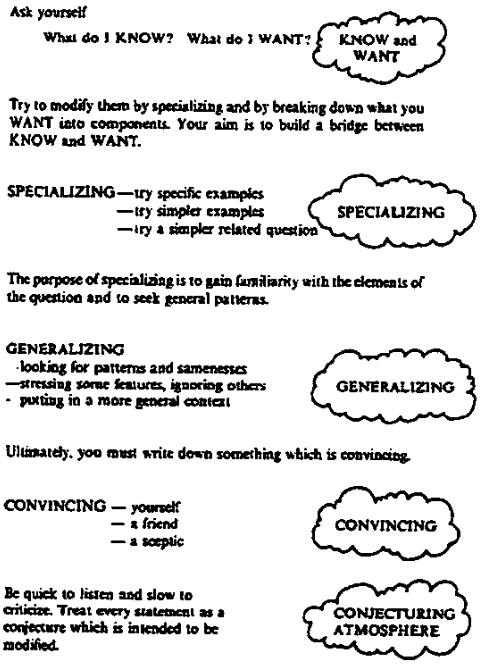
> Specializing, Generalizing and Convincing

which are equally important in studying and doing mathematics.

It is more important to explain ytour thinking than to reach a complete solution.

Take every opportunity to express your conjectures and uncertainties to others. In a truly conjecturing atmosphere, *all* statments are expected to need modifying!

Stuck?



STARTERS

N1 Sequential

Specify a general rule for extending each sequence. Which numbers arise from the calculations?

Convince someone!

$(3 \times 5) + 1,$

$(4 \times 6) + 1,$

$(5 \times 7) + 1, \ldots ?$

$3(5^2 - 4) + 1,$

$4(6^2 - 5) + 1,$

$5(7^2 - 6) + 1, \ldots ?$

$(4^2 - 1)(5^2 - 1) + 1,$

$(6^2 - 1)(7^2 - 1) + 1,$

$(8^2 - 1)(9^2 - 1) + 1, \ldots ?$

Which numbers arise from the following calculations?

> Four more than the product of two numbers differing by four.
>
> One more than the product of four consecutive numbers.

N2 Divide and Conquer

Is it always true that a large number divided by a small number must be larger than a small number divided by a larger number? Convince someone!

G1 Painted Cube

A wooden cube is painted red, and then sliced into a lot of little cubes all the same size. How many of the little cubes have paint on one, two, ..., faces?

G2 Paper Strip

A long thin strip of paper is folded so that it is half as long, and creased. The action of folding in half is repeated several times. How many creases will be in the strip when it is unfolded?

MP1 Socked Out

My sock drawer contains 10 identical loose brown socks and 10 identical loose black socks. Rising early one morning and not wishing to disturb my spouse by putting on a light, I reach into the drawer in search of a matching pair. Since I cannot see them, how many must I take out to be sure that when I get into the light I will have a pair? What if I want to be sure of a pair of brown?

Generalize (e.g. more colours, more feet, ...).

MP2 Painted Tyres

Once, while riding my bicycle along a path, I went through a patch of wet paint that had been recently spilled. A short while later, I looked back at the marks on the path left by the paint on my tyres. What did I see?

NUMERICAL

N3 Drop Out

$40^2 = 1600$; drop out the 6 to get $100 = 10^2$

$39^2 = 1521$; drop out the 5 to get $121 = 11^2$

$38^2 = 1444$; drop out the 4 to get $144 = 12^2$

... ?

What about working in different bases?

N4 One Sum (1982 p22)

N5 Productive Difference

Is it always true that if the difference of two numbers is even, then their product is a difference of two squares? Convince!

N6 Rooted

Generalize the statement that because 60 is closer to 64 than it is to 49, $\sqrt{60}$ is closer to 8 than it is to 7. Is your generalization true? Convince!

N7 Divisible

Consider carefully the following statements.

$n^2 - n$ is always divisible by 2.
$n^3 - n$ is always divisible by 3.
$n^4 - n$ is always divisible by 4
...

Conjecture, modify and convince!

N8 Two Cubes

Place the following facts in a general context, and prove your conjecture.

$$2 + 3 + 4 = 1 + 8$$

$$5 + 6 + 7 + 8 + 9 = 8 + 27$$

$$10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64$$

N9 Square Root Button

Investigate the iterative process $x_{n+1} = \sqrt{\sqrt{\lambda x_n}}$ for various values of λ. Generalize to other processes which use primarily the $\sqrt{\ }$ button on your calculator.

N10 Exponential

Investigate the iterative process $x_{n+1} = \lambda e^{x_n}$ for various values of λ.

N11 Medieval Eggs (1979 p 20)

N12 Integral

I know that

$$\int \frac{dx}{x} = \log_e(x) + C$$

and that $\int \frac{dx}{x^2+1} = \tan^{-1}(x) + C.$

Investigate which integrals of the form polynomial over polynomial can be done explicitly. Give a recipe for doing them.

N13 Consecutive Sums

Some numbers can be expressed in several ways as the sum of consecutive numbers. For example:

$9 = 2 + 3 + 4$
$15 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5$
$21 = 6 + 7 + 8 = 1 + 2 + 3 + 4 + 5 + 6$

Which numbers can be so expressed, and in how many ways?

N14 Fibonacci Matrices

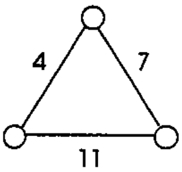
Successive members of the iterative process $x_0 = 1, x_1 = 1, x_{n+2} = x_n + x_{n+1}$ can be represented by

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$$

Call the square matrix **M**. Investigate the meaning and entries of $\mathbf{M}^t$ for t = 1, 2, 3, ... Investigate and interpret

$$\mathbf{M}^t + (-\mathbf{M})^t; \quad \mathbf{M}^{2t}, \mathbf{M}^{3t}, \ldots$$

G3 Arithmagons (1982 p22)



Hidden at each vertex of the triangle is a number.

The edge numbers are the sums of the numbers of adjacent vertices.

Can you reconstruct the numbers?

Generalize to a quick rule for recovering the hidden numbers. When will one or more of the hidden numbers be negative or fractional?

Generalize (e.g. more vertices, more complicated networks, other operations, ...).

G4 Dotty

On square dotted paper, can you find an equilateral triangle? Can you find a square of area 10? Generalize.

G5 Reflective

Select three points, *A*, *B* and *C*.

Reflect *A* in *B*. Reflect that in *C*. Reflect that in *A*. Keep going, reflecting in *A*, *B*, and *C* in turn. What happens and why?

Now reflect *B* in *C* and keep going.

Formulate some questions. Select one and make some conjectures. Convince!

Generalize (e.g. more points, three dimensions, line segments in place of points, ...).

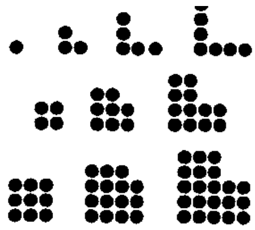
G6 Similitude

What shapes of paper have the property that they can be cut in half by a straight line to yield two pieces, each similar to the original? Convince someone that you have not omitted any! Generalize!

G7 Right-angled (1982 p22)

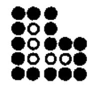
G8 Dots and Frames

For each sequence, write down a rule for generating further members of the sequence. Find an expression for the number of dots needed to make up a general member of each sequence.

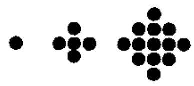


Generalize! Convince someone that your formula is correct by direct reference to the diagrams.

How many circles do you need to make a frame around each member in a sequence? For example:



Now try the sequence



and its frames.

Make up your own! Swap with other people!

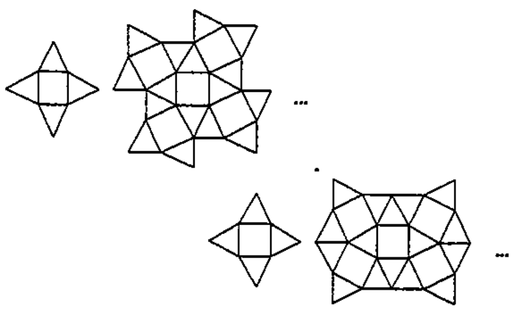
G9 Polygons (1978 p14,)

G10 Regional

A circle has a number of chords drawn on it. What is the maximum number of regions there can be? What about maximum and minimum numbers of crossings?

G11 Tri-square Tess

Formulate a rule for extending these sequences of patterns indefinitely. How many squares, triangles, edges and vertices will there be at the *n*th stage?

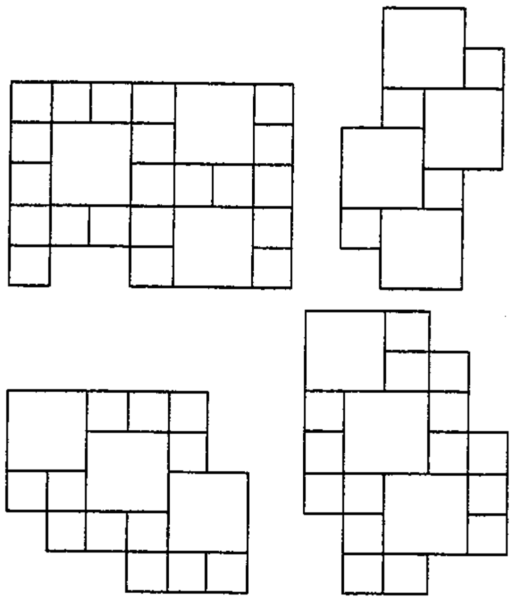


Find other rules for covering the plane with squares and triangles.

G12 Square Tess

The diagrams below suggest four ways to glue squares by size 1 by 1 and 2 by 2 together along their edges. In each case, formulate a rule for continuing the pattern to cover the plane. Convince someone that your rule will work.

How many other ways can you find? Develop a notation to enable you to describe them succinctly.



G13 Deltahedra

A deltahedron is an object made from equilateral triangles glued together so that each edge of each triangle is glued to the edge of exactly one other triangle.

Can a deltahedron be made from exactly 13 triangles? Generalize!

How many distinct deltahedra can be made with the same number of triangles?

How many triangles are needed to make a deltahedron with a hole?

What happens if every vertex is surrounded by the same number of triangles?

Are you convinced that your deltahedra actually exists (and are not the results of inaccurate gluing)?

MARKET PLACE

MP3 Car Hire

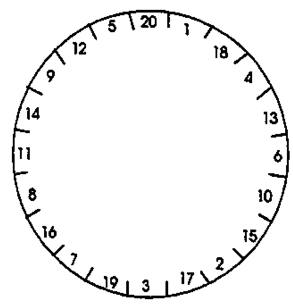
You need to hire a car for a period. Which option should you choose under which circumstance?

Suggest, with convincing reasons, what rates should be put in the blank spaces. Each row is a different type of car.

1-2 days per day; 200 free miles per day	+ mpd	Fri 5pm-Mon 9am; 500 free miles	+ mpd	3days +, per day; free miles	Per week; free miles
£14.00	11p	£30.00	11p	£14.00	£77.00
£16.00	13p	£33.00	13p	£16.00	£87.50
£18.00	14p	£39.00	14p	£18.00	£99.75
£19.00	15p	£42.00	15p	£19.00	£105.
£24.50	19p		19p		£143.
	21p	£66.00	21p		£168.
£47.5	33p	£102.	33p		

MP4 Dartboard

What should be the criteria for the design of a fair and interesting dartboard? The usual one is show below. Can you design one that better fits your criteria?



MP5 Potatoes (1979 p20)

For those who can still face them!

MP6 Oranges

Special price! Oranges at 9p each. Then I noticed that they were rather smaller, about 6 cm, compared to my usual 7.5 cm size which cost 16p. Is the price special? Can you develop a rule of thumb for distinguishing a bargain?

MP7 Shifty (1979 p14)

MP8 Cat's-eyes

The thump of a tyre as it passes over a cat's-eye between lanes in a motorway can be annoying and is probably not helpful to the tyre. Advise someone, with convincing reasons, how to change lanes without the thump.

MP9 Stagecoach (1978 p13)

MP10 Fare's Fair

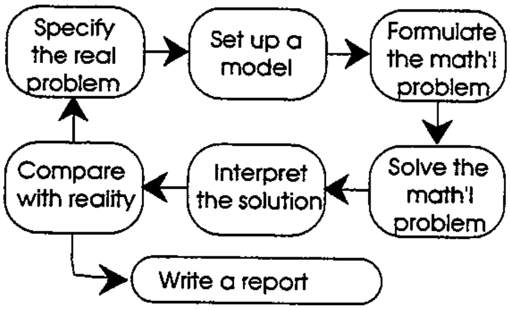
I wish to divide 18 identical chocolate bars equally amongst 30 children. How many cuts must be made, and how many pieces must there be? Generalize!

MP11 Sugar

I wish to weigh out 2 kg of sugar using a 1 kg weight and some old balance-scales with two pans, but the fulcrum is no longer in the middle. I put the weight on one side and sugar to balance on the other, which I put into a fresh bag, then transfer the weight to the other side and put sugar to balance. Putting that also in the bag, do I have less than, more than or exactly 2 kg of sugar?

MP12 Ceilings (1978 p12)

Modelling Diagram



M101 1992 INVESTIGATIONS

INTRODUCTION

The purpose of the Investigations sessions

Each activity in this booklet offers an opportunity for investigation. Some activities require only pencil and paper work, some involve using your calculator, some involve working with a spreadsheet package on the computer.

Each activity is potentially interesting in its own right. Collectively however, the primary purpose for working on them is to emphasize the fundamental mathematical processes of

- specializing, generalizing, conjecturing and convincing

which are so important in studying and doing mathematics. During each of the Investigations sessions you will be encouraged to make explicit use of these processes and to reflect on the insight gained as a result. You should not expect this to be easy. Remember above all that in these sessions

- it is more important to explain your thinking than to reach a complete resolution.

Take every opportunity to express your conjectures and uncertainties out loud to others. In a truly conjecturing atmosphere all statements are expected to need modifying! This is reflected in the mark-scheme of the TMA that you will be tackling towards the end of the week.

Some suggestions for getting started

Specializing

The purpose of specializing is to gain familiarity with the elements of the question and to seek patterns.

How do I specialize?

- Try specific examples.
- Try simpler examples.
- Try a simpler, related question.
- Try to be systematic in order to reveal patterns.

Generalizing

How do I generalize?

- Look for patterns and sameness.
- Stress some features and ignore others.
- Try to put things in a more general context.

Conjecturing

At some stage you will want to express your findings in symbols or perhaps words. Such a statement forms a conjecture. In working towards a resolution you may well go from conjecture to (modified) conjecture. Each contributes something towards your understanding of the problem (even if it is subsequently discarded) so do remember to record them.

How do I form a conjecture?

- The act of generalizing often suggests conjectures.
- Be quick to listen and slow to criticize. Treat every statement as a conjecture which is intended to be modified.

Convincing

Ultimately, you must try to write down something which is convincing, and it is very important to be aware of the need to convince even if the task of doing so proves to be too difficult. There are two types of convincing.

- Convincing using an informal argument (convincing, say, yourself).
- Convincing using a more formal argument (convincing, say, your tutor).

You should always aim to convince yourself and you should always be prepared at least to try to convince your tutor.

What happens if I'm STUCK?

A common plea!

- Ask yourself 'What do I KNOW?' and 'What do I WANT?'
- Your aim is to build a bridge between KNOW and WANT. Try to modify KNOW and WANT by specializing and by breaking down what you WANT into components.

When should I move on to another activity?

- When I have formulated a satisfying resolution.
- When I can explain this to someone else.
- When I have exhausted my interest in the one that I'm working on now (I can always come back to this later).

STARTERS FOR SUNDAY

N1 Sequential (1983 p24)

N2 Divide and Conquer (1983 p24)

N3 Fractions

Notice that

$$2\left(\tfrac{1}{3}\right) + 1 = 2 - \left(\tfrac{1}{3}\right)$$

$$3\left(\tfrac{2}{4}\right) + 1 = 3 - \left(\tfrac{2}{4}\right)$$

$$4\left(\tfrac{3}{5}\right) + 1 = 4 - \left(\tfrac{3}{5}\right)$$

Generalize and form a conjecture. Do you need to modify your conjecture to take into account the following observations?

$$3\left(\tfrac{1}{5}\right) + 2 = 3 - 2\left(\tfrac{1}{5}\right)$$

$$4\left(\tfrac{2}{6}\right) + 2 = 4 - 2\left(\tfrac{2}{6}\right)$$

$$5\left(\tfrac{3}{7}\right) + 2 = 5 - 2\left(\tfrac{3}{7}\right)$$

N4 Smaller and Smaller

Carry on the following sequence:

$7 \times 8 > 6 \times 9 > 5 \times 10 > \ldots$

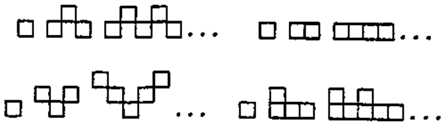
Will it always work? Generalize and convince someone.

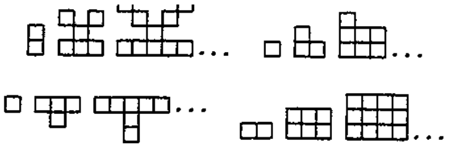
G1 Painted Cube (1983 p24)

G2 Paper Strip (1983 p24)

G3 Beginnings

Choose one of the following sequences. Describe in words how to make the next, and the next, and the next ... terms in your sequence.





Express the number of squares or edges or vertices needed to make up a general term. Convince someone that your conjecture always holds.

Make up some more sequences to explore in the same way.

MP1 Socked Out (1983 p22)

MP2 Painted Tyres (1983 p42)

MP3 Junctions

How many junctions will there be in a network of roads which cross each other two at a time?

What is the maximum (minimum) number of crossings obtainable with a given number of straight lines? Do you permit more that two lines to cross at a point?

SPREADSHEETS FOR MONDAY

Tabled

Construct a multiplication table using fill-down and fill-right.

Create a spreadsheet multiplication table using modular arithmetic.

Do not just treat this as a programming exercise – try to develop your sense of conjecturing by identifying what remains the same and what alters as you move to a new modulus. Work especially hard at making notes, either mentally or preferably on paper, of what you conjecture. It is often the case that a mathematician will feel quite strongly about some conjecture, which then fails on being tested. It is a great pity to lose the insight into the strength of your conviction, just because it is 'wrong' and you forget about it!.

Accumulated Interest

Create a spreadsheet to display the payments, interest and outstanding debt on a loan of £5000 at 15% annual interest with 24 monthly payments of £200.

Extend it to show monthly interest. each month, one-twelfth of the annual interest is added to the outstanding debt, then that month's payment is subtracted.

Compare the two repayment schemes. How much can you reduce the monthly repayment and still pay of f in exactly 24 months? In 'real life', do these differences matter to you? Are there any surprises?

Compare these with the straightforward repayment mortgage in which each year is treated separately. At the start if a year, the amount owing is subject to interest. For the rest of the year, although the amount owing reduces, this is not reflected in the interest calculation, which has been done at the beginning of the year. Compare this method with the other two for a mortgage of £40,000 at 15% over 25 years.

> What is the effect of paying an extra £10 per month over and above the required repayment?
>
> How do the three models compare? What are the total costs and the hidden interest rates?
>
> If the interest rate rises, how much leeway do you have in each of the three models before the loan becomes impossible to pay off?
>
> What is the true rate of interest (the real of interest required in model 1) of the repayment mortgage?
>
> Can you design and model a fairer way of borrowing money?

Dicing With Chance

Create a spreadsheet which will simulate the throw of two dice. Make one of them a regular die, and give the other three 6's and three 0's. Use the spreadsheet to explore:

> What scores are possible (a throw means one of each die thrown and the scores added).
>
> Which score is the most likely?
>
> What do you expect the distribution to be?

Can you design a pair of 20-sided dice with the same probabilities for the integers 1 – 40. What about a pair of 12-sided dice? A pair of 5-sided dice?

Calculus by Numbers

Start with the specific function $f(x) = x^3 - x$. Create a spreadsheet to display the values of x in a given interval $[a, b]$ with a given step size, together with the associated values of $f(x)$ and $f(x + h)$ for a specified value of h, together with the slope of the chord between $(x, f(x))$ and $(x + h, f(x + h))$, the slope of f at x, that is, the derivative of f at x, $f'(x)$ and the difference between the slope and the derivative.

Where is the approximation best?

How does this depend on the interval chosen for x?

How small would h have to be to achieve, say, four decimal place agreement between chord slope and slope at the point?

Try other functions.

One important skill in mathematics is learning to ask the right questions. This investigation is an opportunity to practice that skill. Try to be explicit about what your questions are and make conjectures about what you expect. Then experiment with the spreadsheet in order to convince yourself. You may need to refine your questions after the experiments, and the accuracy of your conjectures should improve.

INVESTIGATIONS FOR TUESDAY

Unchanged from 1983

M101 1994 INVESTIGATIONS

Unchanged from 1992.

SOME LEFT OVERS!

Quadratic Forms

Construct a spreadsheet table to compute the value of $ax^2 + bxy + cy^2$ modulo m as a generalized 'multiplication' table for x and $y = 0, 1, 2, \ldots$. You will of course have to choose specific values for a, b, c, and m. Try varying m and seeing what patterns emerge in your table.

Digital Roots

The digital root of a number expressed in base ten is found by adding up the digits, then adding up the digits of the answer, and continuing this process until there is a single digit answer.

How are the digital root of the sum of two numbers and the sum of their digital roots related?

How are the digital root of a product of two numbers and the product of the digital roots related?

Do digital roots have any other properties that are not deducible from these two? For what k is it the case that the digital root of the kth power of a number is also the digital root of the number?

What happens if you change the base of the representation?

What other functions have the properties of a digital root?

Street Lamps

Why are lamp standards so high on motorways? More precisely, what height, power of lamp and distance apart optimizes a given minimum level of light on a roadway?

Gear Configurations

You are given access to a large supply of gears, all mutually meshable. Which polygons can have gears centred on their vertices so that adjacent gears mesh, and all the gears rotate when one rotates? What planar configurations of gears are such that rotating any one gear will rotate all the others?
Generalize to frictionfull spheres with centres located at fixed points in space.

Eight Digit Construction

Construct an eight digit number (base ten) which is divisible by 999, has leading digit 1, and fifth digit less than 8, or show that it cannot be done.

Integral Sided Triangles

How many integral sided triangles can be made with perimeter p units?

Switches

Given $2n$ points in a row, with the first n coloured red, and the second n coloured blue, a move consists of selecting k consecutive points (k is fixed for any one problem) and reversing their order. Under what conditions can you interchange the reds and the blues? Under what conditions can you arrange the reds and blues to be alternating in colour? (Start with $k = 2$)

Suppose that there is a square array of dots, with half coloured red and half blue. A move consist of selecting four adjacent dots in a two-by-two square and rotating that square through a multiple of 90 degrees about its centre. Can you arrange for all the reds to be in one half of the square and all the blues in the other? Or can you get a checkerboard arrangement?

Mirrors

If you approach a mirror, how fast does your image appear to approach you? If the mirror moves towards you, same question.

How small can a mirror be and still enable a 6 ft person to see their whole length in it?

The Lecture

A lecturer informs her audience that the lecture will last for a micro-century. Could you sit still that long?

Sticks and Angles

Given an unlimited supply of sticks of fixed length and angle-makers at a fixed angle, can you arrange to make a closed polygon? Does confining yourself to the plane matter?

Figurate Differences

Which numbers are the differences of two triangular numbers? Of two square numbers? Of two pentagonal numbers? Generalize.

Sequence Reduction

You are given a finite sequence of 0's and 1's. Underneath it you write down a 0 between two consecutive terms if they are the same, and a 1 if they are different. Your task is to predict from the original sequence what single symbol will appear in the last row.

Snowplow

It is snowing uniformly. At noon a snowplow sets out and covers two miles in the first hour and one mile in the following hour. When did it start snowing?

Dominoes and Attribute Blocks

How many dominoes are there in a set which has up to p pips? What about tri-minoes (three active edges) etc.?

How many different rings of dominoes are possible with a set having up to p pips?

Attribute blocks are shapes which offer, say, two sizes, two colours, and two thicknesses. In how many ways can these blocks be arranged in a circle so that adjacent blocks differ in exactly one attribute? In exactly two attributes? Generalize.

In how many ways can the attribute blocks be arranged in a rectangle so that adjacent blocks differ in exactly one attribute (and going off one edge brings you back on the opposite edge)?

Among

Among the poets there is just one painter. How many painters are there among the poets?

Is it true that among any three whole numbers, the sum of some two of them must be even? Generalize!

INDEX OF PROBLEM NAMES

Some problems were slightly modified and sometimes renamed. I have tried to list variants under their different names, so a given page number may have only a similar problem with a different name to the one you seek.